Limiting Analysis of Outage Probabilities for Diversity Schemes in Fading Channels

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***Abstract*— Diversity schemes that exploit wireless channel variations in time, frequency, and space are an essential component for combating multipath fading in modern wireless communications systems. To evaluate performance and thereby design reliable and resource-efficient systems, designers require analytical tools that capture the salient, if not precise, characteristics of the fading channel model and diversity scheme employed. In this paper, we develop a simple and powerful way of characterizing performance of diversity schemes via limiting analysis of outage probabilities. As in other approaches to such an analysis, the two key parameters in our analysis are diversity order and coding gain, corresponding to the slope and intercept, respectively, in a plot of log-outage versus signal-to-noise ratio (SNR) in decibels (dB). Our approach allows for the characterization of a wide variety of diversity schemes operating over a broad class of fading channels, especially non-repetition diversity schemes such as parallel channel coding as well as multiuser diversity schemes such as cooperative diversity.**

I. Introduction

A central issue in wireless communications is designing techniques that mitigate, or even exploit, the inherent variability of the channel across time, frequency, and space. This variability, known as *fading*, arises because of multipath propagation in the medium and several time-varying effects.[1] To combat fading, system designers often create signal redundancy through particular signaling schemes at the transmitter and/or processing algorithms at the receiver. These approaches, known collectively as *diversity schemes*, are well developed and currently utilized in one form another in all wireless communications systems. Classical diversity techniques come in a variety of related forms [1]:

- channel coding and signal processing to exploit temporal variations of the channel and achieve time diversity;
- wideband channel coding and signal processing to exploit spectral variations of the channel and achieve frequency diversity; and
- multi-antenna receivers (and, with suitable space-time/frequency codes, transmitters) to exploit spatial variations in the channel and achieve space diversity.

The underlying idea behind all of these diversity techniques is to essentially repeat the information across multiple, independent channel realizations in order to allow the receiver to experience the average channel effect rather than an instantaneous fade. Numerous combinations of the above techniques have also been developed, increasingly more so in the context of multi-antenna systems.

Channel coding schemes for achieving time diversity require increased delay, and spread-spectrum methods for achieving frequency diversity require excess bandwidth—resources that are often scarce in wireless environments. By contrast, systems with multiple transmit or receive antennas can achieve spatial diversity only at the cost of additional hardware and algorithmic complexity, making them very appealing for emerging systems with sufficiently powerful radios. Even when radios individually employ a single antenna, they can pool their resources to emulate the performance of an antenna array by relaying signals for each other to achieve cooperative diversity. (See [2]–[8] and the references therein.) When several of these forms of diversity are available, it is necessary to determine which forms should be exploited and to what extent, especially when resources such as delay and bandwidth are constrained by the application.

A. Limiting Analysis of Outage Probability

As a first step in addressing these important practical questions in a comprehensive way, this paper develops a simple and powerful approach for characterizing performance of diversity schemes via limiting analysis of outage probabilities. As in other approaches to such an analysis, the two key parameters in our analysis are diversity order and coding gain, corresponding to the slope and intercept, respectively, in a plot of log-outage versus signal-to-noise ratio (SNR) in decibels (dB). Our approach allows for the characterization of a wide variety of diversity schemes operating over a broad class of fading channels.

More specifically, due to limited resources such as delay, bandwidth, and antenna elements, we focus on non-ergodic settings and consider channels with a finite number of fading realizations. Taking an information-theoretic point of view, the channel mutual information [9], as a function of the fading coefficients, can be viewed as a random variable. To emphasize the dependence of the mutual information on the channel SNR, we denote the mutual information by $I(\mathrm{SNR})$. We define outage probability for a diversity scheme operating over a particular fading channel to be the probability of the event that

[1] These time-varying effects might include, for example, mobility of the transmitter and/or receiver, environment changes, and so forth.

the mutual information falls below a pre-specified transmission rate R, *i.e.*,

$$P_{\mathrm{out}}(\mathrm{SNR},\mathrm{R}) = \Pr\left[I(\mathrm{SNR}) < \mathrm{R}\right] \ . \quad (1)$$

Our results develop approximations to the outage probability of the form[2]

$$P_{\mathrm{out}}(\mathrm{SNR},\mathrm{R}) \sim (c(\mathrm{R})\cdot\mathrm{SNR})^{-d} \ , \quad (2)$$

for large SNR, where $d > 0$ corresponds to the *diversity order* of the scheme, and $c(R)$ corresponds to the *coding gain* of the scheme.

B. Related Work

Outage probabilities for coded systems were developed from an information-theoretic point of view in [10]. This point of view relies upon random coding arguments and defines outage probability as in (1). This *information-theoretic outage probability*, referred to simply as outage probability throughout the paper, can be minimized through choice of the channel input distribution.

For uncoded modulations, or particular channel codes, outage probability is often defined as the probability of the event that the channel SNR falls below some pre-specified (and generic) threshold [11]. This *communication-theoretic outage probability* can often be related to the bit- or block-error probability of the candidate modulation or coding scheme.

The results in this paper are extensions of the tools developed in [6], [7] for characterizing the limiting information-theoretic outage performance of cooperative diversity. These tools can also be applied to the scenarios considered in [10], where exact results are obtained but only for small diversity orders. Recent work in [12], [13] develops a limiting analysis for communication-theoretic outage performance. For additive white Gaussian noise (AWGN) channels with fading and certain kinds of diversity schemes, the two notions of outage are of course related. Indeed, in such cases, our results are essentially the same as those in [12], [13]. However, for more general channel models and diversity schemes, particularly multiuser diversity schemes such as cooperative diversity, the information-theoretic outage approach can be more readily generalized. An additional advantage of the information-theoretic outage approach is that it explicitly captures the dependence of the performance on the transmission rate R in addition to the channel SNR.

II. Abstract Channel Model and Main Result

In this section, we develop our channel model and main result.

A. Fading Channel Model

We consider a frequency non-selective, time-selective block fading channel model with AWGN. In this model, the fading coefficients remain fixed for N channel uses, and the delay constraints allow for coding over K blocks of N channel uses each. In block $k = 1, 2, \dots, K$, we model the channel in baseband-equivalent, discrete-time form as

$$y[n] = a_k\, x[n] + z[n] \ , \quad n = (k-1)N+1, \dots, kN \ , \quad (3)$$

where: $y[n]$ is the received signal; a_k captures the effects of path-loss and multipath fading; $x[n]$ is the transmitted signal of power $\mathrm{E}\left[|x[n]|^2\right] = P$; and $z[n]$ captures the effects of additive receiver noise and other interference of power $\mathrm{E}\left[|z[n]|^2\right] = N_0$. Throughout the paper, $\mathrm{SNR} = P/N_0$, the *transmit* power divided by the noise power. Our results in the sequel are based upon properties of the distribution of the *received* signal-to-noise ratio random variables $\mathrm{SNR}\cdot|a_k|^2$ (or certain functions of them) for large SNR.

We consider the scenario in which the fading coefficients a_k are known to, *i.e.*, accurately measured by, the receiver, but not fully known to (or not exploited by) the transmitter. Statistically, we model a_k as mutually independent, circularly-symmetric complex random variables with variances $1/\lambda_k$. For example, if a_k are circularly-symmetric complex Gaussian, the magnitudes $|a_k|$ are Rayleigh distributed, and $|a_k|^2$ are exponentially distributed with parameter λ_k. In other cases of interest, we specify the distributions of the effective fading coefficients, or else needed properties of them. Furthermore, we model $z[n]$ as a zero-mean independent, circularly-symmetric, complex Gaussian random sequence with variance N_0.

B. Main Result

As we will see in Section III, the result of many diversity schemes is that the received signal-to-noise ratios $\mathrm{SNR}\cdot|a_k|$ (or certain functions of them) sum through appropriate diversity combining at the receiver. Our main result characterizes large SNR properties of the distribution of the sum given large SNR properties of the distribution of the individual terms.

Theorem 1: Let u_s and v_s be two independent random variables with the property that

$$\lim_{s\to\infty} s\cdot\Pr\left[u_s < t\right] = f(t)$$
$$\lim_{s\to\infty} s^d\cdot\Pr\left[v_s < t\right] = g(t) \ ,$$

where $f(t)$ and $g(t)$ are monotone increasing and integrable, and $f'(t)$ is integrable. Then

$$\lim_{s\to\infty} s^{d+1}\Pr\left[u_s + v_s < t\right] = \int_0^t g(t-x)f'(x)dx \ . \quad (4)$$

Proof: Due to space considerations, we give an outline of the proof. First, we let $\mathcal{U} = \{u_0, u_1, \dots, u_L\}$, for some finite L, be any partition of the interval $[0, t]$ with $u_0 = 0$ and $u_L = t$. We can then obtain inner and outer bounds on the event $u_s + v_s < t$ as

$$\{u_s + v_s < t\} \subseteq \bigcup_{i=1}^{L}\{u_{i-1} \le u_s < u_i\} \cap \{v_s < t - u_{i-1}\}$$

$$\{u_s + v_s < t\} \supseteq \bigcup_{i=1}^{L}\{u_{i-1} \le u_s < u_i\} \cap \{v_s < t - u_i\} \ .$$

[2]The approximation in (1) is in the sense of $(c(\mathrm{R})\cdot\mathrm{SNR})^d \cdot P_{\mathrm{out}}(\mathrm{SNR},\mathrm{R}) \to 1$ as $\mathrm{SNR}\to\infty$.

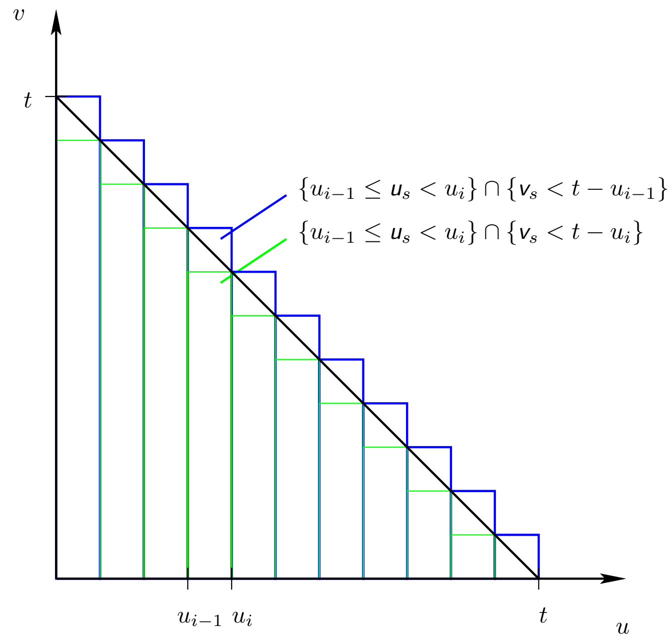

Fig. 1. Inner and outer bounds to the event $u_s + v_s < t$.

Fig. 1 illustrates these bounds. For the outer bound we cover the event with a union of rectangles, and for the inner bound we cover a union of rectangles with the event.

The probabilities for these inner and outer bound events are easy to compute because the random variables u_s and v_s are independent. Specifically, for the terms in the upper bound, we have

$$\Pr\left[u_{i-1} \leq u_s < u_i, v_s < t - u_{i-1}\right] = \left(\Pr\left[u_s < u_i\right] - \Pr\left[u_s < u_{i-1}\right]\right) \cdot \Pr\left[v_s < t - u_{i-1}\right] \ . \quad (5)$$

Taking limits and applying the conditions of the theorem, we have

$$\lim_{s\to\infty} s^{d+1} \Pr\left[u_{i-1} \leq u_s < u_i, v_s < t - u_{i-1}\right] = g(t - u_{i-1})\left(f(u_i) - f(u_{i-1})\right) \ , \quad (6)$$

so that

$$\limsup_{s\to\infty} s^{d+1} \Pr\left[u_s + v_s < t\right] \leq \sum_{i=1}^{L} g(t - u_{i-1})\left(f(u_i) - f(u_{i-1})\right) \ . \quad (7)$$

Now (7) holds for all partitions $\mathcal{U}$ of the interval $[0, t)$; thus, it holds for the infimum of the right-hand side of (7) over all such partitions. A similar argument applies to the lower bound, taking the supremum over all partitions. Since $f(t)$, $f'(t)$, and $g(t)$ are all integrable, the infimum and supremum of the right-hand side of (7) become the integral in (4). ■

III. Applications

In this section, we illustrate applications of Theorem 1 in the context of comparing repetition coding to parallel channel coding as well as evaluating performance of amplify-and-forward cooperative diversity.

A. Comparison of Repetition and Parallel Channel Coding

As one application of our result, we consider the abstract channel model of Section II-A with two diversity schemes, namely, repetition coding and parallel channel coding.

1) Repetition Coding: For repetition coding, the transmitter uses the same codeword in each of the K blocks. The destination performs maximum-ratio combining and accumulates mutual information[3]

$$I(\mathrm{SNR}) = \frac{1}{K} \log\left(1 + \mathrm{SNR} \sum_{k=1}^{K} |a_k|^2\right) \quad (8)$$

for Gaussian codebooks. Thus, the outage probability is, after some manipulation,

$$P_{\mathrm{out}}(\mathrm{SNR}, \mathrm{R}) = \Pr\left[\sum_{k=1}^{K} \mathrm{SNR}|a_k|^2 < 2^{K\mathrm{R}} - 1\right] \ . \quad (9)$$

Using properties of the distribution of $\mathrm{SNR} \cdot |a_k|^2$ for large SNR, we repeatedly apply Theorem 1 to approximate the outage probability (9) for large SNR. For example, if $|a_k|^2$ are independent exponential random variables with parameters λ_k, then

$$\lim_{s\to\infty} s \cdot \Pr\left[s|a_k|^2 < t\right] = \lambda_k t \ .$$

This result utilized in Theorem 1 $(K-1)$ times, with $t = 2^{K\mathrm{R}} - 1$, yields the approximation

$$P_{\mathrm{out}}(\mathrm{SNR}, \mathrm{R}) \sim \left(c_{\mathrm{rep}}(\mathrm{R}) \cdot \mathrm{SNR}\right)^{-d_{\mathrm{rep}}} \ , \quad (10)$$

where $d_{\mathrm{rep}} = K$ and

$$c_{\mathrm{rep}}(\mathrm{R}) = \left(\frac{K!}{\prod_{k=1}^{K} \lambda_k}\right)^{1/K} \cdot \frac{1}{2^{K\mathrm{R}} - 1} \ . \quad (11)$$

Fig. 2 demonstrates the accuracy of the approximation (10) for the case of $\lambda_k = 1$ and $\mathrm{R} = 1/2$. In this symmetric case, the sum of exponentials in (9) has the Erlang distribution, and an exact expression for (9) is

$$P_{\mathrm{out}}(\mathrm{SNR}, \mathrm{R}) = \gamma(K, (2^{K\mathrm{R}} - 1)/\mathrm{SNR}) \ , \quad (12)$$

where $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$ is the incomplete Gamma function. More generally, an exact form for (9) is substantially more complex. For λ_k distinct, the sum of exponentials in (9) has the hypoexponential distribution. A closed form is also available in this case [14], but can be numerically sensitive to compute. In the most general case, only some of the λ_k are distinct, and appropriate limits of the hypoexponential result can be employed. On the other hand, due to its accuracy and ease of computation, the approximation in (10) can be used in all cases.

[3]All logarithms in the paper are base-2.

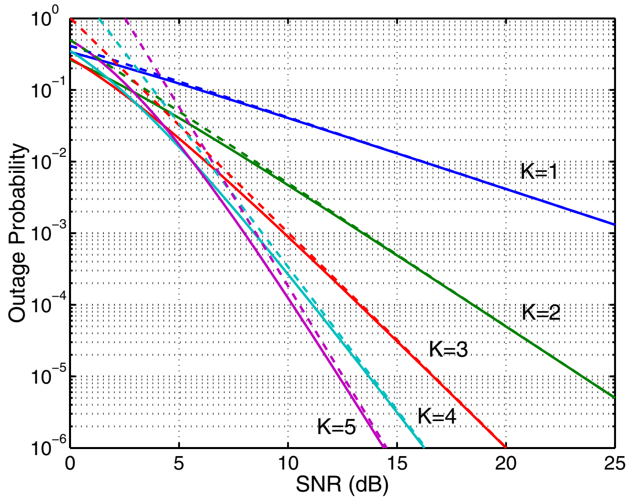

Fig. 2. Outage probabilities for repetition coding over identically distributed Rayleigh fading channels. Successively lower solid lines correspond to the exact expression (12), and dashed curves correspond to the approximation (10), for $K = 1, 2, \ldots, 5$, respectively. These example results are for $\mathrm{R} = 1/2$

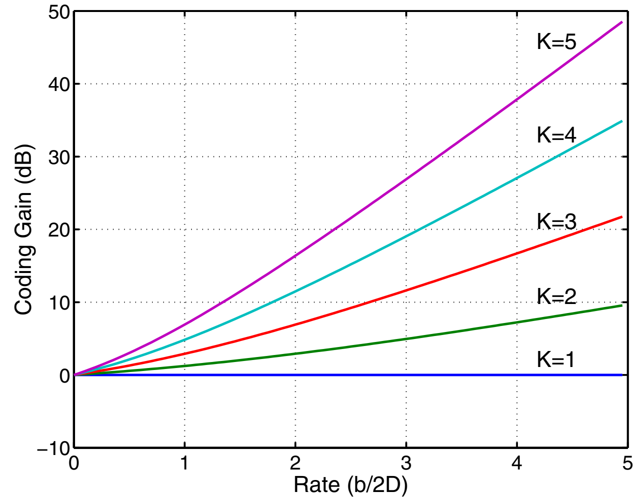

Fig. 3. Relative coding gains $c_{\text{par}}/c_{\text{rep}}$ for parallel channel coding over repetition coding. Successively higher curves correspond to diversity orders $K = 1, 2, \ldots, 5$, respectively.

2) Parallel Channel Coding: For parallel channel coding, the transmitter uses different and jointly designed codewords for each of the K blocks. The receiver accumulates mutual information

$$I(\mathrm{SNR}) = \frac{1}{K}\sum_{k=1}^{K} \log\left(1 + \mathrm{SNR}|a_k|^2\right) \tag{13}$$

for Gaussian codebooks. Thus, the outage probability is, after some manipulation,

$$P_{\text{out}}(\mathrm{SNR}, \mathrm{R}) = \Pr\left[\sum_{k=1}^{K} \log\left(1 + \mathrm{SNR}|a_k|^2\right) < K\mathrm{R}\right] . \tag{14}$$

Letting $u_k = \log\left(1 + s|a_k|^2\right)$, it is straightforward to show that, if $|a_k|^2$ is exponential with parameter λ_k, then

$$\Pr\left[u_k < t\right] = 1 - \exp\left[-\lambda_k \frac{2^t - 1}{s}\right] , \tag{15}$$

and

$$\lim_{s\to\infty} s \cdot \Pr\left[u_k < t\right] = \lambda_k(2^t - 1) . \tag{16}$$

Letting $g_1(t) = f(t) = (2^t - 1)$, so that $f'(t) = 2^t \ln(2)$, repeated application of Theorem 1 with $t = K\mathrm{R}$ yields, for example,

$$\lim_{s\to\infty} s^2 \Pr\left[\sum_{k=1}^{2} u_k < t\right] = \lambda_1\lambda_2 \int_0^t g_1(t-x) f'(x)dx = \lambda_1\lambda_2 \underbrace{\left(2^t(\ln(2)t - 1) + 1\right)}_{g_2(t)}$$

$$\lim_{s\to\infty} s^3 \Pr\left[\sum_{k=1}^{3} u_k < t\right] = \lambda_1\lambda_2\lambda_3 \underbrace{\int_0^t g_2(t-x) f'(x)dx}_{g_3(t)} ,$$

and so forth. These results provide approximations to the outage probability for parallel channel coding of the form

$$P_{\text{out}}(\mathrm{SNR}, \mathrm{R}) \sim \left(c_{\text{par}}(\mathrm{R}) \cdot \mathrm{SNR}\right)^{-d_{\text{par}}} \tag{17}$$

with $d_{\text{par}} = K$ and

$$c_{\text{par}}(\mathrm{R}) = \left(\frac{1}{g_K(K\mathrm{R}) \prod_{k=1}^{K} \lambda_k}\right)^{1/K} . \tag{18}$$

We note that [10] computes an exact, but rather involved, expression for (14) for $K = 2$ only. Our approximation in (17) allows us to compare repetition and parallel channel coding for general values of K.

Examining (10) and (17), we see that both diversity schemes achieve full diversity K. However, the coding gain of parallel channel coding (18) can be much larger than that of repetition coding (11) for moderate to large spectral efficiencies. Fig. 3 illustrates the relative gain $c_{\text{par}}/c_{\text{rep}}$ for diversity orders $K = 1, 2, \ldots, 5$. As we might expect, the improvements for parallel channel coding grow with increasing R or K. Although repetition coding is frequently employed in practice for its ease of implementation, these results outline the additional power or energy efficiency that can be obtained through parallel channel coding. The most dramatic gains arise for scenarios in which either (1) delay constraints allow for coding across multiple fading realizations, or (2) application demands require high spectral efficiencies.

B. Amplify-and-Forward Cooperative Diversity

As another application of our result, we consider amplify-and-forward cooperative diversity. As depicted in Fig. 4 and developed in [6], [7], amplify-and-forward cooperative diversity corresponds to our abstract model of Section II-A for $K = 2$ in the following way. In the first block, the source transmits to the relay and the destination. In the second

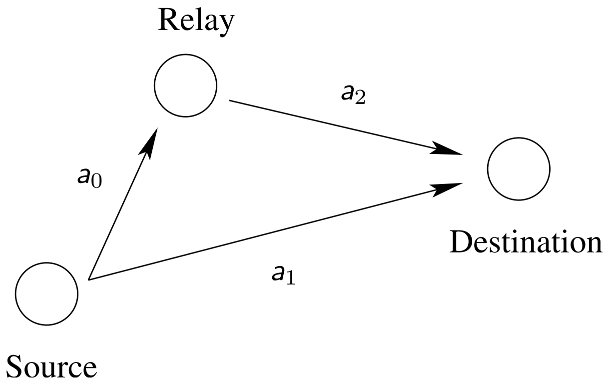

Fig. 4. Cooperative diversity model.

block, the relay amplifies its received signal subject to a power constraint, corresponding to repetition coding but with noise amplification. The destination performs maximum-ratio combining in order to accumulate mutual information

$$I(\mathrm{SNR}) = \frac{1}{2}\log\left(1+\mathrm{SNR}|a_1|^2+h(\mathrm{SNR}|a_0|^2, \mathrm{SNR}|a_2|^2)\right) \ , \quad (19)$$

again for Gaussian codebooks, where

$$h(x, y) = \frac{xy}{x+y+1} \ .$$

Slight manipulation of (19) yields outage probability

$$\Pr\left[\mathrm{SNR}|a_1|^2+h(\mathrm{SNR}|a_0|^2, \mathrm{SNR}|a_2|^2) < 2^{2\mathrm{R}}-1\right] . \quad (20)$$

Since, for $|a_k|^2$ independent exponentials with parameters λ_k, $k = 0, 1, 2$, we have [6]

$$\lim_{s\to\infty} s \cdot \Pr\left[s|a_1|^2 < t\right] = \lambda_1 t$$
$$\lim_{s\to\infty} s \cdot \Pr\left[h(s|a_0|^2, s|a_2|^2) < t\right] = (\lambda_0+\lambda_2)t \ ,$$

Theorem 1 applies so that (20) can be approximated as

$$P_{\mathrm{out}}(\mathrm{SNR}, \mathrm{R}) \sim (c_{\mathrm{af}}(\mathrm{R}) \cdot \mathrm{SNR})^{-d_{\mathrm{af}}}$$

with $d_{\mathrm{af}} = 2$ and

$$c_{\mathrm{af}}(\mathrm{R}) = \sqrt{\frac{2}{\lambda_1(\lambda_0+\lambda_2)}} \cdot \frac{1}{2^{2\mathrm{R}}-1}$$

for large SNR. These results are identical to those in [6]; however, Theorem 1 in this paper substantially simplifies the proof in [6] and can be readily extended to many relays.

IV. Conclusion

Limiting analysis of outage probabilities for diversity schemes provides a compact way (*cf.* (2)) of approximating performance in terms of the diversity order d and the coding gain c. Our general result in Theorem 1, in combination with results from [6], [7], [12], [13] among others, provides a convenient framework for analyzing tradeoffs between performance and resource-efficiency for a variety of diversity schemes operating over a broad class of fading channels. Our future work will focus on these tradeoffs for scenarios in which cooperative diversity can be exploited along with other forms of diversity.

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