1. Jackson Prob. 5.15: Shielded Bifilar Circuit: Two wires carrying oppositely directed currents are surrounded by a cylindrical shell of inner radius a, outer radius b, and relative permeability μ_r.

   (a) Determine the magnetic potential for two wires; the first is located at $x = d/2$ and carries current I in the -z direction and the second is located at $x = -d/2$ and carries current I in the z direction.

   $$\Phi_m = \frac{I}{2\pi} \ln \left(\frac{\sqrt{\rho^2 + d^2/4 - d\rho\cos\phi}}{\sqrt{\rho^2 + d^2/4 + d\rho\cos\phi}} \right) \approx -\frac{I}{2\pi}\frac{\cos\phi}{\rho},$$

   where ϕ is measured counter-clockwise from the x axis.

   (b) Find the potential in the three regions. We may assume that only terms in the expansion of the potential in cylindrical coordinates proportional to $\cos\phi$ contribute:

   $$\begin{aligned} \Phi_m &= \left[A\rho + \frac{\kappa}{\rho} \right] \cos\phi \qquad \rho < a \\ &= \left[B\rho + \frac{C}{\rho} \right] \cos\phi \qquad a < \rho < b \\ &= \frac{D}{\rho}\cos\phi \qquad\qquad b < \rho, \end{aligned}$$

   where $\kappa = -I/(2\pi)$ and where (A, B, C, D) are unknown expansion coefficients to be determined by boundary coefficients on the two surfaces $\rho = a$ and $\rho = b$. These conditions; Φ_m continuous and normal component of $\boldsymbol{B}$ continuous, lead to the equations:

   $$\begin{aligned} A + \frac{\kappa}{a^2} &= B + \frac{C}{a^2} \\ -A + \frac{\kappa}{a^2} &= \mu_r \left(-B + \frac{C}{a^2} \right) \\ B + \frac{C}{b^2} &= \frac{D}{b^2} \\ \mu_r \left(-B + \frac{C}{b^2} \right) &= \frac{D}{b^2} \end{aligned}$$

   Solving, we find

   $$\begin{aligned} \Phi_a &= \left[\frac{(a^2 - b^2)\kappa\left(\mu_r^2 - 1\right)}{a^2\left(b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2\right)}\,\rho + \frac{\kappa}{\rho} \right] \cos\phi \\ \Phi_b &= \left[\frac{2\kappa(\mu_r - 1)}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\,\rho + \frac{2b^2\kappa(\mu_r+1)}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\,\frac{1}{\rho} \right] \cos\phi \\ \Phi_c &= \left[\frac{4b^2\kappa\mu_r}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}\,\frac{1}{\rho} \right] \cos\phi \end{aligned}$$

Substituting into the earlier expression gives explicit results for Φ_m in each region. In particular, outside the shield we find a dipole potential with coefficient proportional to that of the two wires (κ); the coefficient of proportionality is

$$F = \frac{4b^2\mu_r}{b^2(\mu_r+1)^2 - a^2(\mu_r-1)^2}.$$

In problem 5.14, a uniform external field maintained its form but was reduced in strength inside a cylindrical shield. Here an internal dipole field maintains it's form but is reduced in strength outside a cylindrical shield.

(c) For $\mu_r \gg 1$ and $b = a+t$ with $t \ll b$, we find ($\mu_r = 200$, $b = 1.25$ cm, $t = 0.3$ mm)

$$F \approx \frac{2b}{t\mu_r} = 0.417$$

2. Jackson: Prob. 5.24: For a conducting plane with a circular hole and a tangential field $\boldsymbol{H}_0$ on one side:

   (a) Determine $\boldsymbol{H}^{(1)}$ on the side with $\boldsymbol{H}_0$ for $\rho > a$. We have for $z = 0$

$$\Phi^{(1)}(\rho,\phi) = \frac{2aH_0}{\pi}\int_0^\infty j_1(ka)J_1(k\rho)dk\,\sin\phi$$
$$= \frac{H_0}{\pi}\left(\rho\sin^{-1}\left(\frac{a}{\rho}\right) - a\sqrt{1-\frac{a^2}{\rho^2}}\right)\sin\phi. \qquad \rho > a.$$

