Multiple Access Channel with Feedback

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Abstract

Multiple access channel with feedback is an extension of classical MAC, by providing senders with ability to observe the channel output. The capacity region in the general case is not known. This report presents the main results of [1] by Cover and Leung, which is a region of achievable rates exceeding the capacity region of classical MAC (section 3). Since the coding technique used in [1] relies on partial cooperation between senders, the discussion of MAC with feedback is preceded by an introduction to MAC with correlated sources, based on results from [2] by Slepian and Wolf (section 2).

1 Introduction

Consider a classical discrete memoryless multiple access channel with two transmitters. Such channel is fully described by the two input alphabets $\mathcal{X}_1$, $\mathcal{X}_2$, output alphabet $\mathcal{Y}$ and fixed transition probabilities $p(y|x_1,x_2)$. As described in [4] and discussed in Wenyi's summary, if the two transmitters have no means of exchanging information, their capacity region is the convex closure of rate pairs (R_1,R_2) satisfying

$$\begin{aligned} R_1 &< I(X_1;Y|X_2) \\ R_2 &< I(X_2;Y|X_1) \\ R_1+R_2 &< I(X_1,X_2;Y) \end{aligned} \tag{1}$$

over all independent distributions $p(x_1)$, $p(x_2)$, $p(x_1,x_2)=p(x_1)\cdot p(x_2)$.

For most channels the above rate region is considerably smaller than the region achievable by cooperating transmitters. In fact, if both transmitters fully knew each other's messages, the region of achievable rates would be

$$R_1+R_2 < \max_{p(x_1,x_2)} I(X_1,X_2;Y), \tag{2}$$

where the maximization is over arbitrary joint distributions of X_1 and X_2. This is of course the capacity region of a single user channel with input (X_1,X_2), where the two data streams are sharing the transmission bandwidth.

Between these two extreme scenarios there are several variants of MAC channel permitting a limited cooperation between transmitters. As can be expected, their rate regions will exceed the region (1) while remaining bounded by (2), as illustrated in Figure 1. Within this report, two such scenarios will be discussed, namely MAC with correlated sources (based on a paper [2] by Slepian and Wolf) and MAC with feedback (based on [1] by Cover and Leung). Although the latter is the true focus of this report, the discussion of the former might be helpful to understand the concept of partial cooperation applied in [1].

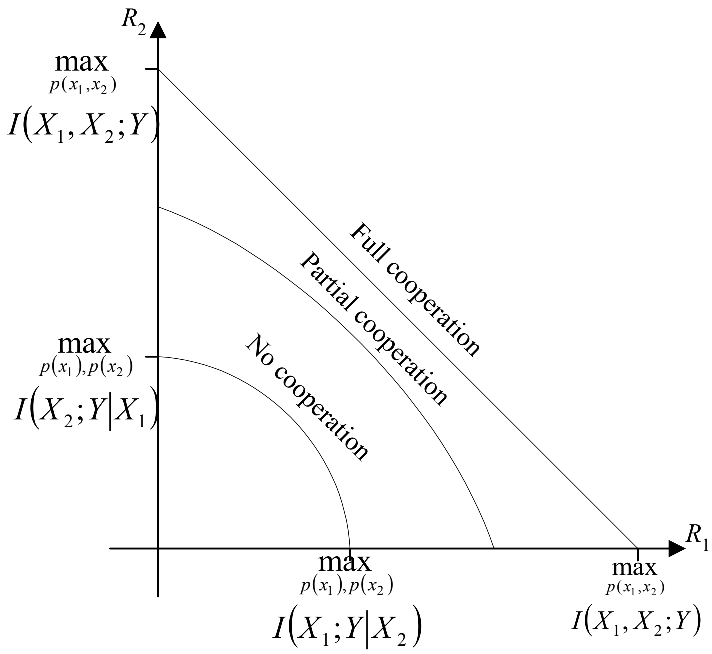

Figure 1. Capacity regions for MAC channel with different degree of transmitter cooperation.

