1. Jackson Prob. 6.4: A uniformly magnetized spherical conductor of radius R and magnetic moment $\boldsymbol{m} = (4\pi R^3/3)\boldsymbol{M}$ rotates about its magnetization axis (z) with angular velocity ω. In the steady state, no current flows in the conductor. There is no excess charge on the sphere.

   (a) Show that the motion induces an electric field and a uniform charge density $\varrho = -m\omega/\pi c^2 R^3$.

   Inside the magnetized sphere there is uniform magnetic induction

$$B_z = \frac{2\mu_0}{3} M.$$

   A unit charge moving in this field experiences an electromotive force $\boldsymbol{f} = \boldsymbol{v} \times \boldsymbol{B}$ leading to a charge separation in the conductor; positive charges move in the direction of $\boldsymbol{f}$ negative charges move in the opposite direction. In equilibrium, this charge separation leads to an electric field that precisely cancels f. Thus, we expect an electric field

$$\boldsymbol{E} = -[\boldsymbol{v} \times \boldsymbol{B}] = -\omega r \sin\theta B \hat{\rho} = -\omega r \sin\theta B \left[\sin\theta \hat{r} + \cos\theta \hat{\theta}\right]$$

   to arise in the conductor. Here $\hat{\rho}$ is the unit vector directed radially outward from the axis. From Gauss's law, we find that the charge density inside the conductor is

$$\varrho = \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \epsilon_0 \left[\frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta}\right] = -2\epsilon_0 \omega B$$

   This charge density is uniform and may be rewritten in the form

$$\varrho = -2\epsilon_0\omega B = -\frac{4\epsilon_0\mu_0\omega}{3} M = -\frac{4\omega}{3c^2}\frac{3m}{4\pi R^3} = -\frac{\omega m}{\pi c^2 R^3}.$$

   Note: This gives a negative volume charge. There must be a compensating positive charge on the surface since the sphere has no excess charge.

   (b) Show that the electric field outside the sphere has quadrupole symmetry.

   Let's set up a boundary value problem to determine the outside field. Outside the sphere, the field can be expanded in spherical harmonics. Inside the sphere, we can no longer assume a spherical harmonic expansion, since the potential no longer satisfies the Laplace equation ($\varrho \neq 0$). However, we can still find the potential inside:

$$\Phi^{\text{in}} = \Phi_0 - \int^\rho E_\rho d\rho = \Phi_0 + \omega B \int^\rho \rho d\rho = \Phi_0 + \frac{\omega B}{2}\rho^2 = \Phi_0 + \frac{\omega B}{2} r^2 \sin^2\theta$$

Using the fact that $\sin^2\theta = (2/3)[1 - P_2(\cos\theta)]$, we may write

$$\Phi^{\text{in}} = \Phi_0 + \frac{\omega B}{3} r^2 - \frac{\omega B}{3} r^2 P_2(\cos\theta)$$

The first two terms will match with a monopole potential outside the sphere and third will match with a quadrupole potential outside. Owing to the fact that the sphere is electrically neutral, there is no monopole term outside. Therefore, outside the sphere

$$\Phi^{\text{out}} = \frac{A}{r^3} P_2(\cos\theta)$$

Matching terms on the the boundary,

$$\Phi_0 + \frac{\omega B}{3} R^2 = 0 \qquad \text{and} \qquad A = -\frac{\omega B}{3} R^5$$

It should be noted that there is only one component of the quadrupole potential. That is only possible if all off-diagonal terms vanish and if the diagonal terms are related by $Q_{33} = -2Q_{11} = -2Q_{22}$. In that case we find

$$\begin{aligned}
\Phi &= \frac{1}{4\pi}\frac{1}{2}\left[\frac{\cos^2\theta}{r^3}Q_{33} - \frac{\sin^2\theta\cos^2\phi}{r^3}\frac{Q_{33}}{2} - \frac{\sin^2\theta\sin^2\phi}{r^3}\frac{Q_{33}}{2}\right] \\
&= \frac{1}{8\pi}\frac{Q_{33}}{r^3}P_2(\cos\theta)
\end{aligned}$$

It follows that

$$Q_{33} = 8\pi A = -\frac{8\pi\omega B}{3}R^5 = -\frac{4}{3}\mu_0\omega m R^2$$

(c) Show that the surface charge density is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4\omega m}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