For $\rho > a$, we find

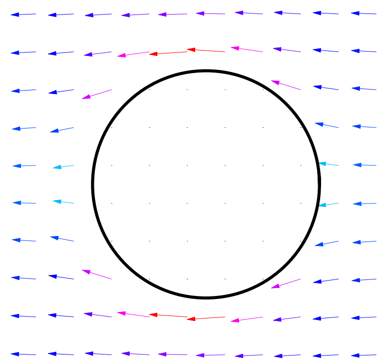
$$H_\rho = -\frac{\partial\Phi^{(1)}}{\partial\rho}$$
$$= \frac{H_0}{\pi}\left[\frac{a^3}{\sqrt{1-\frac{a^2}{\rho^2}}\,\rho^3} + \frac{a}{\sqrt{1-\frac{a^2}{\rho^2}}\,\rho} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\sin\phi$$
$$H_\phi = \frac{\partial\Phi^{(1)}}{\rho\phi}$$
$$= \frac{H_0}{\pi}\left[\frac{a}{\rho}\sqrt{1-\frac{a^2}{\rho^2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\cos\phi$$

$$H_x = H_\rho\cos\phi - H_\phi\sin\phi = \frac{H_0}{\pi}\frac{a^3}{\rho^2}\frac{\sin 2\phi}{\sqrt{\rho^2-a^2}}$$
$$= \frac{2H_0}{\pi}\frac{a^3}{\rho^4}\frac{xy}{\sqrt{\rho^2-a^2}}$$

$$H_y = H_\rho \sin\phi + H_\phi \cos\phi = \frac{H_0}{\pi}\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right) - \frac{a^3}{\rho^2}\frac{\cos 2\phi}{\sqrt{\rho^2 - a^2}}\right]$$
$$= \frac{2H_0}{\pi}\frac{a^3}{\rho^4}\frac{y^2}{\sqrt{\rho^2 - a^2}} + \frac{H_0}{\pi}\left[\frac{a}{\rho^2}\sqrt{\rho^2 - a^2} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]$$

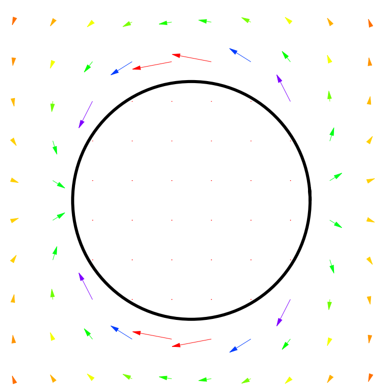
(b) Sketch the surface currents above and below the plane. Above the plane both $\mathbf{H_0}$ and $\mathbf{H^{(1)}}$ contribute to the current:

$$(K_x,\, K_y) = (-H_0 - H_y^{(1)},\, H_x^{(1)})$$



while below, only $\mathbf{H^{(1)}}$ contributes:

$$(K_x,\, K_y) = (-H_y^{(1)},\, H_x^{(1)})$$



3. Jackson Prob. 5.25: A rectangular loop carrying current I_1 interacts with a wire carrying current I_2. The center of the loop is a distance d from the wire and two sides of the loop of length a are parallel to the wire; the sides of length b make angle α with the plane of the wire and the line from the wire to the center of the loop The direction of the current in the side nearest the wire is in the same direction as I_2. Set up a coordinate system with the loop in the xy plane and center of the loop at the origin;

the sides a run parallel to y and are located at $x = \pm b/2$; the sides b are parallel to the x axis. The wire located at $z = d\sin\alpha$, $x = d\cos\alpha$ and I_2 flows along $+y$. In this coordinate system, the vector potential of the wire has only a y component and

$$\boldsymbol{A}_2 = -\frac{I_2}{4\pi}\ln[(x - d\cos\alpha)^2 + (z - d\sin\alpha)^2]\,\hat{y}$$

(a) The interaction energy is

$$\begin{aligned} W_{12} &= I_1 \oint d\boldsymbol{l}_1 \cdot \boldsymbol{A}_2 \\ &= \frac{\mu_0 I_1 I_2}{4\pi} a \ln\left[\frac{(-b/2 - d\cos\alpha)^2 + (-d\sin\alpha)^2}{(b/2 - d\cos\alpha)^2 + (-d\sin\alpha)^2}\right] \\ &= \frac{\mu_0 I_1 I_2}{4\pi} a \ln\left[\frac{4d^2 + b^2 + 4db\cos\alpha}{4d^2 + b^2 - 4db\cos\alpha}\right] \end{aligned}$$

where only the two sides parallel to y contribute.

(b) Calculate the force on the loop We have in the xy plane

$$\begin{aligned} B_x(x,0) &= -\left.\frac{\partial A_y}{\partial z}\right|_{z=0} = \frac{-d\sin\alpha}{(x - d\cos\alpha)^2 + d^2\sin^2\alpha} \\ B_z(x,0) &= \left.\frac{\partial A_y}{\partial x}\right|_{z=0} = \frac{(x - d\cos\alpha)}{(x - d\cos\alpha)^2 + d^2\sin^2\alpha} \end{aligned}$$

The force on the two sides of the rectangle of length b precisely cancel. The x and z components of the force on the two sides of length a are

$$\begin{aligned} F_x &= I_1[B_z(b/2,0) - B_z(-b/2,0)] = \frac{2\mu_0 I_1 I_2\, ab\left(4d^2\cos(2\alpha) - b^2\right)}{\pi\left(b^4 - 8d^2\cos(2\alpha)b^2 + 16d^4\right)} \\ F_z &= -I_2[B_x(b/2) - B_z(-b/2,0) = -\frac{8\mu_0 I_1 I_2\, ab\, d^2\sin(2\alpha)}{\pi\left(b^4 - 8d^2\cos(2\alpha)b^2 + 16d^4\right)} \end{aligned}$$

(c) Repeat for the case where the rectangle of sides a, b is replaced by a circle of radius a. In this case, we write

$$W_{12} = I_1 \int_0^{2\pi} a\cos\phi\, A_y(a\cos\phi, 0)\, d\phi,$$

where we have used the fact that $x = a\cos\phi$ and $dl_y = a\cos\phi\, d\phi$ along the circle. We expand the A_y in a series in powers of $1/d$ and carry out the integral term by term to find

$$W_{12} = I_1 I_2 \mu_0 a\left(\frac{\cos(\alpha)a}{2d} + \frac{\cos(3\alpha)a^3}{8d^3} + \frac{\cos(5\alpha)a^5}{16d^5} + \frac{5\cos(7\alpha)a^7}{128d^7} + \frac{7\cos(9\alpha)a^9}{256d^9}\right)$$

The same series results if we evaluate $W_{12} = I_1\Phi_2$, where Φ_2 is the magnetic flux through the circle. Note that the term in parentheses above can be written

$$\Big(\cdots\Big) = \Re\left\{\frac{z}{2} + \frac{z^3}{8} + \frac{z^5}{16} + \frac{5z^7}{128} + \frac{7z^9}{256}\right\},$$

Where

$$z = \frac{a}{d}e^{i\alpha}$$

Moreover,

$$\frac{1-\sqrt{1-z^2}}{z} = \frac{z}{2} + \frac{z^3}{8} + \frac{z^5}{16} + \frac{5z^7}{128} + \frac{7z^9}{256}$$

Thus, we may write

$$W_{12} = I_1 I_2 \mu_0 a\,\Re\left\{\frac{1-\sqrt{1-z^2}}{z}\right\} \quad \text{with} \quad z = \frac{a}{d}e^{i\alpha}$$

This *correct* answer is close (but not identical) to the answer given in the text. Indeed, if we assumed

$$\Re\left\{\frac{1-\sqrt{1-z^2}}{z}\right\} = \frac{1-\sqrt{1-(\Re z)^2}}{\Re z},$$

then we would recover the result in the text.
Find the force.

$$\boldsymbol{F} = \hat{\imath}\, I_1 \int_0^{2\pi} a\cos\phi\, B_z(a\cos\phi, 0)\, d\phi - \hat{k}\, I_1 \int_0^{2\pi} a\cos\phi\, B_x(a\cos\phi, 0)\, d\phi$$

Again, expanding the potential and carrying out the integrations leads to We find

$$F_x = \mu_0 I_1 I_2 \left(\frac{\cos(2\alpha)a^2}{2d^2} + \frac{3\cos(4\alpha)a^4}{8d^4} + \frac{5\cos(6\alpha)a^6}{16d^6} + \frac{35\cos(8\alpha)a^8}{128d^8} + \frac{63\cos(10\alpha)a^{10}}{256d^{10}}\right)$$

$$F_z = \mu_0 I_1 I_2 \left(\frac{\sin(2\alpha)a^2}{2d^2} + \frac{3\sin(4\alpha)a^4}{8d^4} + \frac{5\sin(6\alpha)a^6}{16d^6} + \frac{35\sin(8\alpha)a^8}{128d^8} + \frac{63\sin(10\alpha)a^{10}}{256d^{10}}\right)$$

Again, we can identify the two series: Consider the function

$$G(z) = \frac{1}{\sqrt{1-z^2}} - 1 = \frac{z^2}{2} + \frac{3z^4}{8} + \frac{5z^6}{16} + \frac{35z^8}{128} + \frac{63z^{10}}{256}$$

Comparing, we find

$$F_x = \mu_0 I_1 I_2 \Re G\left(\frac{a}{d}e^{i\alpha}\right)$$
$$F_z = \mu_0 I_1 I_2 \Im G\left(\frac{a}{d}e^{i\alpha}\right)$$

(d) Express the energies for large d in terms of moments of loops. For the rectangular loop:

$$W_{12} = \frac{\mu_0 I_1 I_2}{4\pi} a \ln\left[\frac{4d^2+b^2+4db\cos\alpha}{4d^2+b^2-4db\cos\alpha}\right]$$
$$\to \frac{\mu_0 I_1 I_2}{4\pi} a\ \frac{2b\cos\alpha}{d} = (I_1 ab)\frac{\mu_0 I_2}{2\pi d}\cos\alpha = m_1 B_{2z}$$

For the circular loop:

$$W_{12} = I_1 I_2 \mu_0 a\left(\frac{\cos(\alpha)a}{2d} + \frac{\cos(3\alpha)a^3}{8d^3} + \frac{\cos(5\alpha)a^5}{16d^5} + \cdots\right.$$
$$\to I_1 I_2 \mu_0 a\ \frac{\cos(\alpha)a}{2d} = (I_1 \pi a^2)\frac{\mu_0 I_2}{2\pi d}\cos\alpha = m_1 B_{2z}$$

In both cases, the + sign is a result of the fact that the moment and the normal component of the field are in opposite directions.

4. Jackson Prob. 5.34: Two identical circular loops are located a distance R apart on a common axis,

(a) Find M_{12} using A_ϕ from Prob. 5.10b:

$$A_\phi(\rho,z) = \frac{\mu_0 I_1 a}{2}\int_0^\infty J_1(ka)J_1(k\rho)e^{-k|z|}dk$$

$$W_{12} = I_2\int_0^{2\pi} a\, d\phi A_\phi(a,R) = \mu_0 I_1 I_2 \pi a^2\int_0^\infty J_1(ka)J_1(ka)e^{-kR}dk$$

Leading to the result

$$M_{12} = \mu_0 \pi a^2\int_0^\infty [J_1(ka)]^2 e^{-kR}dk$$

(b) Assuming $a << R$, we obtain an asymptotic series in R by expanding $J_(ka)$ in a power series

$$[J_1(ka)]^2 \approx \frac{a^2k^2}{4} - \frac{a^4k^4}{16} + \frac{5a^6k^6}{768} - \frac{7a^8k^8}{18432}$$

Integrating, we find

$$M_{12} = \frac{\mu_0 a\pi}{2}\left(\frac{a^3}{R^3} - \frac{3a^5}{R^5} + \frac{75a^7}{8R^7} - \frac{245a^9}{8R^9} + \cdots\right)$$

(c) Find the mutual inductance for co-planer loops with centers separated by R. The axial B_z field from the loop centered at the origin is

$$B_z(z) = \frac{\mu_0 I_1}{2} \frac{a^2}{[a^2+z^2]^{3/2}}$$

This field can be derived from a scalar potential

$$\begin{aligned}\Phi_m(z) &= \frac{\mu_0 I_1}{2}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right] \\ &= \frac{\mu_0 I_1}{2}\left(\frac{a^2}{2z^2} - \frac{3a^4}{8z^4} + \frac{5a^6}{16z^6} - \frac{35a^8}{128z^8} + \frac{63a^{10}}{256z^{10}} + \cdots\right).\end{aligned}$$

Analytically continuing the potential leads to

$$\begin{aligned}\Phi_m(r,\theta) = \frac{\mu_0 I_1}{2}\Bigg(&\frac{a^2}{2r^2}P_1(\cos\theta) - \frac{3a^4}{8r^4}P_3(\cos\theta) + \frac{5a^6}{16r^6}P_5(\cos\theta) \\ &- \frac{35a^8}{128r^8}P_7(\cos\theta) + \frac{63a^{10}}{256r^{10}}P_9(\cos\theta) + \cdots\Bigg)\end{aligned}$$

We need B_z at large values of r and $\theta = \pi/2$. We find

$$\begin{aligned}B_z(r,\pi/2) &= \left.\frac{1}{r}\frac{\partial\Phi_m}{\partial\theta}\right|_{\theta=\pi/2} \\ &= \frac{\mu_0 I_1}{2}\left(\frac{a^2}{2r^3} + \frac{9a^4}{16r^5} + \frac{75a^6}{128r^7} + \frac{1225a^8}{2048r^9} + \frac{19845a^{10}}{32768r^{11}}\right)\end{aligned}$$

Introduce the vector ρ centered on the second loop. Then we may write $bmr = \boldsymbol{R} + \boldsymbol{\rho}$, where R is the vecror from the center of the first loop to the center of the second. We may replace

$$r \to \sqrt{R^2 + \rho^2 + 2R\rho\cos\phi}$$

where ϕ is the polar angle with respect to the center of the second loop and carry out a second expansion of B_z with respect to R. With this in hand, we calculate the flux Φ_2 the through the second loop. First, integrating B_z over the polar angle ϕ, we obtain

$$\begin{aligned}\int_0^{2\pi} B_z d\phi = \frac{\mu_0 I_1 \pi}{2}\Bigg\{&\frac{a^2}{R^3} + \left(\frac{9a^4}{8} + \frac{9\rho^2 a^2}{4}\right)\frac{1}{R^5} \\ &+ \left(\frac{75a^6}{64} + \frac{225\rho^2 a^4}{32} + \frac{225\rho^4 a^2}{64}\right)\frac{1}{R^7} \\ &+ \left(\frac{1225a^8}{1024} + \frac{3675\rho^2 a^6}{256} + \frac{11025\rho^4 n a^4}{512} + \frac{1225\rho^6 a^2}{256}\right)\frac{1}{R^9}\Bigg\}\end{aligned}$$

To evaluate the flux through the second loop, we integrate the previous result over ρ

$$\Phi_2 = \int_0^a \rho d\rho \int_0^{2\pi} B_z d\phi = \frac{\mu_0 I_1 \pi a}{4} \left(\frac{a^3}{R^3} + \frac{9a^5}{4R^5} + \frac{375a^7}{64R^7} + \frac{8575a^9}{512R^9} \right)$$

Given that $\Phi_2 = M_{12} I_1$, we may write

$$M_{12} = \frac{\mu_0 \pi a}{4} \left(\frac{a^3}{R^3} + \frac{9a^5}{4R^5} + \frac{375a^7}{64R^7} + \frac{8575a^9}{512R^9} \right)$$

(d) calculate the force in each case. For the co-planar loops, the only non-vanishing component of the force on the second loop is

$$\begin{aligned} F_x &= I_2 \int_0^\infty a \cos\phi B_z(\rho = a, \phi) d\phi \\ &= \frac{\mu_0 I_1 I_2 \pi}{2} \left(\frac{3a^4}{2R^4} + \frac{45a^6}{8R^6} + \frac{2625a^8}{128R^8} + \frac{77175a^{10}}{1024R^{10}} \right) \end{aligned}$$

The force is repulsive and along the line joining the centers of the loops.

In the case of the co-axial loops, only the component F_z of the force on the second loop contributes:

$$F_z = \cos\theta F_r - \sin\theta F_\theta.$$

Now, at the location of the second loop, components of the force are

$$\begin{aligned} F_r &= -2\pi a\, I_2\, B_\theta(r,\theta) \\ F_\theta &= 2\pi a\, I_2\, B_r(r,\theta), \end{aligned}$$

where $r = \sqrt{a^2 + z^2}$ and $\theta = \arccos(z/\sqrt{a^2+z^2})$. Therefore,

$$F_z = -2\pi a\, I_2 \left(\cos\theta B_\theta(r,\theta) - \sin\theta B_r(r,\theta) \right)$$

Substituting and expanding the fields in z, one obtains

$$F_z = I_1 I_2 \mu_0 \pi \left(-\frac{3a^4}{2R^4} + \frac{15a^6}{2R^6} - \frac{525a^8}{16R^8} + \frac{2205a^{10}}{16R^{10}} \right)$$

The force is attractive and along z.