MIMO Broadcast Channel

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I. INTRODUCTION

Let us consider the following discrete, memoryless communication channel model,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}, \tag{1}$$

where $\mathbf{x} \in \mathcal{C}^{\mathbf{t}}$ is the input vector, $\mathbf{y} \in \mathcal{C}^{\mathbf{r}}$ is the output vector, $H \in \mathcal{C}^{r \times t}$ is the channel matrix, $\mathbf{z} \in \mathcal{C}^{\mathbf{r}}$ is the Gaussian random noise vector $z \sim \mathcal{N}_{\mathcal{C}}(0,1)$. Without making any further assumptions, this channel is a general Multiple Input Multiple Output (MIMO) channel. We say that the transmitters can cooperate, if the messages is jointly encoded into the input vector $\mathbf{x}$, instead of each entries of $\mathbf{x}$. We say the receivers can cooperate if the whole output vector $\mathbf{y}$ instead of each individual entries of $\mathbf{y}$ is used to decode the message.The transmitter cooperation usually means the elements in $\mathbf{x}$ are somehow correlated, while the receiver cooperation usually means all entries of $\mathbf{y}$ are used for decoding each message. If no partial cooperation in any subset is allowed, the channel can have four different interpretations [1] according to the whether either side cooperates,

1. If both transmitters and receivers are allowed to cooperate, it represents a single point to point channel, which is called single user MIMO channel. In the context of multiple antenna systems, the capacity region is found in Telatar's famous paper [2].

2. If neither transmitters nor receivers are not allowed to cooperate, it represents a multi-point to multi-point interference channel.

3. If only receivers can cooperate, it represents a multi-point to point channel which can be a multiple-access channel (MAC) with each user has one antenna.

4. If only transmitters can cooperate, it represents a point to multi-point channel which can be a broadcast channel (BC) with each user has one antenna.

The capacity of the channel in the last scenario, i.e. broadcast channel is what will be discussed in this summary.

II. HISTORY AND STATE OF THE ART

Before we delve ourself into the search for capacity of MIMO BC. Let's review some major results and the method used. The capacity region of the degraded broadcast channel is found by Cover and Bergmans and Gallager. The capacity region of the general non-degraded BC is still unknown. However, the inner bound and outer bound of BC does exist due to Marton and Sato respectively.

Since the beginning of this century, with the trend of studying the multiple antenna system i.e. MIMO system, more research interests and activities were drawn to the study of a broadcast channel equipped with MIMO. This is also motivated by the need for a better design for the CDMA downlink and digital subscriber line downstream direction. For the MIMO BC, since it's in general non-degraded, the capacity region is unknown and still an open problem. However the optimal sum rate, or throughput and the suggestive precoding scheme is now completely known due to the work of Caire and Shamai, Yu and Cioffi, Viswanath and Tse, and Vishwanath and Jindal and Goldsmith.

Caire and Shamai [3] were the first to give the optimal sum capacity of the MIMO BC by explicitly maximizing the Dirty Paper lower bound and minimizing the Sato's upper bound in the $t \times 2$ case which, as it turns out, is equal. The computation is complex and hard to extend to more general cases [4]. Caire and Shamai also gives the asymptotic optimal sum capacity in the high SNR region and low SNR region and their coding strategy respectively.

Yu and Cioffi use a Generalized Decision Feedback Equalizer structure for precoding at the transmitter. The optimal sum capacity for general case under some power constraint is found to be the saddle-point of the mutual information expression maximizing over signal covariance and minimizing over noise covariance. The results only hold for the non-singular noise covariance [5]. This condition is not always true as we can see in some example later.

Viswanath and Tse, and Vishwanath and Jindal and Goldsmith also get the optimal sum capacity on the general $t \times r$ case. Viswanath and Tse prove the tightness of the Marton inner bound and Sato's upper bound by showing the intimate downlink-uplink duality and point-to-point reciprocity [6]. Vishwanath and Jindal and Goldsmith uses the results of the MAC channel to simplify the calculation of the achievable region of the MIMO BC [4].

III. Problems of MIMO BC

From here on, we will try to solve the problems of the MIMO BC. But first, what exactly are the problems ?

A. From BC to MIMO BC: an "unfortunate" upgrade

Recall that the capacity region for the scalar Gaussian broadcast channel is fully known. The key is that a Gaussian broadcast channel is statistically degraded ! When the Gaussian BC is upgraded to MIMO BC, this degradedness is unfortunately lost in most case. The reason is simple, while scalar Gaussian noise channels can be ordered according to the variance of the noise, the vector Gaussian noise channels can not ! This is because the vector channels $\mathbf{h}^i$ which is the ith row of the channel matix $\mathbf{H}$ can not be ordered.

There's one exception though. When $\mathbf{H}$ is a rank 1 matrix, the MIMO Gaussian BC is degraded again since $\mathbf{h}^i = k_i \mathbf{h}^0$ for $i = 0, \cdots, r-1$. The vector channel thus can be ordered according to k.

Another important implication of the non-degradedness of the MIMO BC is that the superposition encoding or successive decoding is no longer capacity achieving.

B. From MIMO to MIMO BC: value of cooperation

From a point to point MIMO to a MIMO BC, we lose the cooperation in the receivers, how much and why that costs in bits ?

Given a MIMO channel [17] by singular value decomposition

$$\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{V}^H \tag{2}$$

The MIMO channel can by transformed into some parallel channels

$$\tilde{\mathbf{y}} = \mathbf{D}\tilde{\mathbf{x}} + \tilde{\mathbf{n}} \tag{3}$$

by the following substitution

$$\begin{aligned} \tilde{\mathbf{y}} &= \mathbf{U}^H \mathbf{y} & (4) \\ \tilde{\mathbf{x}} &= \mathbf{V}^H \mathbf{x} & (5) \\ \tilde{\mathbf{n}} &= \mathbf{U}^H \mathbf{n} & (6) \end{aligned}$$

The substitution in (4) is a linear transformation of the channel output vector $\mathbf{y}$ which can not be done in BC since this obviously needs cooperation at receivers.

IV. More Setup

We will begin to derive the optimal sum capacity for the MIMO BC following the steps in [1]. The channel model again is defined in [17], with the assumption that t transmitters can cooperate, r receivers must process their signals separately. This channel is a vector Gaussian broadcast channel (GBC).

The input is power constrained as

$$\frac{1}{n}\sum_{i=1}^{n}|\mathbf{x}_i|^2 \leq A \tag{7}$$

where A is the maximum allowed total transmit energy per channel use. $\mathbf{x}_i$ is the input vector at time instance i.

For kth receiver, the rate R_k is defined in the usual information theoretical sense i.e. as the code rate approach R_k the decoding error shall approaching to 0. The system throughput is defined to be the sum of all users' rate R_k in bit/s/Hz

$$R = \sum_{k=1}^{r} R_k \tag{8}$$

We will consider the scenario that $\mathbf{H}$ is deterministic, fixed and perfectly known at both receivers and transmitters.

V. Dirty Paper Coding

We will review Costa's Dirty Paper Coding in the context of broadcast channel. For a degraded BC, the capacity can be achieved by either successive decoding [7] or by Costa's Dirty Paper Coding which is a successive encoding scheme. Due to the non-degradedness of MIMO BC, only dirty paper coding is applicable.

Dirty Paper coding states that the capacity of a single-user memoryless channel $P_{Y|X,S}$ with input X, output Y and interference S, where the interference sequence $\mathbf{s}$ is non-causally known by transmitter and unknown by the receiver, was given by

$$\sup_{P_{X|U,S}} \{I(U;Y) - I(U;S)\} \tag{9}$$

where $P(S)$ is assumed to be known and supremum is taken over all possible $P(U|S)$ and $P(X|U,S)$.

In the additive Gaussian channel case where $Y = X + S + Z$, Z is Gaussian noise, S is Gaussian Interference, X is under some power constraint $E[X^2] \leq P$. The surprising result using dirty paper coding is that, the capacity is the same as if interference was not present.

Since in the broadcast channel, the signal for each user is non-causally known. Thus users can be ordered and encoded successively. For a given user, by treating the all previous users' signal as the known interference state, he can achieve the capacity as if they are non-exist.

VI. Marton's Inner bound

As we mentioned, the MIMO GBC is general non-degraged whose best known achievable region is given by Marton [8].

$$co \bigcup_{P_{U_1,U_2,X,Y_1,Y_2 \in \mathcal{P}}} \left\{ \begin{array}{l} 0 \leq R_1 \leq I(U_1;Y_1) \\ 0 \leq R_2 \leq I(U_2;Y_2) \\ R_1 + R_2 \leq I(U_1;Y_1) + I(U_2;Y_2) - I(U_1;U_2) \end{array} \right\} \tag{10}$$

where "*co*" denotes the convex closure, and $\mathcal{P}$ is the set of all joint probability distributions on U_1, U_2, X, Y_1, Y_2 such that $(U_1, U_2) \to X \to Y_1$, $(U_1, U_2) \to X \to Y_2$ and such that the marginal conditional distribution of Y_1 and Y_2 given X are equal to $P_{Y_1|X}$ and $P_{Y_2|X}$ respectively.

A corner point in Marton's inner bound is the following rate pair

$$R_1 = I(U_1;Y_1) - I(U_1;U_2), R_2 = I(U_2;Y_2)$$

for any joint distribution P_{X,U_1,U_2}. The special case can be achieved by dirty paper encoding. User 2's sequence $\mathbf{u}_2$ is generated i.i.d. $\sim U_2$. Then $\mathbf{u}_2$ is treated as a known interference. User 1 can achieve the capacity of dirty paper coding, i.e. $I(U_1;Y_1) - I(U_1;U_2)$

VII. Sato's upperbound

First we will show briefly the capacity of BC only depends on the marginal distribution $P_{Y_1|X}$ and $P_{Y_2|X}$. Without the loss of generality, we will use 2 user case.

Given any code $\mathcal{C}^n$ with length n. For user i, Let $w_i \in \{1, 2, \cdots, 2^{nR_i}\}$ be the message, $\mathbf{y}_i = (y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(n)})^T$ the recieved signal vector and ψ_i be the decoding function i, such that $\psi_i : \mathcal{C}^n \rightarrow \{1, 2, \cdots, 2^{nR_i}\}$, the error probability is given by $\epsilon_i = Pr(\psi_i(\mathbf{y}_i) \neq w_i)$.

Lemma 1. The error probability for user 1 is same for any channel distribution $P(y_1, y_2|\mathbf{x})$ if it has the given marginal distribution

$$\int_{y_2 \in \mathcal{Y}} P(y_1, y_2|\mathbf{x}) dy_2 = P(y_1|\mathbf{x})$$

Proof: Let $\mathcal{E}_1 = \{\mathbf{y}_1 \in \mathcal{Y}^n, \mathbf{y}_2 \in \mathcal{Y}^n, \mathbf{x} \in \mathcal{X}^n : \psi_1(\mathbf{y}_1, \mathbf{y}_2) \neq w_i\}$ be the set that will cause a decoder error. Since actually $\psi_1(\mathbf{y}_1, \mathbf{y}_2) = \psi_1(\mathbf{y}_1)$, the decoder is not a function of $\mathbf{y}_2$, $\mathcal{E}_1 = \{\mathbf{y}_1, \mathbf{x} : \psi_1(\mathbf{y}_1) \neq w_i\} \times \mathcal{Y}^n = \mathcal{E}_1' \times \mathcal{Y}^n$

$$\begin{aligned}
\epsilon_i &= P(\psi_i(\mathbf{y}_i) \neq w_i) \\
&= \int_{\mathbf{y}_1, \mathbf{y}_2, \mathbf{x} \in \mathcal{E}_1} P(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}) P(\mathbf{x}) d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{x} \\
&= \int_{\mathbf{y}_1, \mathbf{x} \in \mathcal{E}_1'} d\mathbf{y}_1 d\mathbf{x} \int_{\mathbf{y}_2 \in \mathcal{Y}^n} P(\mathbf{y}_1, \mathbf{y}_2|\mathbf{x}) P(\mathbf{x}) d\mathbf{y}_2 \\
&= \int_{\mathbf{y}_1, \mathbf{x} \in \mathcal{E}_1'} P(\mathbf{y}_1|\mathbf{x}) d\mathbf{y}_1 d\mathbf{x} \\
&= \int_{\mathbf{y}_1, \mathbf{x} \in \mathcal{E}_1'} \prod_k P(y_1^{(k)}|\mathbf{x}) dy_1^{(k)} d\mathbf{x}
\end{aligned}$$

which is only dependent on the marginal distribution. **Q.E.D.**

Thus the sum capacity of a general BC is upper bounded by letting the receivers cooperate. The capacity of broadcast channel only depends on the marginal distribution $P_{Y_1|X}$ and $P_{Y_2|X}$ while the cooperative capacity depends on the joint distribution $P_{Y_1,Y_2|X}$. Any sum rate that can be achieved by given marginal $P_{Y_1|X}$ and $P_{Y_2|X}$ must also be achieved by any joint distribution $P_{Y_1,Y_2|X}$ which has the same marginal. Thus we have the upperbound for 2 users case,

$$R_1 + R_2 \leq \inf_{P_{Y_1,Y_2|X} \in \mathcal{U}} \sup_{P_X \in \mathcal{A}} I(X; Y_1, Y_2)$$

where $\mathcal{U}$ is the set of joint transition probabilities with fixed marginals $P_{Y_1|X}$ and $P_{Y_2|X}$ and $\mathcal{A}$ is the set of input distribution which satisfies the power constraint A.

Now we will argue that in our Gaussian case minimizing over all joint $P_{Y_1,Y_2|X}$ which has marginal $P_{Y_1|X}$ and $P_{Y_2|X}$ is equivalent to the following mini-max problem over the covariance matrix of noise $\mathbf{\Sigma_z}$ and signal covariance matrix $\mathbf{\Sigma_x}$ over certain sets.

Thus we has the following:

Lemma 2. For any channel matrix $\mathbf{H}$,

$$R \leq \min_{\sum_z \in \mathcal{U}} \max_{\sum_x \in \mathcal{A}} \log \frac{\det(\mathbf{H} \sum_{\mathbf{x}} \mathbf{H}^H + \sum_z)}{\det \sum_z}$$

where $\mathcal{U}$ is the set of all noise covariance matrices satisfying the sub-unit diagonal constraint which means the diagonal elements are not greater than 1. $\mathcal{A}$ is the set of input matrices satisfying the input constraint.

VIII. Solution for 2 users case

The optimal sum capacity of the MIMO BC is abtained by proving that Marton's lowerbound on throughput are tight, i.e. it can coincide with Sato's upperbound.

In [1], the above idea is used to calculate the 2 users case. Lemma 1 is used to find an upperbound. Dirty Paper coding as a special case to achieve Marton's bound is used to find a lower bound. Both bounds are then shown to be identical thus find the optimal sum capacity which is of the following form.

Theorem 1. The maximum achievable throughput of the $t \times 2$ Gaussian Broadcast channel is given by

$$R = \begin{cases} \log(1 + |\mathbf{h}^1|^2 A) & A \leq A_1 \\ \log \frac{(A \det(\mathbf{H}\mathbf{H}^H) + tr(\mathbf{H}\mathbf{H}^H))^2 - 4|\mathbf{h}^2(\mathbf{h}^1)^H|^2}{4 \det(\mathbf{H}\mathbf{H}^H)} & A > A_1 \end{cases}$$

where without loss of generality we assume $|\mathbf{h}^1| \geq |\mathbf{h}^2|$ and where

$$A_1 = \frac{|\mathbf{h}^1|^2 - |\mathbf{h}^2|^2}{\det(\mathbf{H}\mathbf{H}^H)}.$$

IX. Some more general results

For a more general case of more than 2 users. the optimal throughput is given in [5], [9] and [4]. Some asymptotic results and the downlink transmission strategies are discussed in [1].

Let R^{zfdp} denotes the rate achieved by the Zero-Forcing Dirty-Paper coding. and R^{zf} denotes the Zero-Forcing linear beamforming rate, R^{coop} be the full cooperate rate and R

the optimal rate. Also define

$$R_{max}^{zfdp} = \max_{\mathcal{S}} R^{zfdp} \tag{11}$$

$$R_{max}^{zf} = \max_{\mathcal{S}} R^{zf} \tag{12}$$

where $\mathcal{S}$ is the user set. $\mathcal{S}$ is unordered in the Zero-Forcing beamforming case and ordered in the Zero-Forcing Dirty-Paper case.

We have the following result

If $\mathbf{H}$ has full row-rank, then

$$\lim_{A\to\infty} (R^{coop} - R_{max}^{zfdp}) = 0 \tag{13}$$

$$\lim_{A\to 0} R_{max}^{zfbf} / R_{max}^{zf} = 1 \tag{14}$$

$$\lim_{A\to\infty} (R - R_{max}^{zfdp}) = 0 \tag{15}$$

$$\lim_{A\to 0} R / R_{max}^{zfdp} = 1 \tag{16}$$

The above results can be interpreted as follows

- In the low SNR region, as available transmission power goes to 0, the capacity of both Zero-Forcing beamforming and Zero-Forcing Dirty-Paper coding converges to the true capacity.
- In the high SNR region, as available transmission power goes to infinity. Only the capacity of Zero-Forcing Dirty-Paper coding converges to the true capcity. The Sato's upper bound also converges to the true capacity.

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