On the Duality of Gaussian Multiple-Access and Broadcast Channels

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I. Introduction

Although T. Cover has been pointed out in [1] that "one would have expected a duality between the broadcast channel(BC) and the multiple access channel (MAC)", this problem is not partially addressed until recently [2]. As the capacity region of general broadcast channel is still not known, the duality between BC and MAC remains open. In [2], it shows that the Gaussian MAC and the Gaussian BC are essentially duals of each other. The same argument can easily be generalized to the fading channel case under ergodic capacity. The duality also holds for outage capacity and minimum rate capacity in fading channels. Further more, the duality between the achievable rate for Gaussian MIMO BC and the capacity region for Gaussian MIMO MAC is addressed in [3]. The practical significance of duality between the two channels is that the solution to one channel may give interesting intuition about the other channel and vice versa. In this summer, I will cover most of the parts in [2] and focus on Gaussian channels. All proofs in this paper can be found in [4]. [3] can be served as further reading after MIMO MAC and MIMO BC are covered in the further of this class.

The rest of the summer is organized as follows. In section II, system models of dual Gaussian MAC and Gaussian BC are described. Capacity region for both channels are also reviewed. Section III states the fundamental theorem that connects Gaussian MAC and Gaussian BC. Section IV shows that the Gaussian BC and MAC are duals. Section V generalizes the same idea to fading channels and MIMO channels. Finally, section VI connects the research in MAC and BC with network information theory.

II. System Model

The dual Gaussian MAC and BC are showed in Fig. 1. The MAC has M transmitters and one receiver. The received signal waveform is

$$Y[i] = \sum_{j=1}^{M} h_j[i] X_j[i] + n[i]. \tag{1}$$

For BC channel,

$$Y_j[i] = h_j[i] X[i] + n_j[i]. \tag{2}$$

Here $h_j[i]$ are channel gains. For the Gaussian channel, $h_j[i]$ are constant for all i and known to both the transmitters and the receivers. $n_j[i]$ and $n[i]$ are additive Gaussian noise with power equal to σ^2.

The capacity region of a Gaussian multiple-access channel with channel gains $\mathbf{h} = (h_1, \cdots, h_M)$ and power constrain $\mathbf{P} = (\mathbf{P_1}, \cdots, \mathbf{P_M})$ is

$$\mathcal{C}_{MAC}(\mathbf{P}; \mathbf{h}) = \left\{ \mathbf{R} : \sum_{\mathbf{j} \in \mathbf{S}} \mathbf{R_j} \leq \frac{\mathbf{1}}{\mathbf{2}} \log \left(\mathbf{1} + \frac{\mathbf{1}}{\sigma^{\mathbf{2}}} \sum_{\mathbf{j} \in \mathbf{S}} \mathbf{h_j^2 P_j} \right) \forall \mathbf{S} \subseteq \{\mathbf{1}, \cdots, \mathbf{M}\} \right\}. \tag{3}$$

The capacity region of the Gaussian MAC is an M-dimensional polyhedron. All corner points of the capacity region can be achieved by successive decoding with interference cancellation. Every decoding order