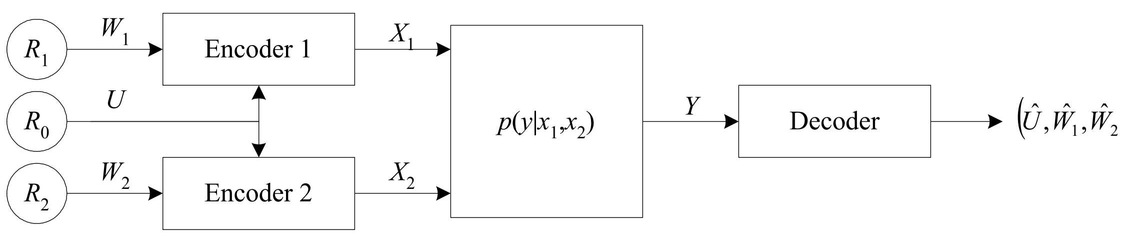

Figure 2. Multiple access channel with correlated sources.

2 MAC with Correlated Sources

The multiple access channel with correlated sources is illustrated in Figure 2. Contrary to what the name suggests, MAC with correlated sources involves three statistically independent sources producing information at rates R_0, R_1, and R_2. It is the task of the two encoders to perform the mappings $f:\mathcal{U}\times\mathcal{W}_1\to\mathcal{X}_1^N$ and $g:\mathcal{U}\times\mathcal{W}_2\to\mathcal{X}_2^N$ given by

$$\begin{aligned}\mathbf{x}_1 &= f(u,w_1)\\ \mathbf{x}_2 &= g(u,w_2)\end{aligned}, \tag{3}$$

where $\mathcal{U}=\{1,2,\ldots,2^{NR_0}\}$, $\mathcal{W}_1=\{1,2,\ldots,2^{NR_1}\}$, $\mathcal{W}_2=\{1,2,\ldots,2^{NR_2}\}$ are the sets of equiprobable values of discrete RV U, W_1, W_2 associated with the sources. Furthermore, N is a positive integer denoting the block length. After the vector of all N channel outputs has been collected, the decoder estimates the source outputs by performing a mapping $h: \mathcal{Y}^N \to \mathcal{U} \times \mathcal{W}_1 \times \mathcal{W}_2$ given as

$$(\hat{u}, \hat{w}_1, \hat{w}_2) = h(\mathbf{y}). \tag{4}$$

The decoder tries to minimize the probability of error event, which occurs whenever $U \neq \hat{U}$ or $W_1 \neq \hat{W}_1$ or $W_2 \neq \hat{W}_2$. We call the rate triple (R_0,R_1,R_2) achievable if there exists a sequence of codes with larger and larger N for which the probability of error tends to zero.

Theorem 1 *(coding theorem for MAC with correlated sources) The capacity region of MAC with correlated sources is a convex closure of a union of rate triples* (R_0,R_1,R_2) *satisfying*

$$\begin{aligned} R_1 &< I(X_1;Y|X_2,Z) \\ R_2 &< I(X_2;Y|X_1,Z) \\ R_1 + R_2 &< I(X_1,X_2;Y|Z) \\ R_0 + R_1 + R_2 &< I(X_1,X_2;Y) \end{aligned} \tag{5}$$

over all distributions $p(z,x_1,x_2)=p(z)\cdot p(x_1|z)\cdot p(x_2|z)$*, where* Z *is an auxiliary RV taking on values from* $\mathcal{Z}$, $\mathcal{Z} = \{1,2,\ldots,\min(|\mathcal{X}_1|\cdot|\mathcal{X}_2|,|\mathcal{Y}_1|)\}$.

The outline of the achievability part of the proof is presented below. The complete proof can be found in [2]. The proof uses a random coding argument. The main feature of proposed coding scheme is a clever utilization of an auxiliary discrete random variable Z to create three "virtual inputs" to the channel. These inputs are Z itself, X_1 given Z, and X_2 given Z, intended to carry messages from sources U, W_1, and W_2 respectively.

