A Summary of Multiple Access Channels*

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February 24, 2003

Abstract

In this summary we attempt to present a brief overview of the classical results on the information-theoretic aspects of multiple access channels (MACs thereafter). We confine our discussion to the basic MAC model which is discrete, memoryless, without any channel side information or feedback, and where the two (or more) senders are synchronized. In Section 1 we introduce the channel model and formulate the problem. In Section 2 this general problem is solved, i.e., the capacity region for a general MAC is established. Corresponding results for the Gaussian MAC is presented in Section 3. Finally some references are listed for further reading.

1 The MAC Model and Problem Formulation

Intuitively MAC is a channel via which two (or more) senders send information to a common receiver. To make the point precise, we introduce the following definitions.

Definition 1 *(Discrete memoryless MAC) A discrete memoryless MAC consists of three alphabets,* $\mathcal{X}_1$, $\mathcal{X}_2$ *and* $\mathcal{Y}$, *and a probability transition matrix* $p(y|x_1, x_2)$.

As shown in Figure 1.

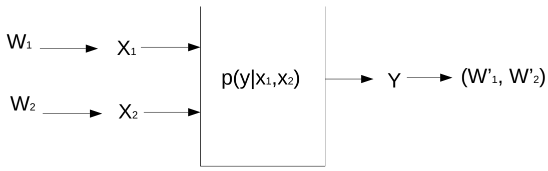

Figure 1: The discrete memoryless MAC model

*This summary is mainly based on the material in [1], the classical textbook by T. Cover and J. Thomas.

Definition 2 *(Encoding/decoding of MAC) A* $((2^{nR_1}, 2^{nR_2}), n)$ *code for the MAC consists of two sets of integers* $\mathcal{W}_1 = \{1, 2, \ldots, 2^{nR_1}\}$ *and* $\mathcal{W}_2 = \{1, 2, \ldots, 2^{nR_2}\}$ *called the message sets, two encoding functions,*

$$X_1 : \mathcal{W}_1 \to \mathcal{X}_1^n, \tag{1}$$

$$X_2 : \mathcal{W}_2 \to \mathcal{X}_2^n \tag{2}$$

and a decoding function

$$g : \mathcal{Y}^n \to \mathcal{W}_1 \times \mathcal{W}_2. \tag{3}$$

Here we see that, first, the MAC model is discrete, furthermore, an implicit and inherent assumption is that the two senders are synchronized. One might imagine that there is a global clock available to both senders and the receiver. This reduces to the discrete model here, which is simpler for analysis.[1]

Second, this model does not take into any consideration of bursty characteristic of the two senders. Sender 1 always chooses an index W_1 uniformly from $\mathcal{W}_1$ and sends the corresponding codeword over the channel. What sender 2 does is likewise.

Finally, note that the model is memoryless, thus we have

$$p(y^n | x_1^n, x_2^n) = \prod_{i=1}^{n} p(y_i | x_{1i}, x_{2i}) \tag{4}$$

Now let us introduce the definitions of achievable rate pair and capacity region for MAC.

Definition 3 *(achievable rate pair) A rate pair* (R_1, R_2) *is achievable for the MAC if there exists a sequence of* $((2^{nR_1}, 2^{nR_2}), n)$ *codes with* $P_e^{(n)} \to 0$*. Here* $P_e^{(n)}$ *is the average probability of error defined as*

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(w_1,w_2)\in\mathcal{W}_1\times\mathcal{W}_2} Pr\{g(Y^n) \neq (w_1, w_2) | (w_1, w_2) sent\}. \tag{5}$$

Definition 4 *(capacity region) The capacity region of the MAC is the closure of the set of achievable* (R_1, R_2) *rate pairs.*

2 Capacity Region for General MACs

The theory on the capacity region of MAC is a bit involved. In this section we present the established results following the logical order as in [3], discuss their interpretations, along with the outline of their proofs (those looking for rigorous mathematics can refer to [1] or [3]).

First the following lemma holds true

[1] In [2] (pp. 132) there is a discussion on whether such phase synchronization a reasonable model, and references dealing with asynchronous MAC will be given in Section 4.

Lemma 1 *(Time sharing principle) The capacity region $\mathcal{C}$ of the MAC is a closed convex set. Convexity here refers to that, if $(R_1, R_2) \in \mathcal{C}$ and $(R_1', R_2') \in \mathcal{C}$, then $(\lambda R_1 + (1-\lambda)R_1', \lambda R_2 + (1-\lambda)R_2') \in \mathcal{C}$ for $0 \leq \lambda \leq 1$.*

Then in the next step, we assume that the two encoders satisfy

$$p(x_1, x_2) = p_1(x_1)p_2(x_2) \tag{6}$$

Given any two distributions $p_1(x_1)$ and $p_2(x_2)$ over $\mathcal{X}_1$ and $\mathcal{X}_2$, we have

Theorem 1 *(MAC capacity region for given $p_1(x_1)$ and $p_2(x_2)$) For any given $p_1(x_1)$ and $p_2(x_2)$, all pairs (R_1, R_2) satisfying*

$$R_1 < I(X_1; Y|X_2), \tag{7}$$

$$R_2 < I(X_2; Y|X_1), \tag{8}$$

$$R_1 + R_2 < I(X_1, X_2; Y) \tag{9}$$

belong to the capacity region of the MAC.

Theorem 1, together with Lemma 1, establishes the MAC capacity region:

Theorem 2 *(MAC capacity region) The capacity region $\mathcal{C}$ of the MAC is the convex closure of the regions established in Theorem 1 for every possible product distribution $p_1(x_1)p_2(x_2)$ over $\mathcal{X}_1 \times \mathcal{X}_2$.*

Furthermore, the capacity region $\mathcal{C}$ established above is equivalent to the region specified in the following corollary:

Corollary 1 *The capacity region of the MAC is also given by the convex closure of all (R_1, R_2) pairs satisfying*

$$R_1 < I(X_1; Y|X_2, Q), \tag{10}$$

$$R_2 < I(X_2; Y|X_1, Q), \tag{11}$$

$$R_1 + R_2 < I(X_1, X_2; Y|Q) \tag{12}$$

Here Q is some auxiliary random variable such that $X_1 \leftrightarrow Q \leftrightarrow X_2$ and $Q \leftrightarrow (X_1, X_2) \leftrightarrow Y$ are two Markov chains. Furthermore, $|Q|$ can be as small as 2.

Now let us attempt to interpret the above results. The time sharing principle tells us two things. First, $\mathcal{C}$ is a closed set, this is obvious since its complement $\overline{\mathcal{C}}$ is open due to the definition of $\mathcal{C}$. Second, $\mathcal{C}$ is convex. This property is illustrated via constructing a new timesharing code from any two existing codes with achievable rate pairs $\mathbf{R} = (R_1, R_2)$ and $\mathbf{R}' = (R_1', R_2')$. The new codebook is constructed by using codebook of $\mathbf{R}$ for the first λn channel symbols, and using codebook of $\mathbf{R}'$ for the last $(1-\lambda)n$ channel symbols. The rate of this new code is $\lambda\mathbf{R} + (1-\lambda)\mathbf{R}'$, and this new code rate is achievable, so the capacity region is convex.

Theorem 1 is famous, and occasionally its region, a beautiful pentagon (Figure 2), is incorrectly interpreted as the final capacity region of the MAC. But generally it is only a subset of the actual capacity region $\mathcal{C}$, since it is achieved when the distribution of (X_1, X_2) is fixed as some $p_1(x_1)p_2(x_2)$ over $\mathcal{X}_1 \times \mathcal{X}_2$, rather than for every possible distribution of (X_1, X_2).

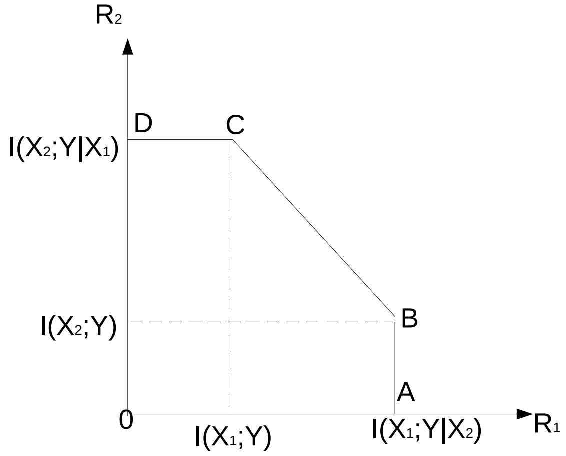

Figure 2: The achievable rate region for fixed $p_1(x_1)p_2(x_2)$

For a given $p_1(x_1)p_2(x_2)$, it is of interest to investigate the boundary of the pentagon region. As shown in Figure 2, the point A corresponds to the maximum rate achievable from sender 1 to the receiver while sender 2 is not sending any information, i.e.,

$$R_1 = \max R_1 = \max_{p_1(x_1)p_2(x_2)} I(X_1; Y|X_2), \tag{13}$$

$$R_2 = 0. \tag{14}$$

Now since

$$I(X_1; Y|X_2) = \sum_{\mathcal{X}_2} p_2(x_2) I(X_1; Y|X_2 = x_2) \tag{15}$$

$$\leq \max_{\mathcal{X}_2} I(X_1; Y|X_2 = x_2), \tag{16}$$

therefore $\max R_1$ is attained when X_2 is fixed to the paricular $x_2^\star$ which maximizes the conditional mutual information between X_1 and Y.

Next an interesting and anti-intuitive result is that sender 2 can actually send at some non-zero rate while the rate of sender 1 is still kept the same as at point A. In particular, point B

$(R_1 = I(X_1;Y|X_2), R_2 = I(X_2;Y))$ corresponds to such maximum rate of sender 2, i.e., sender 2 cannot send at rate greater than $I(X_2;Y)$ while sender 1 is still sending at rate $I(X_1;Y|X_2)$, its maximum rate. Point B is achieved when sender 2 sends as if over a single-user channel, at rate $I(X_2;Y)$. The receiver can decode the codewords from sender 2 with arbitrarily small error probability, and "substract" its effect from the compound received symbol sequence. Now as we have decoded W_2, we can index the MAC channel as a series of single-user channels, where the index is the X_2 symbol used. Then the rate of sender 1 is achieved by taking the average mutual information over all these indexed channels, and each channel occurs as many times as the corresponding X_2 symbol appears in the codewords. Hence the rate R_1 is

$$R_1 = \sum_{\mathcal{X}_2} p(x_2) I(X_1;Y|X_2 = x_2) = I(X_1;Y|X_2). \tag{17}$$

The points C and D correspond to B and A, respectively, with the roles of sender 1 and 2 reversed.

Actually the time sharing principle already plays a role in Theorem 1, all the non-corner points on the boundary can be achieved via timesharing.

For segment AB (DC similarly), one can also imagine that there is a "genie" at the decoder which knows exactly (with arbitrarily small error probability) what codeword encoder 2 sends, assisting the decoding process. Such a "genie" can actually be obtained for free if R_2 is less than $I(X_2;Y)$, the capacity of the virtual single-user channel between X_2 and Y.

As stated in Thereom 2, the capacity region for the MAC is the convex closure of all the possible pentagons in Theorem 1, as illustrated in Figure 3. First we solve the regions (pentagons) for all the possible $p_1(x_1)p_2(x_2)$, then take their union, finally take this union's convex closure, as the capacity region $\mathcal{C}$. Usually it suffices to stop after taking the union, but there actually exist some occasions where the union is nonconvex, then we need to take its convex closure. One such example is given in [2], for the collision channel.

Corollary 1 is actually a recast of Theorem 2 with the formal statement of timesharing using an auxiliary random variable Q. The idea is similar to the argument in the interpretation of point B. The MAC channel is indexed by random variable Q, for each $Q = q$ there corresponds a code with rate $\mathbf{R}_q$, by taking the average over all the possible $Q = q$, the time sharing principle asserts that this average rate is also achievable.[2]

The achievability part of Theorem 1 can be proved following the standard path in [1]. Fix $p_1(x_1)p_2(x_2)$.

Codebook generation. Generate 2^{nR_1} independent codewords $\mathbf{X}_1(i), i \in \{1, 2, \ldots, 2^{nR_1}\}$, of length n, generating each element i.i.d. with $\prod_{i=1}^n p_1(x_{1i})$. Similarly, generate 2^{nR_2} independent codewords $\mathbf{X}_2(j), j \in \{1, 2, \ldots, 2^{nR_2}\}$, generating each element i.i.d. with $\prod_{i=1}^n p_2(x_{2i})$. These codewords form the codebook, which is revealed to the senders and the receiver.

[2]Using this type of argument, it seems that the cardinality of Q can be as small as 2, as stated in [3], rather than 4, as in [1].

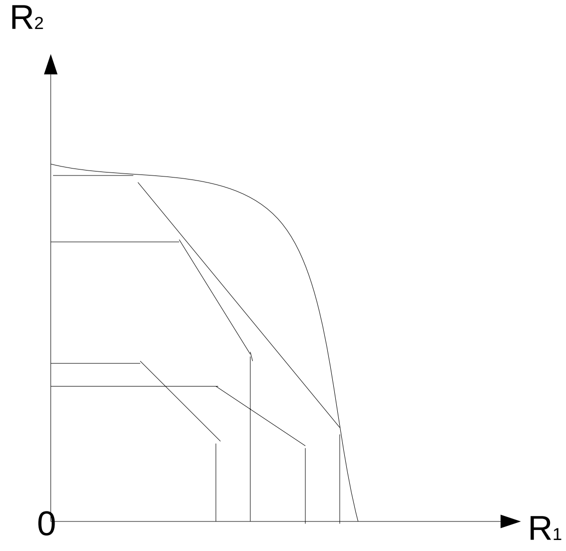

Figure 3: Capacity region $\mathcal{C}$ of the MAC

Encoding. To send index i, sender 1 sends the codeword $\mathbf{X}_1(i)$. Similarly, to send j, sender 2 sends $\mathbf{X}_2(j)$.

Decoding. Let $A_{\epsilon}^{(}n)$ denote the set of typical $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$ sequences. The receiver chooses the pair (i, j) such that

$$(\mathbf{x}_1(i), \mathbf{x}_2(j), \mathbf{y}) \in A_{epsilon}^{(}n) \tag{18}$$

if such a pair (i, j) exists and is unique; otherwise, an error is declared.

Analysis of the probability of error. By the symmetry of the random code construction, the conditional probability of error does not depend on which pair of indices is sent. Thus the conditional probability of error is the same as the unconditional probability of error. So, without loss of generality, we can assume that $(i, j) = (1, 1)$ was sent.

The details of the analysis is omitted here, and can be found in [1].

On the other hand, to prove the converse part of Theorem 2, as in the single-user channel case, the key is still Fano's inequality, along with other manipulations which finally upper bound

$$R_1 \leq \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}; Y_i | X_{2i}) + \epsilon_n, \tag{19}$$

$$R_2 \leq \frac{1}{n} \sum_{i=1}^{n} I(X_{2i}; Y_i | X_{1i}) + \epsilon_n, \tag{20}$$

$$R_1 + R_2 \leq \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_i) + \epsilon_n. \tag{21}$$

The details can also be found in [1].

Here for the integrity of presentation, we give the capacity region result for generalized m-user MAC.

Theorem 3 *(Capacity region for m-user MAC) The capacity region of the m-user MAC is the convex closure of the rate vectors satisfying*

$$R(S) \leq I(X(S);Y|X(S^c)) \; \forall S \quad \subseteq \quad \{1,2,\ldots,m\} \tag{22}$$

for every possible product distribution $p_1(x_1)p_2(x_2)\ldots p_m(x_m)$.

3 The Gaussian MAC

In the Gaussian MAC, there are two (or more) senders, X_1 and X_2, sending to the single receiver Y. The received signal at time i is

$$Y_i = X_{1i} + X_{2i} + Z_i, \tag{23}$$

where $\{Z_i\}$ is a sequence of i.i.d. zero Gaussian random variables with variance N. For each sender j there is a power constraint P_j, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} x_{ji}^2(w_j) \leq P_j, \; w_j \in \{1,2,\ldots,2^{nR_j}\}, \; j=1,2. \tag{24}$$

A nice property of the Gaussian MAC is that its capacity region can be fully attained using one single optimal product distribution, without regard to any operation such as union or convex closure. This is formalized in Theorem 4.

Theorem 4 *(Capacity region of the Gaussian MAC) The capacity region of the Gaussian MAC is defined as*

$$R_1 \quad < \quad C(\frac{P_1}{N}), \tag{25}$$

$$R_2 \quad < \quad C(\frac{P_2}{N}), \tag{26}$$

$$R_1 + R_2 \quad < \quad C(\frac{P_1+P_2}{N}). \tag{27}$$

where $C(x)$ *is defined as* $\frac{1}{2}\log(1+x)$.

These upper bounds are achieved when X_1 *is* $\mathcal{N}(0,P_1)$ *and* X_2 *is* $\mathcal{N}(0,P_2)$*, independently.*

Almost parallel to the case in discrete-valued MAC (Section 2), Theorem 4 also corresponds to a two-stage decoding process. For example, consider point B in Figure 4, this corner point corresponds to the case that the decoder first decodes the codeword from sender 2, then "substracts" it from the compound received symbol before decoding the codeword from sender 1.

This two-stage decoding philosophy is the root of CDMA, in which the decoder decodes each sender one by one, considering the remaining as noise. But sometimes it is of interest (e.g., with lower complexity) to use other types of schemes, such as TDMA or FDMA.

In TDMA scheme, for each channel symbol slot, only one of the sender can use the channel, so the boundary of its capacity region is achieved via timesharing between points A and D (in Figure 4). I.e., the capacity region of TDMA is the triangle area OAD as shown in Figure 4.

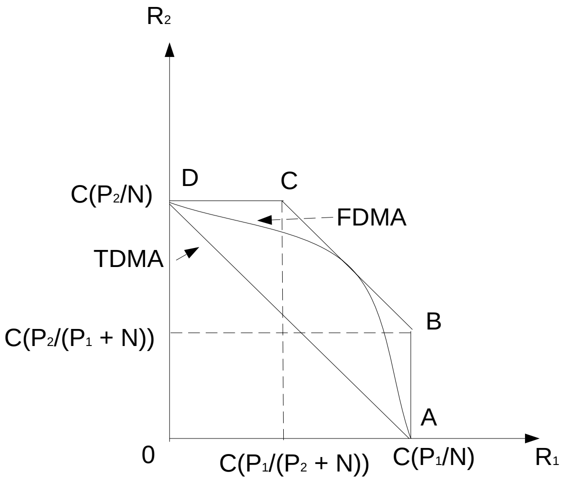

Figure 4: Capacity region of the Gaussian MAC

In FDMA scheme, the whole bandwidth W is divided into two non-intersecting subbands W_1 and W_2 where $W_1 + W_2 = W$, then the MAC appears as two independent single-user channels. So using the well-known result for band-limited Gaussian channel, we have