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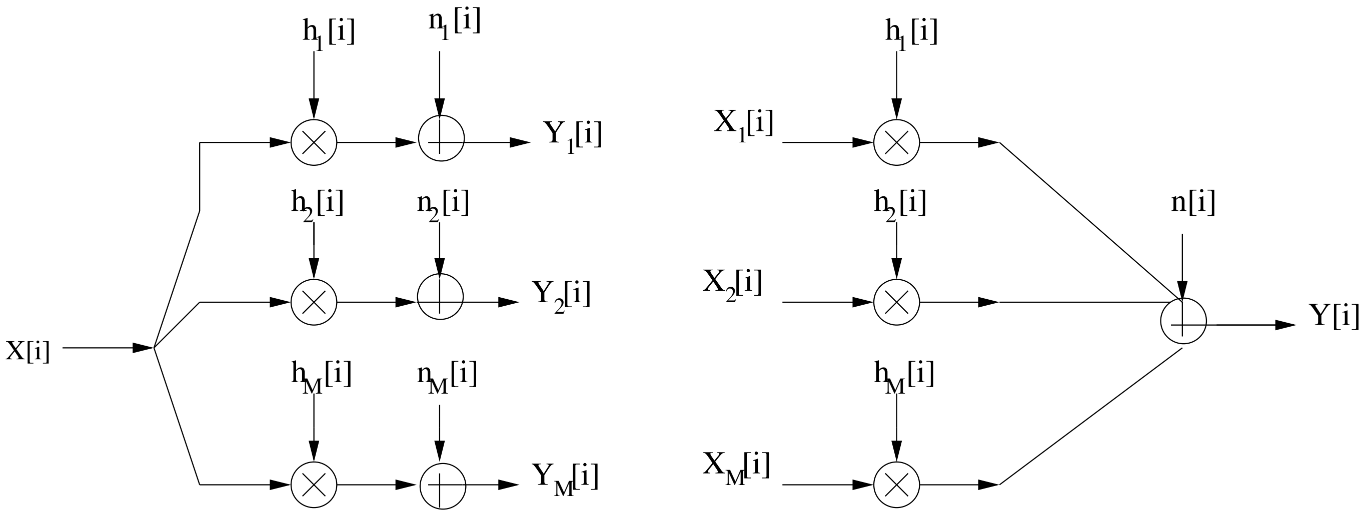

Fig. 1. System Models for the Multiple Access Channel and the Broadcast Channel

corresponds to a different corner point in the capacity region. There are $M!$ corner points in the capacity region. For a decoding order $(\pi(1), \pi(2), \cdots, \pi(M))$, the rates of the corresponding corner point are:

$$R_{\pi(j)} = \frac{1}{2}\log\left(1 + \frac{h^2_{\pi(j)}P_{\pi(j)}}{\sigma^2 + \sum_{i=j+1}^{M} h^2_{\pi(j)}P_{\pi(i)}}\right) j = 1, \cdots, M. \tag{4}$$

The capacity region of a Gaussian BC with channel gains $\mathbf{h} = (h_1, \cdots, h_M)$ and power constrain P is

$$\mathcal{C}_{BC}\left(P; h\right) = \left\{\mathbf{R} : R_j \leq \frac{1}{2}\log\left(1 + \frac{h_j^2 P_j{}^B}{\sigma^2 + h_j^2\sum_{h_k > h_j} P_k^B}\right) j = \{1, \cdots, M\}, \forall \sum_{j=1}^{M} P_j^B = P\right\}. \tag{5}$$

The boundary of the capacity region is achieved by optimal order of successive decoding with interference cancellation. The message is encoded such that the strongest user can decoding all users' messages, the second strongest user can decode all users' message except for the strongest user's message. The "strength" of the user is measured by the channel gain magnitude. Note that stronger here means bigger channel gains, not bigger power. At each receiver, the decoder decodes and subtract out messages intended for other users before decoding its own signal.

Although any points in the capacity region of Gaussian BC can be achieved by interference subtraction at the receiver [1], an alternative way of viewing the capacity region of Gaussian BC is using the result know as "writing-on -dirty paper" [5]. Later we will show that this can simplify the argument of the MAC-BC duality.

Recall that in the study of channel with transmitter side information,

$$y_k = x_k + s_k + n_k, \tag{6}$$

Where x_k and y_k are the transmitted and the received signals respectively, and s_k is an interfering signal known to the transmitter but not to the receiver. In a surprising result by Costa [6], the optimum coding scheme involves the pre-subtraction of s_k at the transmitter. The receiver can decode x_k as if s_k does not exist given that s_k and n_k are both i. i. d. Generalize this result to the vector case when s_k and n_k are independent i. i. d. Gaussian vector processes, we can get the capacity region of Gaussian BC. If we view the transmitter at the BC trying to send $\{x_1, \cdots, x_M\}$ codewords draw from M different codebooks to M different users, thus, the transmitting signal can be viewed as $x = x_1 + x_2 + \cdots + x_M$. Now for each receiver i, $y_i = x_1 + x_2 + \cdots + c_M + n_i \;\; i = 1, 2, \cdots, M$. Then we can apply the dirty-paper coding strategy to get the capacity region. First pick a codeword for any user, the codeword of the next user is selected such that the codeword of the first user will not interfere with the second user. Since the codeword for the first user is already known, this can be done by pre-subtract (using dirty paper coding technique) the first codeword from the second codeword. Repeating this procedure for all users. We can see that any encoding order with any power allocation lie in the capacity region. Assuming encoding order $\pi(1), \pi(2), \cdots, \pi(M)$ in which the codeword of user $\pi(1)$ is encoded first, the rates achieved by this encoding/decoding order are

