A Survey of The Multiple Access Channel with a discussion of the capacity regions for the Memoryless MAC and MAC with Fading

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1 Introduction

The Multiple Access Channel, or the MAC channel, is a simple multi-terminal channel model where m users attempt to communicate independent information over a noisy channel to a single receiver. This is a common model for different communication systems including the reverse link of the cellular channel, the uplink of a satellite channel, a computer network with a central monitoring hub/server etc. In its simplest form, the memoryless MAC channel without feedback, where the output depends only on the current inputs of all the users and a random noise variable, described usually using a probability transition matrix, is well understood, and this document discusses the channel coding theorem for this channel, following the development in [1]. The addition of complexity to this model, while making it more realistic, also makes it intractable to a great degree. Thus, the assumption of a fading model for the channel, which makes the MAC model more in tune with real-world wireless multi-user networks, renders the problem of finding the capacity region extremely difficult except in certain specific cases [2]-[4] etc. Some aspects of this extended problem will be discussed whereas the more interesting, and hence more involved, model of MAC with feedback [5], which, along with fading in the system, would model a real-world cellular reverse link with great accuracy, is left alone with only the comment that feedback does improve the capacity of the MAC channel and the capacity region of MAC channels with noiseless/noisy feedback is not known completely.

The rest of the summary is organized as follows. Section II discusses the channel model for the discrete memoryless MAC and describes the proof of the achievability and the converse parts of the channel coding theorem to obtain the capacity region for a 2-user MAC. Section III provides a brief argument extending the results of a 2-user MAC to a general m-user MAC and also to the special case of the Gaussian MAC channel. Section IV describes the fading models used in MAC channels and provides a brief summary of some interesting results for the fading MAC channel for different cases.

2 The Discrete Memoryless MAC with 2 users - Channel Model and Capacity Regions

The discrete memoryless MAC with 2 users is the simplest form of the MAC channel. It consists of three alphabets $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{Y}$ and a probability law $p(y|x_1, x_2)$. The inputs, X_1 and X_2, are obtained by encoding the messages to be sent, W_1 and W_2, respectively. The channel block diagram is shown in Fig. 1. Note that this can be extended to the general case of m users but here $m = 2$ is considered to make the proofs tractable. Also, much of the following discussion can be found in [1].

Definition A $((\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil), n)$ code for the MAC consists of two sets of integers $\mathcal{W}_1 = \{1, 2, ..., \lceil 2^{nR_1} \rceil\}$ and $\mathcal{W}_2 = \{1, 2, ..., \lceil 2^{nR_2} \rceil\}$ called the message sets, two encoding functions, $X_1 : \mathcal{W}_1 \rightarrow \mathcal{X}_1^n$, $X_2 : \mathcal{W}_2 \rightarrow \mathcal{X}_2^n$ and a decoding function $g : \mathcal{Y}^n \rightarrow \mathcal{W}_1 \times \mathcal{W}_2$

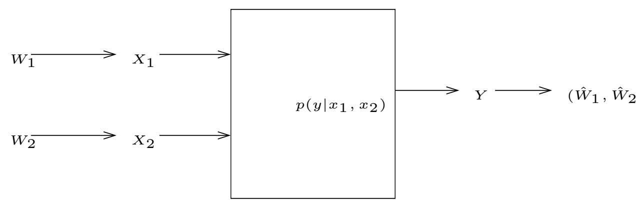

Figure 1: The multiple access channel

Sender 1 chooses an index W_1 uniformly from $\mathcal{W}_1$ and sends the corresponding codeword along the channel and sender 2 does likewise, independently. The average probability of error for the code is then given by

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(w_1,w_2) \in \mathcal{W}_1 \times \mathcal{W}_2} Prg(Y^n \neq (w_1, w_2)|(w_1, w_2)sent)$$

The notions of achievability of a rate pair and the capacity region are defined in the usual sense. A rate pair (R_1, R_2) is said to be achievable if there exists a sequence of $((\lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil), n)$ codes with $P_e^{(n)} \rightarrow 0$. The capacity region for the MAC is defined as the closure of the set of achievable rate pairs. Given these definitions, the achievability and the converse parts of the coding theorem are proved to obtain the capacity region.