We know that the surface charge density is

$$\begin{aligned}
\sigma = &\; -\epsilon_0\left[\frac{\partial\Phi^{\text{out}}}{\partial r} - \frac{\partial\Phi^{\text{in}}}{\partial r}\right]_{r=a} \\
&= \frac{2\epsilon_0\omega BR}{3} - \frac{5\epsilon_0\omega BR}{3}P_2(\cos\theta) \\
&= \frac{2\epsilon_0\omega BR}{3}\left[1 - \frac{5}{2}P_2(\cos\theta)\right] \\
&= \frac{1}{4\pi R^2}\frac{4\omega m}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]
\end{aligned}$$

Note that the integrated surface charge is

$$Q^{\rm surf} = \frac{4\omega m}{3c^2}$$

and the integrated volume charge is

$$Q^{\rm vol} = \frac{4\pi R^3}{3}\varrho = -\frac{4\pi R^3}{3}\frac{\omega m}{\pi c^2 R^3} = -\frac{4\omega m}{3c^2}.$$

Therefore the total charge on the sphere vanishes, as it should.

2. Jackson Prob. 6.5: A localized charge distribution produces a field $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$. A small localized time-independent current $\boldsymbol{J}$ is introduced into the field.

   (a) Show that

   $$\boldsymbol{P} = \frac{1}{c^2}\int d^3r\,\Phi(\boldsymbol{r})\,\boldsymbol{J}(\boldsymbol{r}).$$

   Proof:

   $$\begin{aligned} P_i &= \frac{1}{c^2}\,\epsilon_{ijk}\int d^3r E_j H_k = -\frac{1}{c^2}\,\epsilon_{ijk}\int d^3r\,[\nabla_j\Phi]H_k \\ &= -\frac{1}{c^2}\,\epsilon_{ijk}\int d^3r\left[\frac{\partial(\Phi H_k)}{\partial r_j} + \Phi\frac{\partial H_k}{\partial r_k}\right] \\ &= -\frac{1}{c^2}\int_S da\,\Phi\,[\hat{n}\times H]_i + \frac{1}{c^2}\int d^3r\,\Phi J_i \end{aligned}$$

   The first integral vanishes for localized charge and current distributions. For such cases, Φ falls off at least as fast as $1/R$ and H falls of at least as fast as $1/R^2$ for large R. Since the surface area grows as R^2, the first term approaches 0 as $1/R$ in the limit $R\to\infty$ and the identity is proved.

   (b) If the region containing $\boldsymbol{J}$ is small compared to the scale of variation of Φ, show that

   $$\boldsymbol{P} = \frac{1}{c^2}\,[\boldsymbol{E}(0)\times\boldsymbol{m}]$$

   Proof: Expand the potential about the center of the current distribution

   $$\Phi = \Phi(0) - \boldsymbol{r}\cdot\boldsymbol{E}(0) + \cdots$$

   The first term does not contribute so

   $$P_k = -\frac{1}{c^2}\,E_l(0)\int d^2r\,r_l J_k = -\frac{1}{c^2}\,\epsilon_{lkm}\,E_l(0)\,m_m = \frac{1}{c^2}\,[\boldsymbol{E}(0)\times\boldsymbol{m}]_k\,,$$

   as was to be proved.

(c) Show (two ways) that for a uniform E field,

$$\boldsymbol{P} = \frac{2}{3c^2}\,[\boldsymbol{E}\times\boldsymbol{m}]$$

Proof: 1st method: In this case the surface integral in the first part of this question does not vanish. Its value is

$$P_i^{\rm surf} = -\frac{1}{c^2}\,\epsilon_{ijk}\int_S da\,\Phi\,\hat{r}_j\,H_k = +\frac{1}{c^2}\,\epsilon_{ijk}E_l\int_S da\,r_l\,\hat{r}_j\,H_k$$

Further, in the dipole approximation

$$H_k = \frac{1}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{r})\hat{r}_k - m_k}{r^3}$$

Since the product $\hat{r}_j\hat{r}_k$ is symmetric with respect to interchange of j and k, while ϵ_{ijk} is antisymmetric, the first term in the numerator does not contribute. Thus,

$$P_i^{\rm surf} = -\frac{1}{4\pi c^2}\,\epsilon_{ijk}E_l\,m_k\int_S\frac{da}{r^4}\,r_l\,r_j.$$

It is easy to show that

$$\int_S\frac{da}{r^4}\,r_l\,r_j = \int\frac{d\Omega}{r^2}\,r_l\,r_j = \frac{4\pi}{3}\,\delta_{lj}$$

Therefore,

$$P_i^{\rm surf} = -\frac{1}{3c^2}\,[\boldsymbol{E}\times\boldsymbol{m}]_i$$

and

$$\boldsymbol{P} = \boldsymbol{P}^{\rm surf} + \boldsymbol{P}^{\rm vol} = \frac{2}{3c^2}\,[\boldsymbol{E}(0)\times\boldsymbol{m}]\,.$$

as was to be shown.