$$R_1 = \frac{W_1}{2}\log(1+\frac{P_1}{NW_1}), \tag{28}$$

$$R_2 = \frac{W_2}{2}\log(1+\frac{P_2}{NW_2}). \tag{29}$$

Via adjusting the ratio between W_1 and W_2, we obtain the whole achievable capacity curve for FDMA, as shown in Figure 4, which is better than TDMA, but still suboptimal, except at three points: $(\frac{W}{2}\log(1+\frac{P_1}{NW}), 0)$ when $(W_1, W_2) = (W, 0)$, $(0, \frac{W}{2}\log(1+\frac{P_2}{NW}))$ when $(W_1, W_2) = (0, W)$, and $(\frac{P_1 W}{2(P_1+P_2)}\log(1+\frac{P_1+P_2}{NW}), \frac{P_2 W}{2(P_1+P_2)}\log(1+\frac{P_1+P_2}{NW}))$ when $(W_1, W_2) = (\frac{P_1}{P_1+P_2}W, \frac{P_2}{P_1+P_2}W)$. The last case implies that, to optimally exploit the capacity of FDMA, one should allotting bandwidth to each sender proportional to its power constraint, or conversely, one should allotting power to each sender proportional to its allotted bandwidth. Further discussions on FDMA in Gaussian MAC can be found in the first few chapters of [4].

Here for the completeness we also present the capacity region result for m-user Gaussian MAC.

Theorem 5 *(m-user Gaussian MAC) For the Gaussian MAC where there are m users with power constraints $(P_1, P_2, \ldots, P_m)$, and AWGN noise which is $\mathcal{N}(0, N)$, we have for any set $S \subseteq \{1, 2, \ldots, m\}$,*

$$\sum_{i \in S} R_i < C(\frac{\sum_{i \in S} P_i}{N}). \tag{30}$$

As a special case, when the m users' power constraints are the same P, $\sum_{i=1}^{m} R_i < C(\frac{mP}{N})$, which goes unbounded as m gets large!

4 References List for Further Reading

Here some references are listed, for potential further reading on those relevant specified topics. The MAC problem was originally proposed in [5] and [6]. [1], [3] both have excellent chapters on MAC, and a series of further literatures can be found in their historic remarks and references lists. In [2], topics on error exponents and coding techniques were treated.

MAC with feedback was discussed in [7], [8], [9], etc.

Asynchronous MAC was discussed in [10], [11], [12], [13], etc.

Rate-splitting was first introduced in [14].

There are huge literatures dealing with MAC with channel side information (CSI) or over fading channels, those refernces are omitted here since these topics will be discussed in subsequent presentations this semester.

References

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A Summary of Multiple Access Channels: Errata and Some Discussions

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March 18, 2003

- In page 3, *Theorem 2 (MAC capacity region)*, "The capacity region *mathcalC* of the MAC is the **closed convex hull** of the regions established in ...", as well all the "**convex closure**" in the other places in this summary should be replaced by "**closed convex hull**". This is because in *Theorem 1 (MAC capacity region for given* $p_1(x_1)p_2(x_2)$*)* we did not specify the lower bounds on R_1 and R_2, i.e., $R_1 \geq 0$ and $R_2 \geq 0$.

- In page 3, *Corollary 1*, "... $|Q|$ can be as small as 2." It seems that this "2 or 4" still bothers us in our class. The Carath eodory theorem essentially deals with this problem: for an arbitrary point in the convex closure $conv(A)$ induced by set $A \subset \mathcal{R}^n$, at least how many points in A are sufficient to represent it as the convex combination of these chosen points? For a general set A the minimum number is actually $n+1$, but if the set A has almost n connected components, then it can be shown that n points are sufficient[1]. Here in our MAC problem, the set A is the union of all the possible pentagons, and it seems to be connected(?) in $\mathcal{R}^2$, so I myself still insist that the cardinality of the time-sharing random variable Q should be 2.

- In page 4, we discussed a lot in class about the interpretation of operating point A in the Figure 2. There was a bit confusion about the relation between the selected input distributions $p_1(x_1)$ and $p_2(x_2)$ and rates pair chosen. At point A, the input distributions are still $p_1(x_1)$ and $p_2(x_2)$, but the rate R_2 is chosen to be 0.

- In page 6, Figure 3, all the slope segments should be of -45^o, in parallel.

- In page 6, there are two typos in the *Decoding.* part: $A_\epsilon^(n)$ and $A_{epsilon}^(n)$, both of them should be $A_\epsilon^{(n)}$.

References

[1] S.R. Lay. *Convex Sets and Their Applications.* Wiley, 1982.