Achievability is first proved for a product distribution of inputs, ie., $p(x_1, x_2) = p_1(x_1)p_2(x_2)$ for specific $p_1(.)$ and $p_2(.)$. Thus, it is shown that there exist particular input distributions(sequences of codes) which achieve a given rate with the property that these are product distributions. However, these are not the only possible achievable rates and the rate region can be augmented by considering all rates that are convex combinations of the rates achieved by the product distributions, by introducing an auxiliary timesharing random variable Q. The details are worked out in [1]. Here the general theorem statement and a sketch of the proof for the case where the input is a product distribution are given.

Theorem 2.1 ***(Achievability)*** *The set of achievable rates of a discrete memoryless MAC is given by the closure of the set of all (R_1, R_2) pairs satisfying*

$$\begin{aligned} R_1 &\le I(X_1; Y|X_2, Q) && (2.1) \\ R_2 &\le I(X_2; Y|X_1, Q) && (2.2) \\ R_1 + R_2 &\le I(X_1, X_2; Y|Q) && (2.3) \end{aligned}$$

for some choice of the joint distribution $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$ with $|Q| \le 4$

Proof The proof is provided in [1].

To prove that a rate is achievable, we need to construct a sequence of codes mapping W_i to X_i whose error probability goes to zero. This is accomplished using a joint-typical decoding argument. The codewords are generated randomly and i.i.d for both users and the codebook is revealed to both senders and to the receiver. To transmit a particular message, (i, j) the users transmit the codewords, $\mathbf{x}_1(i)$ and $\mathbf{x}_2(j)$ corresponding to the messages. At the decoder, suppose $\mathbf{y}$ is the received sequence. Then, the decoder uses joint-typical decoding and decodes the transmitted message as (i', j') where $(\mathbf{x}_1(i'), \mathbf{x}_2(j'), \mathbf{y})$ is typical. It can be shown that the probability of error using this decoding strategy goes to zero as the blocklength increases and hence the corresponding rates, which are shown to be those given in 2.1, are indeed achievable.

Theorem 2.2 ***(Converse)*** *For any given sequence of $((2^{nR_1}, 2^{nR_2}), n)$ codes with $P_e^{(n)} \to 0$, the rates must satisfy*

$$\begin{aligned} R_1 &\le I(X_1; Y|X_2, Q) && (2.4) \\ R_2 &\le I(X_2; Y|X_1, Q) && (2.5) \\ R_1 + R_2 &\le I(X_1, X_2; Y|Q) && (2.6) \end{aligned}$$

for some choice of random variable Q defined on $1, 2, 3, 4$ and joint distribution $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$.

Proof The proof of the theorem is given in [1].

In the converse part, a sequence of codes and the codebooks are given and it is known that the probability of error in decoding the input goes to zero when these codes are used. It is required to prove that all such codes have a constraint on the rates of the codes as given by the inequalities in 2.4. The basic idea is to use Fano's inequality and the fact that the entropy of a random variable does not increase conditioned on the knowledge of another random variable to obtain an inequality on the rate of each user and also the sum rate. This bound depends on the empirical distributions of the different columns of the codebook used. These empirical distributions are converted to single-letter distributions by using an auxiliary random variable and Caratheodory's theorem [1] is used to further restrict the cardinality of the auxiliary random variable Q to $|Q| \leq 4$.

From the achievability and the converse parts to the coding theorem, we have a complete characterization of the set of achievable rates for any input distribution. An important point to be noted here is that the achievable rates for the MAC are not given only by the set of rates defined by the product distribution of the inputs. For the MAC, the *block* inputs $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent. However, time-sharing over different codebooks for the inputs makes the input *symbol* distributions only conditionally independent, given the time-sharing random variable. Hence the set of joint input distributions which yield the different achievable rates is a superset of the set of all product distributions.

The rate region for a given distribution is given in Fig. 2. In the case of product distributions of the inputs, the corner points in the rate region are given by $R_1 = I(X_1; Y|X_2)$ and $R_2 = I(X_2; Y|X_1)$. An analysis of the decoding procedure using a successive-decoding approach, where the first user is decoded and then canceled off gives an intuitive understanding of the achievability of the corner points. For example, consider the corner point where $R_1 = I(X_1; Y|X_2)$, the second user can then transmit at a rate given by $R_2 = I(X_2; Y)$ as seen from the constraint on the sum rate. This can be achieved in practice using a successive decoder. The second user transmits at $R_2 = I(X_2; Y)$, treating the other user as noise and, since this rate is achievable, the user's data can be "subtracted" from the MAC output, and the first user can be decoded. For perfect decoding of the first user, it is required that $R_1 \leq I(X_1; Y|X_2)$ as this the amount of information that can be transmitted on the equivalent single-user channel between X_1 and Y, conditioned on the knowledge of X_2.