Code generation. Fix discrete probability distributions $p(z)$, $p(x_1|z)$, $p(x_2|z)$ and block length N. For each message in $u \in \mathcal{U}$ generate a "virtual codeword" $\mathbf{z}(u) \in \mathcal{Z}^N$ according to the distribution $p(\mathbf{z}(u)) = \prod_{i=1}^N p(z^{(i)}(u))$. Then generate the actual codebook, i.e. for each $(u,w_k) \in \mathcal{U} \times \mathcal{W}_k$, $k=1,2$ select $\mathbf{x}_k(u,w_k) \in \mathcal{X}_k^N$ according to $p(\mathbf{x}_k(u,w_k)) = \prod_{i=1}^N p(x_k^{(i)}(u,w_k)|z^{(i)}(u))$.

Transmitter and receiver operation. Every N time units the sources U, W_1, W_2 generate a message u, w_1, w_2. The transmitters select the appropriate codewords $\mathbf{x}_1(u,w_1)$, $\mathbf{x}_2(u,w_2)$ and transmit them during the next N channel usages. The receiver acquires the vector of received values $\mathbf{y} \in \mathcal{Y}^N$. Estimation of u, w_1, w_2 is performed by searching for all $\hat{u}$, $\hat{w}_1$, $\hat{w}_2$, for which $(\mathbf{z}(\hat{u}), \mathbf{x}_1(\hat{u},\hat{w}_1), \mathbf{x}_2(\hat{u},\hat{w}_2))$ is jointly typical with $\mathbf{y}$. A decoding error occurs if the correct codeword $(\mathbf{z}(u), \mathbf{x}_1(u,w_1), \mathbf{x}_2(u,w_2))$ is not jointly typical with $\mathbf{y}$, or if any other codeword is jointly typical with $\mathbf{y}$.

Summary of derivation. The notation and structure of this part will be analogous to that used by Cover and Thomas in [4]. Since the codewords are i.i.d., it can be assumed that the first codeword $(\mathbf{x}_1(1,1), \mathbf{x}_2(1,1))$ was transmitted. Furthermore, let $E(u,w_1,w_2)$ denote the event of $(\mathbf{z}(u), \mathbf{x}_1(u,w_1), \mathbf{x}_2(u,w_2))$ is jointly typical with received vector $\mathbf{y}$. The average (over codewords and codes) block error probability can be bounded as follows:

$$
\begin{aligned}
P_e &= P\left(E_{1,1,1}^c \cup \bigcup_{(u,w_1,w_2)\neq(1,1,1)} E_{u,w_1,w_2}\right) \\
&\leq P\left(E_{1,1,1}^c\right) + \sum_{\substack{u=1\\ w_1=1\\ w_2\neq 1}} P\left(E_{1,1,w_2}\right) + \sum_{\substack{u=1\\ w_1\neq 1\\ w_2=1}} P\left(E_{1,w_1,1}\right) + \sum_{\substack{u=1\\ w_1\neq 1\\ w_2\neq 1}} P\left(E_{1,w_1,w_2}\right) \\
&\quad + \sum_{\substack{u\neq 1\\ w_1=1\\ w_2=1}} P\left(E_{u,1,1}\right) + \sum_{\substack{u\neq 1\\ w_1\neq 1\\ w_2=1}} P\left(E_{u,w_1,1}\right) + \sum_{\substack{u\neq 1\\ w_1=1\\ w_2\neq 1}} P\left(E_{u,1,w_2}\right) + \sum_{\substack{u\neq 1\\ w_1\neq 1\\ w_2\neq 1}} P\left(E_{u,w_1,w_2}\right)
\end{aligned} \tag{6}
$$

This probability will approach zero with $N\to\infty$ if each of the eight components in the above sum can be made arbitrarily small. For the last seven components this is equivalent to following rate constraints:

$$
\begin{aligned}
R_1 &< I(X_1;Y|X_2,Z) \\
R_2 &< I(X_2;Y|X_1,Z) \\
R_1 + R_2 &< I(X_1,X_2;Y|Z) \\
R_0 &< I(X_1,X_2;Z) \\
R_0 + R_1 &< I(X_1;Y|X_2) + I(X_2;Z) \\
R_0 + R_2 &< I(X_2;Y|X_1) + I(X_1;Z) \\
R_0 + R_1 + R_2 &< I(X_1,X_2;Y)
\end{aligned} \tag{7}
$$

Removal of the weakest constraints yields (5).