$$R_{\pi(j)} \leq \frac{1}{2} \log \left(1 + \frac{h^2_{\pi(j)} P^B_{\pi(j)}}{\sigma^2 + h^2_{\pi(j)} \sum_{i=j+1}^{M} P^B_{\pi(i)}} \right). \tag{7}$$

The boundary of the capacity region can be achieved by ordering the encoding order according to the user signal strength. i.e., the weakest user's codeword is selected first and so on. Then $h_{\pi(i)} < h_{\pi(j)} \forall i < j$, combine this with (7), we can actually get the rate set in (5) for a given power allocation. It is also interesting to notice the similarity between (7) and (4).

The rate achieved by optimal successive decoding and optimal dirty-paper coding are the same when the strongest-user -last decoding order is used. However, when sub-optimal decoding/encoding orders are used, the two schemes do differ. For dirty paper coding, as we have shown, any encoding order has a corresponding rate set (by (7)). With successive decoding, when a sub-optimal decoding order is used, it has to be ensured that all users what are supposed to decode and subtract out certain user's signal are actually able to do so. This limits the achievable rates of successive decoding with a sub-optimal decoding order.

III. Relationship between Gaussian MAC and BC

The following theorem states the fundamental connection between Gaussian MAC and BC.

Theorem 1: The capacity region of a constant Gaussian MAC with power constrains $\mathbf{P} = (P_1, \cdots, P_M)$ is a subset of the capacity region of the dual BC with power constraint $P = \mathbf{1} \cdot \mathbf{P}$:

$$\mathcal{C}_{MAC}(\mathbf{P}; \mathbf{h}) \subseteq \mathcal{C}_{BC}(\mathbf{1} \cdot \mathbf{P}; \mathbf{h}). \tag{8}$$

Furthermore, the boundaries of the two regions intersect at exactly one point if the channel gains of all M users are distinct, or $h_i^2 \neq h_j^2$ for all $i \neq j$.

proof outline: It is sufficient to show that every successive decoding point in the MAC ($M!$ in total) is also in the dual BC capacity region because of the convexity of capacity regions. For a give decoding order $\pi(1), \pi(2), \cdots, \pi(M)$ with power constraints $({P_1}^M, \cdots, {P_M}^M)$, the rate of user $\pi(j)$ in the MAC at this successive decoding point is

$$R^M_{\pi(j)} = \frac{1}{2} \log \left(1 + \frac{h^2_{\pi(j)} P^M_{\pi(j)}}{\sigma^2 + \sum_{i=j+1}^{M} h^2_{\pi(i)} P^M_{\pi(i)}} \right). \tag{9}$$

Now assuming that in the dual BC, the opposite encoding/decoding order is used. According to (7), the rate of user $\pi(j)$ with power allocation $(P_1^B, \cdots, P_M^B)$ is

$$R^B_{\pi(j)} = \frac{1}{2} \log \left(1 + \frac{h^2_{\pi(j)} P^B_{\pi(j)}}{\sigma^2 + \sum_{i=1}^{j-1} P^B_{\pi(i)}} \right). \tag{10}$$

Define A_j and B_j as

$$A_j = \sigma^2 + h^2_{\pi(j)} \sum_{i=1}^{j-1} P^B_{\pi(i)}, \quad B_j = \sigma^2 + \sum_{i=j+1}^{M} h^2_{\pi(j)} P^M_{\pi(i)}, \tag{11}$$