Proof: 2nd method: Start from the basic relation

$$\boldsymbol{P} = \frac{1}{c^2}\int d^3r\,[\boldsymbol{E}\times\boldsymbol{H}] = \frac{1}{c^2}\left[\boldsymbol{E}\times\int d^3r\boldsymbol{H}\right]$$

As in Chap. 5.6 use

$$\begin{aligned}
\int d^3rH_i &= \frac{1}{\mu_0}\,\epsilon_{ijk}\int d^3r\frac{\partial A_k}{\partial r_j} = \frac{1}{\mu_0}\,\epsilon_{ijk}\int_S da\frac{r_j}{r}A_k \\
&= \frac{1}{4\pi}\,\epsilon_{ijk}\int_S da\frac{r_j}{r}\,\epsilon_{kst}\,\frac{m_s r_t}{r^3} = \frac{1}{4\pi}\,[\delta_{is}\delta_{jt} - \delta_{it}\delta_{js}]\,m_s\int_S\frac{da}{r^4}r_jr_t \\
&= \frac{1}{3}\,[\delta_{is}\delta_{jt} - \delta_{it}\delta_{js}]\,\delta_{jt} = \frac{1}{3}\,[3\delta_{is} - \delta_{is}]\,m_s = \frac{2}{3}m_i
\end{aligned}$$

Thus, once again,

$$\boldsymbol{P} = \frac{2}{3c^2}\,[\boldsymbol{E}\times\boldsymbol{m}]$$

3. Jackson Prob 6.8: A dielectric sphere is in a uniform external field directed along the x axis and rotates with angular velocity ω about the z axis. Show that there is an induced magnetic field that is characterized by the magnetic scalar potential

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\,\omega\left(\frac{a}{r_>}\right)^5 xz$$

where $r_>$ is the larger of r and a.

Proof: Start with the expression for the polarization vector from Chap. 4.5.

$$\boldsymbol{P} = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\,\hat{x}.$$

From Eq. 6.100 in the text, one finds that the polarization vector in a medium with bulk velocity v leads to an effective magnetization vector

$$\boldsymbol{M}^{\text{eff}} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{H} = [\boldsymbol{P}\times\boldsymbol{v}]$$

Now, $\boldsymbol{v} = [\boldsymbol{\omega}\times\boldsymbol{r}] = \omega r\sin\theta\hat{\phi}$. It follows that

$$\boldsymbol{M}^{\text{eff}} = \omega\,P\,r\sin\theta\cos\phi\,\hat{z}$$

Set up the boundary-value problem for Φ_M. As a preliminary, note that the "driving" term is M_r is a linear combination of spherical harmonics $Y_{2,\pm1}(\theta,\phi)$. We therefore assume

$$\begin{aligned}\Phi_M^{\text{out}} &= \sum_m \frac{B_{2m}}{r^3}Y_{2m}(\theta,\phi)\\ \Phi_M^{\text{in}} &= \sum_m A_{2m}\,r^2\,Y_{2m}(\theta,\phi)\end{aligned}$$

Continuity of potential at $r=a$ leads to $B_{2m} = a^5 A_{2m}$. Matching radial components of $\boldsymbol{B}$ at $r=a$ leads to

$$\sum_m 3\frac{B_{2m}}{a^4}Y_{2m} = M_r|_{r=a} - \sum_m 2aA_{2m}Y_{2m}$$

Rearranging, we find

$$5\sum_m A_{2m}Y_{2m} = \frac{1}{a}M_r|_{r=a} = \omega\,P\,\cos\theta\sin\theta\cos\phi$$

It follows that inside the sphere

$$\Phi_M^{\text{in}} = \sum_m r^2 A_{2m}Y_{2m} = \frac{1}{5}\omega\,P\,r^2\cos\theta\sin\theta\cos\phi = \frac{1}{5}\omega\,P\,xz$$

From the relation $B_{2m} = a^5 A_{2m}$, it easily follows that

$$\Phi_M^{\text{out}} = \sum_m \frac{B_{2m}}{r^3} Y_{2m} = \frac{1}{5}\omega\, P\, \left(\frac{a}{r}\right)^5\, xz$$

Substituting the earlier value of P, we obtain

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\,\omega\left(\frac{a}{r_>}\right)^5 xz,$$

which is the desired result.