3 An Achievable Rate Region for MAC with Feedback

The multiple access channel with feedback is depicted in Figure 3. Just like in case of classical MAC, two encoders are attempting to convey information from two statistically independent sources with rates R_1 and R_2 to the common receiver. The main difference comes from the fact that both encoders can observe the output of the channel, just as the decoder does. Such situation might arise in asymmetric communication systems, when large feedback capacities are available to the decoder (e.g., satellite-to-earth communication).

The encoders can choose codewords based not only on the messages generated by their respective sources, but also on the channel outputs seen so far. Formally, the encoders are causal mappings $f:\mathcal{W}_1\times\mathcal{Y}^N\to\mathcal{X}_1^N$ and $g:\mathcal{W}_2\times\mathcal{Y}^N\to\mathcal{X}_2^N$, which generate the k-th codeword letters as

$$
\begin{aligned}
x_{1,k} &= f_k(w_1, y_1, y_2, \ldots, y_{k-1}) \\
x_{2,k} &= g_k(w_2, y_1, y_2, \ldots, y_{k-1})
\end{aligned}. \tag{8}
$$

The capacity region for general MAC with feedback is not known to date, i.e., the existing upper and lower bounds on the achievable rates do not coincide. However, capacity regions have been established for several specific channels (most notably for a Gaussian case in [3]).

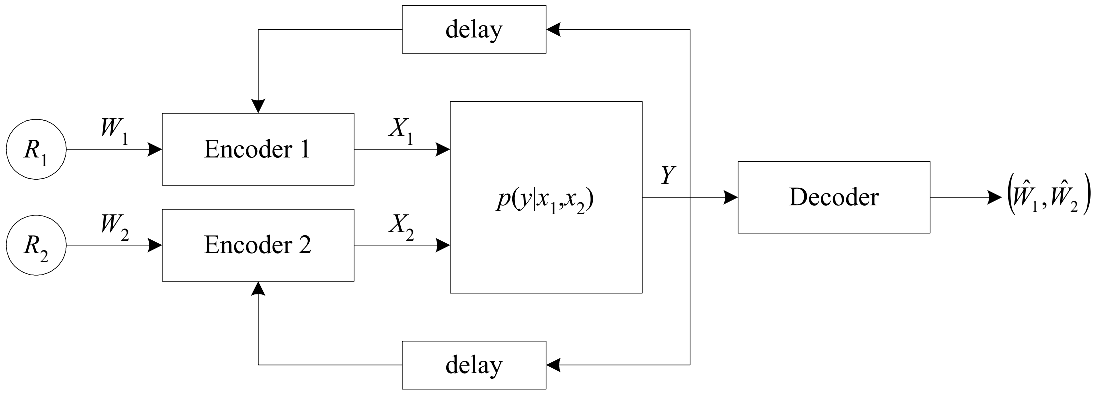

Figure 3. Multiple access channel with feedback.

The above definition assumes feedback delay of one time unit. The following theorem allows for generalization to arbitrary feedback delays by showing that changing feedback delay does not change the region of achievable rates.

Theorem 2 *Rate pair* (R_1,R_2) *is achievable over MAC* $p(y|x_1,x_2)$ *with feedback delay of L time indices iff it is also achievable with feedback delay* 1.

Proof. Assume (R_1,R_2) is achievable with feedback delay 1. Then for arbitrary positive ε, there exists a code achieving $P_e<\varepsilon$ with feedback delay 1 for some block length N. Now, by combining L parallel instances of this code, we can construct a code operating on a channel with feedback delay L. The combining involves using k-th code instance ($0 \leq k < L$) at time indices j for which $k = j \bmod L$. The resulting code has block length $N^* = NL$, same rate, and block error probability $P_e^* = 1\text{-}(1\text{-}P_e)^L$. For $P_e <$ 0.5, $P_e^* < LP_e < L\varepsilon$. This proves that for feedback delay L there exist codes with (R_1,R_2) and arbitrarily small P_e^*. Conversely, any code operating on the channel with feedback delay L can be operated on channel with feedback delay 1, by buffering last L-1 feedback symbols. QED.