We can rewrite the rates in the MAC and BC as

$$R_{\pi(j)}^M = \frac{1}{2}\log\left(1 + \frac{h_{\pi(j)}^2 P_{\pi(j)}^M}{B_j}\right), \quad R_{\pi(j)}^B = \frac{1}{2}\log\left(1 + \frac{h_{\pi(j)}^2 P_{\pi(j)}^M}{A_j}\right), \tag{12}$$

By setting

$$P_{\pi(j)}^B = P_{\pi(j)}^M \frac{A_j}{B_j}, \qquad j = 1, \cdots, M \tag{13}$$

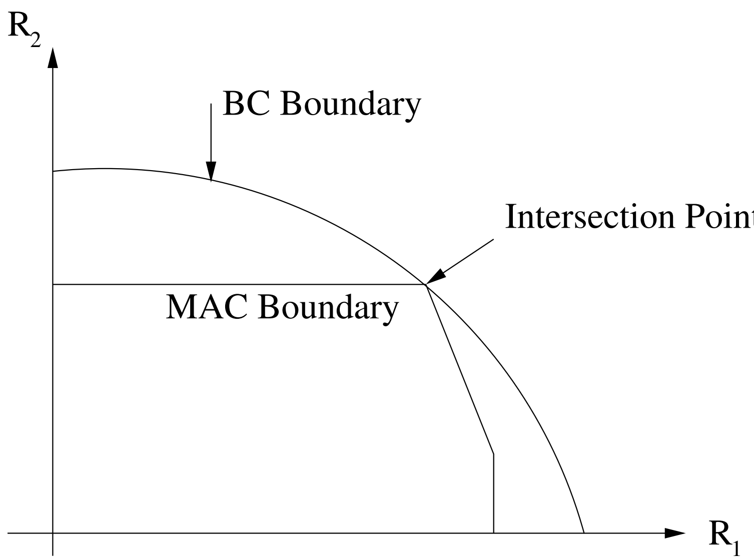

Fig. 2. Capacity regions for Gaussian MAC and its dual BC with $h_1 > h_2$

We will have $R_{\pi(j)}^B = R_{\pi(j)}^M$ and $\sum_{j=1}^M P_j^M = \sum_{j=1}^M P_j^B$. For any decoding order in the MAC, the MAC-BC transformation of (13) holds. This completes the first part of theorem 1. Now consider the decoding order such that $h_{\pi(i)}^2 \geq h_{\pi(i+1)}^2$ for $i = 1, \cdots, M-1$. Since on the dual BC channel, the corresponding rates are (10) by using the opposite decoding order, which is the "strongest-last" optimal order. Thus, the rate point on the dual BC channel is on the boundary. It is interesting to note that in MAC, the successive decoding algorithm decodes the weakest user last, its counterpart in BC decoding the strongest user last. Fig. 2 illustrates the idea on the two user case. Give that $h_1 > h_2$, the intersection point in the MAC channel is achieved by decoding user one (stronger user) first.

IV. Duality between Gaussian MAC and BC

First we will show that the capacity region of a Gaussian BC can be characterized in terms of capacity regions of the dual MAC.

Theorem 2: The capacity region of a constant Gaussian BC with power constraint P is equal to the union of capacity regions of the dual MAC with power constraints $(P_1, \cdots, P_M)$ such that $\sum_{j=1}^M P_j = P$:

$$\mathcal{C}_{BC}(P; \mathbf{h}) = \bigcup_{\{\mathbf{P}:\mathbf{1}\cdot\mathbf{P}=P\}} \mathcal{C}_{MAC}(\mathbf{P}; \mathbf{h}). \tag{14}$$

proof outline: From Theorem 1, we have$\bigcup_{\mathbf{P}:\mathbf{1}\cdot\mathbf{P}=P} \mathcal{C}_{MAC}(\mathbf{P}; \mathbf{h}) \subseteq \mathcal{C}_{BC}(P; \mathbf{h})$. By using the MAC-BC transformations in (13) in the opposite direction, it is easy to show that any points on the boundary of the BC capacity region is in the capacity region of the dual MAC, with power constraints $(P_1, \cdots, P_M)$ that satisfy

$$P_{\pi(j)}^M = P_{\pi(j)}^B \frac{B_j}{A_j}, \quad j = 1, \cdots, M \tag{15}$$

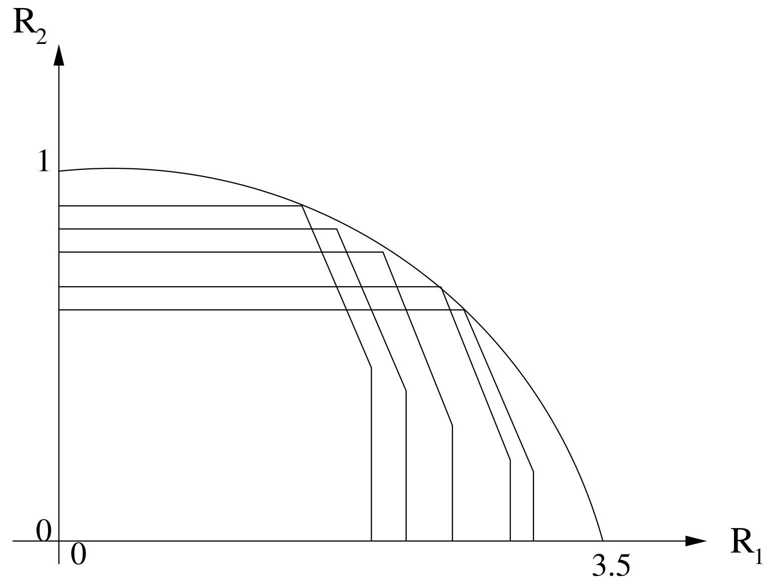

Fig. 3. From MAC capacity region to BC capacity region

We refer to this transformation as the BC-MAC transformation. Then $\mathcal{C}_{BC}(P;\mathbf{h}) \subseteq \bigcup_{\mathbf{P}:\mathbf{1}\cdot\mathbf{P}=P} \mathcal{C}_{MAC}(\mathbf{P};\mathbf{h})$. This completes the proof.

From Gaussian MAC capacity region to Gaussian BC capacity region is showed in Fig. 3 in a two user scenario. One corner point of the MAC capacity region touches the boundary of the BC capacity region, When changing the power constrain on each user while keep the sum power the same, the corner point of the MAC capacity region touches a different point on the boundary of the BC capacity region. Note that the channel state does not change, i.e. $h_1 > h_2$ all the time, so we always decode user one first to achieve the corner point.

Now we will show that the capacity region of the Gaussian MAC can be characterized in terms of the capacity region of the dual BC.

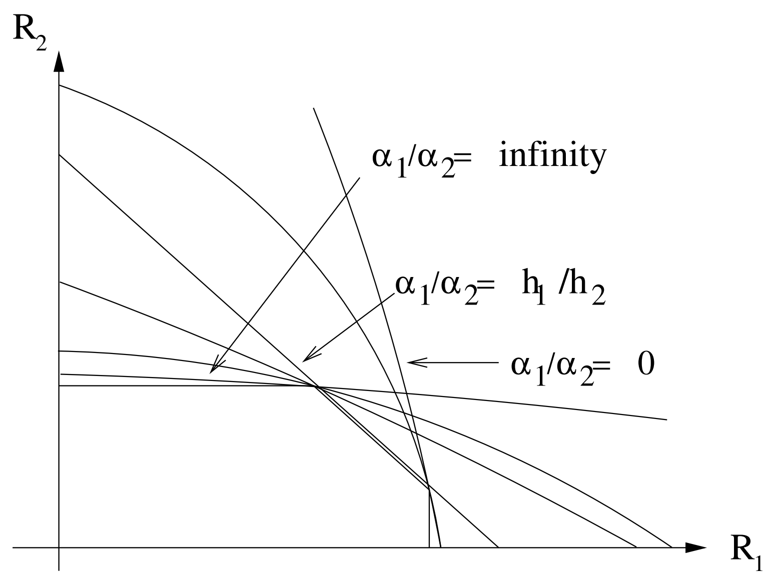

Fig. 4. Intersecting BC capacity region to get MAC capacity region

Theorem 3: The capacity region of a constant Gaussian MAC is equal to the intersection of the capacity regions of the scaled dual BC over all scalings:

$$\mathcal{C}_{MAC}(\mathbf{P};\mathbf{h}) = \bigcap_{\alpha>0} \mathcal{C}_{BC}(\mathbf{1}\cdot\frac{\mathbf{P}}{\alpha};\sqrt{\alpha}\mathbf{h}). \tag{16}$$

proof outline: Note that $\mathcal{C}_{MAC}(\mathbf{P};\mathbf{h}) = \mathcal{C}_{MAC}(\frac{\mathbf{P}}{\alpha}\sqrt{\alpha}\mathbf{h}) \quad \forall \alpha > 0$ This can be seen below,

$$\begin{aligned}
\mathcal{C}_{MAC}(\mathbf{P};\mathbf{h}) &= \left\{\mathbf{R} : \sum_{j\in S} R_j \leq \frac{1}{2}\log\left(1+\frac{1}{\sigma^2}\sum_{j\in S} h_j^2 P_j\right) \forall S \subseteq \{1,\cdots,M\}\right\} \\
&= \left\{\mathbf{R} : \sum_{j\in S} R_j \leq \frac{1}{2}\log\left(1+\frac{1}{\sigma^2}\sum_{j\in S} (\sqrt{\alpha_j}h_j)^2\left(\frac{P_j}{\alpha_j}\right)\right) \forall S \subseteq \{1,\cdots,M\}\right\}. \\
&= \mathcal{C}_{MAC}(\frac{\mathbf{P}}{\alpha}) \quad \forall \alpha > 0.
\end{aligned} \tag{17}$$

Applying Theorem 1, we have:

$$\mathcal{C}_{MAC}(\mathbf{P};\mathbf{h}) \subseteq \mathcal{C}_{BC}(\mathbf{1}\cdot\frac{\mathbf{P}}{\alpha};\sqrt{\alpha}\mathbf{h}) \quad \forall \alpha > 0. \tag{18}$$

A detailed proof can be found in [4]. Fig. 4 showed the illustration for two-user case.

V. Duality of fading/MIMO Gaussian MAC and BC

The previous argument and derivation can be easily extended to fading channels and MIMO channels. Only the capacity region of fading/MIMO MACs and BCs will be introduced in this section. For fading channels, only flat-fading MAC and BC are considered here, also assume perfect CSI at both receivers and transmitters.

A power policy $\mathcal{P}_{MAC}$ over all possible fading states is a function that maps from a joint fading state $\mathbf{h} = (h_1,\cdots,h_M)$ to the transmitted power $(P_1^M(\mathbf{h}),\cdots,P_M^M(\mathbf{h}))$ Let $\mathcal{F}_{MAC} = \{\mathcal{P}_{MAC} : \mathbf{E}_{\mathbf{H}}[P_j^M(\mathbf{h})] \leq P_j, \quad 1 \leq j \leq M\}$. Then the ergodic capacity region of the MAC with perfect CSI and power constraints $\mathbf{P} = (P_1,\cdots,P_M)$ is:

$$\mathcal{C}_{MAC}(\mathbf{P};\mathbf{H}) = \bigcup_{\mathcal{P}_{MAC}\in\mathcal{F}_{MAC}} \mathcal{C}_{MAC}(\mathcal{P}_{MAC};\mathbf{H}) \tag{19}$$

where

$$\mathcal{C}_{MAC}(\mathcal{P}_{MAC};\mathbf{H}) = \left\{\mathbf{R} : \sum_{j\in S} R_j \leq \mathbf{E}_{\mathbf{H}}\left[\frac{1}{2}\log\left(1+\frac{1}{\sigma^2}\sum_{j\in s} h_j^2 P_j^M(\mathbf{h})\right)\right], \ \forall S \subseteq \{1,\cdots,M\}\right\}. \tag{20}$$

For fading BC, define similarly all power policies $\mathcal{F}_{BC}$ as $\mathcal{F}_{BC} = \{\mathcal{P}_{BC} : \mathbf{E}_{\mathbf{H}}[\sum_{j=1}^M P_j^B(\mathbf{h})] \leq P\}$. The ergodic capacity region of the BC with perfect CSI and power constraint P is :

$$\mathcal{C}_{BC}(P;\mathbf{H}) = \bigcup_{\mathcal{P}_{BC}\in\mathcal{F}_{BC}} \mathcal{C}_{BC}(\mathcal{P}_{BC};\mathbf{H}) \tag{21}$$

where

$$\mathcal{C}_{BC}(\mathcal{P}_{BC};\mathbf{H}) = \left\{\mathbf{R} : R_j \leq \mathbf{E}_{\mathbf{H}}\left[\frac{1}{2}\log\left(1+\frac{h_j^2 P_j^B(\mathbf{h})}{\sigma^2 + h_j^2\sum_{h_k^2 > h_j^2} P_k^B(\mathbf{h})}\right)\right], \ \forall S \subseteq \{1,\cdots,M\}\right\}. \tag{22}$$

We can see the similarly between the ergodic capacity region for fading channels and capacity region for Gaussian channels. It is also showed that the duality holds for frequency selective fading channels.

For MIMO MAC and BC, the channel gain h_j is now denoted by a matrix H_j. The capacity region form is in the matrix form consequently. The MIMO MAC capacity region is

$$\mathcal{C}_{MAC}(P_1,\cdots,P_M;\mathbf{H}) = \bigcup_{\{Tr(\Sigma_{\mathbf{x_i}})\leq P_i \ \forall i\}} \left\{(R_1,\cdots,R_M) : \sum_{i\in S} R_i \leq \frac{1}{2}\log|\mathbf{I} + \sum_{i\in S}\mathbf{H_i^T} P_i \mathbf{H_i}| \ \ \forall S \subseteq \{1,\cdots,M\}\right\} \tag{23}$$

where $(P_1, \cdots, P_M)$ are power constrains. For MIMO BC, the capacity region is not known, but the achievable rate can be obtained by extending the dirty paper result to the vector case. For any decoding order $\{\pi(1), \cdots, \pi(M)\}$, the achievable set of rates is

$$R_{\pi(i)} = \frac{1}{2} \log \frac{|\mathbf{I} + \mathbf{H}_{\pi(i)}(\sum_{j \geq i} \Sigma_{\pi(j)})\mathbf{H}^T_{\pi(i)}|}{|\mathbf{I} + \mathbf{H}_{\pi(i)}(\sum_{j > i} \Sigma_{\pi(j)})\mathbf{H}^T_{\pi(i)}|} \;\; i = 1, \cdots, K. \tag{24}$$

Here $\Sigma_i \;\; i = 1, \cdots, M$ are power allocations for the ith-user satisfying $Tr(\sum \Sigma_i) \leq P$. The achievable rate region is all set of rates over all decoding orders and over all power allocations. We can see that this is the matrix form of (7).

VI. Final remarks and network information theory

Gaussian channels are widely used to model real-world communication systems. The duality between Gaussian MAC and BC gives more insight view of both channels. An interesting problem would be to see if the duality holds between Gaussian MAC with feedback and Gaussian BC with feedback. Although the capacity region for general MAC with feedback is not know, the capacity region of Gaussian MAC with feedback is known [7]. A. Gamal showed that feedback does not increase capacity region for physically degraded broadcast channel [8], However, feedback do enlarge the capacity region for physically non-degraded Gaussian BCs [9]. The possible duality of the two channels might shed lights on the achieve rates (or even the capacity region) of Gaussian BC with feedback.

A recent active research area is the network information theory. Both MAC and BC are special cases of multiterminal communication networks. Feedback is usually considered since the random variables of most communication networks are related to each other in a complicated manner. Coding for networks is restricted by *causality*. Thus, *causal conditioning* is more appropriate concept to capture the essential aspects of such coding. In consequence, the *directed information*, first introduced by Massey [10], is used instead to describe the capacity of channels with feedback. In general, the rate region described by *directed information* is bigger than that by traditional mutual information for channels with feedback. In the case of MAC and BC with feedback, Kramer [11] showed that the best known rate regions for both channels are enlarged using *directed information* and code tree.

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