4. Jackson Prob. 6.14: A capacitor in an AC circuit with circular plates (radius a, separation d) is charged by an alternating current $I = I_0 e^{-i\omega t}$.

   (a) Calculate the fields between the plates ignoring fringing:
   Assume that the electric field E is in the z direction and that the magnetic induction B is in the ϕ direction. Two of Maxwell's equations give:

$$\begin{aligned}[\boldsymbol{\nabla}\times\boldsymbol{E}]_\phi &= -\frac{\partial E_z}{\partial\rho} = i\omega B_\phi \\ [\boldsymbol{\nabla}\times\boldsymbol{B}]_z &= \frac{1}{\rho}\frac{\partial\rho B_\phi}{\partial\rho} = -i\frac{\omega}{c^2}E_z\end{aligned}$$

   Substituting from the first into the second, we find

$$\frac{d^2E_z}{d\rho^2} + \frac{1}{\rho}\frac{dE_z}{d\rho} + k^2E_z = 0,$$

   where $k^2 = \omega^2/c^2$. This is Bessel's equation and the solution regular at $\rho = 0$ is

$$E_z(\rho) = AJ_0(k\rho)$$

   Where A is a constant to be determined. The corresponding B field is

$$B_\phi(\rho) = \frac{i}{kc}\frac{dE_z}{d\rho} = -\frac{i}{c}AJ_1(k\rho)$$

   Now, we must determine the constant A. The surface charge density is

$$\sigma(\rho) = \epsilon_0 E_z(\rho) = \epsilon_0 AJ_0(k\rho)$$

   Integrating,we find that the total charge on the plate is

$$Q = 2\pi\epsilon_0 A\int_0^a \rho J_0(k\rho)\,d\rho = 2\pi\epsilon_0\frac{a}{k}AJ_1(ka)$$

   Thus, the constant A is related to the charge $Q_0 = iI_0/\omega$ by

$$A = \frac{kQ_0}{2\pi a\epsilon_0 J_1(ka)}$$

To second order in k, we find

$$E_z^{(2)}(\rho) = \frac{Q_0}{\pi a^2 \epsilon_0}\left[1 + \left(\frac{a^2}{8} - \frac{\rho^2}{4}\right) k^2 + \cdots\right]$$
$$B_\phi^{(2)}(\rho) = \frac{\mu_0 I_0 \rho}{2\pi a^2}\left[1 + \left(\frac{a^2}{8} - \frac{\rho^2}{8}\right) k^2 + \cdots\right],$$

where we have used $I_0 = -i\omega Q_0$.

(b) Now, we can evaluate the electric and magnetic energy stored between the plates in the capacitor. We obtain, through second order

$$w_e^{(2)} = \frac{\epsilon_0}{4}\, 2\pi d \int_0^a \rho\, |E_z|^2\, d\rho = \frac{1}{4\pi\epsilon_0}\frac{|I_0|^2\, d}{\omega^2 a^2}$$
$$w_m^{(2)} = \frac{1}{4\mu_0}\, 2\pi d \int_0^a \rho\, |B_\phi|^2\, d\rho = \frac{\mu_0}{4\pi}\frac{|I_0|^2\, d}{8}\left(1 + \frac{a^2 k^2}{12}\right),$$

as was to be shown.

(c) Find the inductance and capacitance of the capacitor. We use

$$X_L = \omega L = \frac{4\omega}{|I_0|^2}\, w_m = \frac{\omega \mu_0 d}{8\pi}$$
$$X_C = \frac{1}{\omega C} = \frac{4\omega}{|I_0|^2}\, w_e = \frac{d}{\omega \epsilon_0 \pi a^2},$$

where we ignore the order k^2 correction to w_m. It follows that

$$C = \frac{\epsilon_0 \pi a^2}{d}$$
$$L = \frac{\mu_0 d}{8\pi}.$$

The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{8}{\epsilon_0 \mu_0 a^2}} = \sqrt{8}\,\frac{c}{a}$$

The value of $k_0 a = \sqrt{8} \approx 2.83$ differs from the first zero of $J_0(ka)$, which has the value 2.40 by about 20%. This is illustrated in the figure, where we plot $E_z(\rho)$ for a capacitor of radius $a = 1$ at resonance: