

## 1. Jackson Prob. 1.4:

For each case, we assume that the electric field is directed radially outward from the center of the sphere and depends only on the distance from the center:  $\mathbf{E} = E(r)\hat{r}$ . It follows

$$\int \mathbf{E} \cdot \mathbf{n} da = 4\pi r^2 E(r) = \frac{1}{\epsilon_0} Q_{\text{encl}}$$

from which it follows

$$E(r) = \frac{Q_{\text{encl}}}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r}$$

In all cases,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \quad r \geq a$$

For  $r < a$ , we have

(a) Conducting sphere:

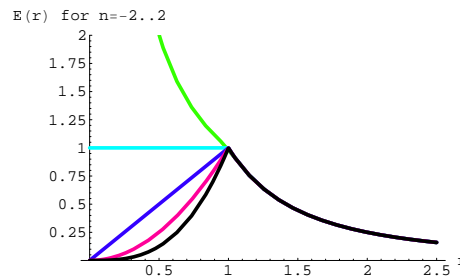
$$\mathbf{E} = 0$$

(b) uniform charge density:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3} \hat{r}$$

(c) charge density proportional to  $r^n$ :

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{r^{n+1}}{a^{n+3}} \hat{r}$$



## 2. Jackson Prob. 1.5:

Solve  $\nabla^2 \Phi = -\rho/\epsilon_0$  where

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{\exp(-2r/a)}{r} \left(1 + \frac{r}{a}\right)$$

to find  $\rho(r)$ . To this end, note that  $\nabla^2\Phi$  in spherical coordinates is

$$\nabla^2\Phi = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d\Phi}{dr} = \frac{1}{r} \frac{d^2}{dr^2} (r\Phi)$$

for a function  $\Phi(r)$  (independent of  $\theta$  and  $\phi$ ). This leads to

$$\rho(r) = -\frac{q}{4\pi\epsilon_0} \frac{1}{r} \frac{d^2}{dr^2} \left[ \exp(-2r/a) \left(1 + \frac{r}{a}\right) \right] = -\frac{q}{\pi a^3} \exp(-2r/a).$$

This is the distributed part of the density associated with the electron. The nuclear contribution is lost since  $\nabla^2\Phi = 0$  for a point charge except at the location of the charge itself. To determine at the point charge contribution let's reverse the process. The contribution to  $\Phi$  from the electron charge  $\rho$  above is

$$\Phi_\rho = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|\mathbf{r}' - \mathbf{r}|} d\tau' = \frac{1}{\epsilon_0} \left( \frac{1}{r} \int_0^r x^2 \rho(x) dx + \int_r^\infty x \rho(x) dx \right)$$

Substituting the electron charge density into the above leads to

$$\Phi_\rho = \frac{q}{4\pi\epsilon_0} \left[ -\frac{1}{r} + \frac{\exp(-2r/a)}{r} \left(1 + \frac{r}{a}\right) \right]$$

To obtain  $\Phi$  above, we must add

$$\Phi_{\text{nucl}} = \frac{q}{4\pi\epsilon_0} \frac{1}{r}$$

to  $\Phi_\rho$ .  $\Phi_{\text{nucl}}$  is precisely the potential due to a point charge  $q$  at the origin!

### 3. Jackson Prob. 1.6:

- (a) Assuming that the plates are parallel to the  $x - y$  plane, and that the upper plate carries charge  $Q$ , the field between the two sheets is

$$\mathbf{E} = -\frac{\sigma}{\epsilon_0} \hat{z} = -\frac{Q}{\epsilon_0 A} \hat{z}$$

The potential in the region between the plates is

$$\Phi = \frac{Q}{\epsilon_0 A} z$$

The potential difference between the plates is

$$V = \frac{Qd}{\epsilon_0 A}$$

The capacitance is

$$C = \frac{Q}{V} = \epsilon_0 \frac{A}{d}$$

- (b) Assume the inner sphere carries charge  $-Q$  and that the outer sphere carries charge  $Q$ . The potential in the region between the spheres is

$$\Phi = -\frac{Q}{4\pi\epsilon_0} \frac{1}{r}$$

The potential difference between the outer and inner sphere is

$$V = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{Q}{4\pi\epsilon_0} \frac{b-a}{ab}$$

It follows that

$$C = \epsilon_0 \frac{4\pi ab}{b-a}$$

- (c) Assume that the outer cylinder carries charge  $Q = \lambda L$ . The potential between the cylinders is

$$\Phi = \frac{\lambda}{2\pi\epsilon_0} \ln(r)$$

and the potential difference between outer and inner cylinder is

$$V = \frac{Q}{2\pi\epsilon_0 L} \ln(b/a)$$

Therefore

$$C = \epsilon_0 \frac{2\pi L}{\ln(b/a)}$$

- (d) Solving for  $b$ , we find

$$b = a \exp\left(\frac{2\pi\epsilon_0}{C/L}\right)$$

For  $C/L = 3 \times 10^{-11}$  we find

$$\frac{2\pi\epsilon_0}{C/L} = \frac{2\pi \cdot 8.854 \times 10^{-12}}{3 \times 10^{-11}} = 1.854$$

and  $b = 6.39$  mm.