3 m-user MAC channels and the Gaussian MAC

The m-user MAC channel is an extension of the 2-user MAC channel with m users and one receiver. The achievability and converse parts of the coding theorem can be extended directly to this case. There are now $2^m - 1$ constraints corresponding to the different possible sums of rates given by $R_{i_1} + R_{i_2} + ... + R_{i_k}$ for some set $S = \{i_1, i_2, ..., i_k\}$ such that $S \subset E$ where $E = \{1, 2, ..., m\}$. This is usually expressed as a polyhedron in the m-dimensional space as $R(S) \leq I(Y; (X_i)_{i \in S} | (X_i)_{i \in S^C}, Q) \ \ \forall \ \ S \subseteq E$, with Q being a random variable defined on $\{1, 2, ..., 2^m - 1\}$. The joint distribution is given in a form similar to the 2-user case. The polyhedron defined by the rates can be shown to be a polymatroid, which is an extension of the polyhedron into a generalized matroid representation as defined in [8], and there are certain mathematical properties

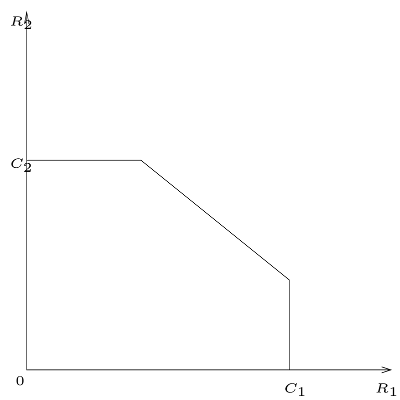

Figure 2: Rate region for a multiple access channel

of the polymatroid that can be used to solve various problems related to the fading MAC channel [3].

The Gaussian MAC is a special case of the MAC, where the channel is assumed to be AWGN with noise variance N. Given block average power constraints P_1 and P_2, we can show that the capacity region is given by [1]

$$\begin{aligned} R_1 &\leq C(\frac{P_1}{N}) \\ R_2 &\leq C(\frac{P_2}{N}) \\ R_1 + R_2 &\leq C(\frac{P_1 + P_2}{N}) \end{aligned}$$

where $C(x) = \frac{1}{2}\log(1 + x)$.

The upper bounds are achieved when the input distributions are zero mean Gaussian with variances P_1 and P_2 respectively. Note that the maximum achievable sum capacity corresponds to the capacity of a single user transmitting with the sum power of the two in a Gaussian channel.

4 Fading MAC channels

Real-world multiple access channels are much more complex than the discrete memoryless and the Gaussian MAC channels considered in the previous sections. One of the most important aspects of practical wireless channels is fading. Here we consider some of the results for the MAC channel with fading. The polymatroidal description of the rate region comes in handy in the description of the rate regions and also in the optimal resource allocation problem considered in [3] for the multiple access fading channel with channel state information at both the transmitters and the receivers. In this section, we will only consider the proof of the coding theorem for the Gaussian MAC fading channel as described in [3] and follow much of the treatment therein.

4.1 Channel Model

The discrete time Gaussian MAC channel with fading is described by the input-output relationship

$$Y(n) = \sum_{i=1}^{m} \sqrt{H_i(n)} X_i(n) + Z(n) \tag{4.1}$$

where m is the number of users, $X_i(n)$ and $H_i(n)$ are the transmitted waveform and fading process for the i^{th} user respectively and $Z(n)$ is white Gaussian noise with variance σ^2. The fading processes for all users are frequency-non-selective, jointly stationary and ergodic and the stationary distribution has a continuous density and is bounded. A block-average power constraint of P_i is assumed for each user.