What kind of information does feedback provide to an encoder? First of all, thanks to feedback the encoder can assess how the receiver perceived his transmission. However, if the feedback was used in the single-user system, it would provide the same benefit, but it is well known (see e.g. [4]) that feedback does not increase capacity in such system. The second benefit of feedback is the ability for encoder 1 to learn about intentions of encoder 2 (and vice versa). In fact, the encoder 1 has much better chances of correctly decoding the transmission of encoder 2 than the final receiver. As a consequence, encoders have a limited possibility to understand one another, and send some information to the decoder in cooperation.

Based on the above principle, Cover and Leung proposed in [1] a transmission scheme, which achieves a rate region exceeding the capacity region of MAC without feedback. This region is specified in the following theorem.

Theorem 3 *The rates* (R_1, R_2) *are achievable over the MAC with feedback, if there exists a discrete RV Z with* $p(z,x_1,x_2)=p(z)\cdot p(x_1|z)\cdot p(x_2|z)$ *such that*

$$\begin{aligned} R_1 &< I(X_1;Y|X_2,Z) \\ R_2 &< I(X_2;Y|X_1,Z) \\ R_1+R_2 &< I(X_1,X_2;Y) \end{aligned} \tag{9}$$

The convex closure of the set of all rate pairs satisfying above conditions forms a region of achievable rates, contained in the capacity region of MAC with feedback.

The coding scheme in [1] is an interesting extension of a scheme used for MAC with correlated sources. Assume encoders T1 and T2 have been able to establish some common information about the messages they intend to send, which still remains unknown to the receiver. Let us represent this information as a random variable U, known to both T1 and T2. During the next transmission block of N time units both transmitters will cooperate in conveying the value of U to the receiver. Additionally T1 will transmit new information from source W_1, and T2 from source W_2.

Up to now the scheme looked almost identically to the one in Section 2. The main difference comes from the fact that we do not require the receiver to perfectly (i.e., with arbitrarily low probability of error) reconstruct W_1 and W_2 (we still require perfect reconstruction of U). Instead, we require that W_1 can be unambiguously determined by T2, and W_2 by T1, so that both encoders know precisely which codewords $\mathbf{x}_1$, $\mathbf{x}_2$ were transmitted.

Since reconstruction of W_2 at T1 and W_1 at T2 is more reliable than decoding of W_1 and W_2 at the receiver, the above procedure leaves the receiver with some residual uncertainty. This uncertainty will manifest itself in an average number of codewords $(\mathbf{x}_1(u,w1),\mathbf{x}_2(u,w_2))$ jointly typical with the received sequence $\mathbf{y}$. Thanks to the feedback, all three communicating parties can determine the set S_y of these codewords, and order it lexicographically. Moreover, T1 and T2 can use their fresh knowledge of the correct $\mathbf{x}_1$ and $\mathbf{x}_2$ to find index of the correct entry in S_y (assuming correct $(\mathbf{x}_1,\mathbf{x}_2)$ actually belongs to S_y). The index of $(\mathbf{x}_1,\mathbf{x}_2)$ in S_y is exactly the information U that will be jointly transmitted in the subsequent block. Of course, the average size of the set of potential codewords cannot be allowed to exceed the cardinality of $\mathcal{U}$.

The informal sketch of derivation of (9) goes as follows. The requirement of reliable reconstruction of W_1 at T2 leads to $R_1 < I(X_1;Y|X_2,Z)$, since T2 knows X_2 and Z. Analogously, reliable reconstruction of W_2 at T1 requires $R_2 < I(X_2;Y|X_1,Z)$, and reconstruction of U at the receiver requires $\frac{1}{N}\log\|\mathcal{U}\| < I(Y;Z)$. Also, with arbitrarily high probability $\frac{1}{N}\log\|S_\mathrm{y}\| \le R_1 + R_2 - I(X_1,X_2;Y|Z)$. Requiring $\|S_\mathrm{y}\| \le \|\mathcal{U}\|$ yields the following condition $R_1 + R_2 < I(X_1,X_2;Y|Z) + I(Y;Z) = I(X_1,X_2;Y)$, which completes the result (9).

References

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