For  $C/L = 3 \times 10^{-12}$ , the argument of the exponential increases by a factor of 10 and  $b$  increases to  $1.13 \times 10^5$  m.

#### 4. Jackson Prob. 1.8:

- (a) Use the expression  $W = \frac{\epsilon_0}{2} \int E^2 d\tau$  to evaluate the energy.

Parallel Plate

$$W = \frac{\epsilon_0}{2} A \int_0^d dx \left( \frac{Q}{\epsilon_0 A} \right)^2 = \frac{1}{2} \frac{Q^2 d}{\epsilon_0 A} = \frac{1}{2} \frac{Q^2}{C}$$

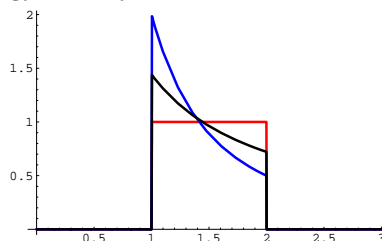
Spherical

$$W = \frac{\epsilon_0}{2} 4\pi \left( \frac{Q}{4\pi\epsilon_0} \right)^2 \int_a^b \frac{dr}{r^2} = \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0} \frac{(b-a)}{ab} = \frac{1}{2} \frac{Q^2}{C}$$

Cylindrical

$$W = \frac{\epsilon_0}{2} 2\pi L \left( \frac{Q}{2\pi\epsilon_0 L} \right)^2 \int_a^b \frac{d\rho}{\rho} = \frac{1}{2} \frac{Q^2}{2\pi\epsilon_0 L} \ln(b/a) = \frac{1}{2} \frac{Q^2}{C}$$

Plot of Energy Density vs. Distance



5. Force between plates of a capacitor.

- (a)  $Q$  fixed: The energy stored in the capacitor is when the plate separation is  $x$  is given by

$$W(x) = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{Q^2}{\epsilon_0 A} x$$

If we increase the plate separation by  $\delta x$ , we increase the energy by

$$\delta W = \frac{1}{2} \frac{Q^2}{\epsilon_0 A} \delta x.$$

This increase is supplied by an external force that pulls the plates apart quasi-statically

$$F_x = \frac{dW}{dx} = \frac{1}{2} \frac{Q^2}{\epsilon_0 A}.$$

This external force is balanced by internal force between the plates. That is to say, the force between the plates of the capacitor is attractive and has the magnitude

$$F_x = \frac{1}{2} \frac{Q^2}{\epsilon_0 A} = \frac{\sigma^2}{2\epsilon_0} A.$$

- (b)  $V$  fixed: In this case the capacitor must be connected to an external source to maintain the fixed potential difference. The energy stored in the capacitor is

$$W(x) = \frac{1}{2} CV^2 = \frac{1}{2} V^2 \frac{\epsilon_0 A}{x}$$

If we increase the plate separation by  $\delta x$ , we change the energy stored in the capacitor by

$$\delta W = \frac{1}{2} V^2 \delta C = -\frac{1}{2} C V^2 \frac{\delta x}{x}.$$

Thus, energy is lost from the capacitor. The loss is caused by fact that an amount of charge

$$\delta Q = V \delta C = -C V \frac{\delta x}{x}$$

must be transferred from the positive plate to the negative plate to maintain a constant potential difference. This charge transfers energy

$$\delta Q V = -C V^2 \frac{\delta x}{x}$$

from the capacitor to the source. This is twice the energy loss from the capacitor! To achieve a energy balance, an additional amount of energy

$$\delta W_{\text{add}} = \frac{1}{2} C V^2 \frac{\delta x}{x}$$

must be supplied to the system. This energy is again supplied by the work done by the external force that separates the plates. Using the previous analysis, we infer that the attractive force between the plates is

$$F = \frac{dW_{\text{add}}}{dx} = \frac{1}{2} \frac{C V^2}{x} = \frac{1}{2} \frac{Q^2}{\epsilon_0 A}$$

As expected, this force is the same as that inferred previously. The force between the plates is a function only on the geometry and charge distribution!