For the case where the fading is a constant attenuation for all n, given by $H_i(n) = h_i$, with the channel being described by a probability law $p(y|x_1, ...x_m)$, the capacity region is given as [6], [7]

$$R(S) \leq I(Y; (X_i)_{i \in S} | (X_i)_{i \in S^C}) \;\; \forall \;\; S \subset \{1, 2, ..., m\} \tag{4.2}$$

which reduces in the Gaussian MAC case to

$$C_g(\mathbf{h}, \mathbf{P}) = \{R : R(S) \leq \frac{1}{2} \log(1 + \frac{\sum_{i \in S} h_i P_i}{\sigma^2}), \; \forall S \subset \{1, 2, ..., m\}\} \tag{4.3}$$

where $\mathbf{h} = (h_1, ..., h_m)$ and $\mathbf{P} = (P_1, ..., P_m)$.
The optimal input distribution is again independent Gaussian inputs in this case. For the time-varying fading case, a similar result holds when the channel state is known at the receiver, and the rate region is given by [4]

$$\{\mathbf{R} : \mathbf{R}(S) \leq \mathcal{E}_{\mathbf{H}}[\frac{1}{2} \log(1 + \frac{\sum_{i \in S} H_i P_i}{\sigma^2})], \; \forall S \subset \{1, 2, ..., m\}\} \tag{4.4}$$

where $\mathbf{H} = (H_1, ...H_m)$ is a random vector with the joint stationary distribution of the fading process.

In the case when the fading process is known at both transmitter and the receiver, a power allocation policy can be used to dynamically allocate power to users depending on the fading state of the corresponding channel to achieve capacity. The capacity region obtained following such an approach is called the throughput capacity region. This power allocation policy gives rise to a particular achievable rate. However, the complete rate region is given by the union of all the achievable rates possible given feasible power allocations to the various users.

A power control policy $\mathcal{P}$ is a mapping from the fading state space to $\mathcal{R}_+^{\Updownarrow}$. Given a joint fading state $\mathbf{h} = (h_1, ..., h_m)$ for the users, $\mathcal{P}_i(\mathbf{h})$ is the transmitted power allocated to user i. For a given power control policy $\mathcal{P}$, define the set of rates

$$C_f(\mathcal{P}) = \{\mathbf{R} : \mathbf{R}(S) \leq \mathcal{E}_{\mathbf{H}}[\frac{1}{2}\log(1 + \frac{\sum_{i \in S} H_i \mathcal{P}_i(H)}{\sigma^2})],\ \forall S \subset \{1, 2, ..., m\}\} \tag{4.5}$$

Then the throughput capacity is given by

$$C(\mathbf{P}) = \cup_{\mathcal{P} \subset \mathcal{F}} C_f(P) \tag{4.6}$$

where $\mathcal{F}$ is the set of all feasible power control policies satisfying the average power constraint

$$\mathcal{F} = \{\mathcal{P} : \mathcal{E}_{\mathbf{H}}[\mathcal{P}_i(\mathbf{H})] \leq P_i\ \forall i\} \tag{4.7}$$

Achievability and converse are proved in [3].

The coding theorem for the fading MAC channel with fading known at both transmitter and receiver is the first step in solving the problem of optimal communication. To achieve the capacity promised by the coding theorem, an efficient power-allocation algorithm needs to be presented. [3] uses a greedy algorithm for optimization over polymatroids, as discussed in [8] to achieve this.

References

[1] T. M. Cover, J. A. Thomas. *Elements of information theory*, John Wiley and Sons, New York, 1991.

[2] E. Biglieri, J. Proakis, and S. Shamai. "Fading channels: Information-theoretic and communications aspects," *IEEE Trans. on Inf. Theory*, vol. 44, no. 6, pp. 2619–2692, Oct 1998.

[3] Tse, D. and S.V. Hanly. "Multi-access fading channels: Part I: Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Transactions on Information Theory*, vol. 44, no.7, pp. 2796-2815, Nov 1998.

[4] S. Shamai and A. D. Wyner. "Information theoretic considerations for symmetric, cellular, multiple-access fading channels - Part I," *IEEE Trans. Inform. Theory*, vol. 43, pp. 1877-1894, Nov. 1997.

[5] T. M. Cover and C. S. K. Leung. "An achievable rate region for the multiple access channel with feedback," *IEEE Trans. Inform. Theory*, vol. 27, pp. 292-298, 1981.

[6] R. Ahlswede, "Multi-way communication channels," *Proc. 2nd. Int. Symp. Information Theory* (Armenian S.S.R., 1971), pp. 23-52.

[7] H. Liao, "Multiple access channels," Ph.D. dissertation, Dep. Elec. Eng., Univ. Hawaii, Honolulu, 1972.

[8] J. Edmonds, "Sub-modular functions, matroids and certain polyhedra," *Proc. Calgary Int. Conf. Combinational Structures and Applications* (Calgary, Alta, June 1969), pp. 69-87.