Interference Channel

Li Guo

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I. Introduction

An *interference channel* is a network consisting N senders and N receivers. There exists a *one-to-one* correspondence between senders and receivers. Each sender only wants to communicate with its corresponding receiver, and each receiver only cares about the information form its corresponding sender. However, each channel interferes the others. So an interference channel has N *principal links* and $N(N-1)$ *interference links.* This scenario often occurs, when several sender-receiver pairs share a common media. For example, in satellite communication, two satellites send information to its corresponding ground station simultaneously. Each ground station can receive the signals from both of the two satellites and its communication is interfered by the other pair's communication.

The study of this kind of channel was initiated by Shannon in 1961. However, this channel has not been solved in general case even in the general Gaussian case. In this summary, I will present some known important results about interference channel, including capacity region for interference channel with strong interference in section III, a inner bound of capacity region for general case by Carleial[4] in section IV and some results on Gaussian interference channel in section V.

II. Preliminaries

In this section, some important definitions are given. In Fig.1. is the model of an interference channel. An interference channel $(\mathcal{X}_1, \mathcal{X}_2, p(y_1, y_2/x_1, x_2), \mathcal{Y}_1, \mathcal{Y}_2)$ consists two finite input sets $\mathcal{X}_1$ and $\mathcal{X}_2$, two infinite output sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and a collection of pmfs $p(y_1, y_2/x_1, x_2)$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$. An $((2^{nR_1}, 2^{nR_2}), n)$ code for the interference channel consists of:

- Two independent message sets $\mathcal{W}_j = \{1, 2, ..., M_j\}, j = 1, 2$. The message (W_1, W_2) are uniformly distributed over $\mathcal{W}_1 \times \mathcal{W}_2$.
- Two encoders $e_j : \mathcal{W}_j \rightarrow \mathcal{X}_j^n, j = 1, 2$. e_1 maps each message $w_1 \in \mathcal{W}_1$ into a codeword $x_1^n(w_1)$,and e_2 maps each message $w_2 \in \mathcal{W}_2$ into a codeword $x_2^n(w_2)$
- Two decoders $d_j : \mathcal{Y}_j^n \rightarrow \mathcal{W}_j, j = 1, 2$. d_1 maps each received sequence y_1^n into a message $\hat{w}_1 \in \mathcal{W}_1$ or an error message, and d_2 maps each received sequence y_2^n into a message $\hat{w}_2 \in \mathcal{W}_2$ or an error message

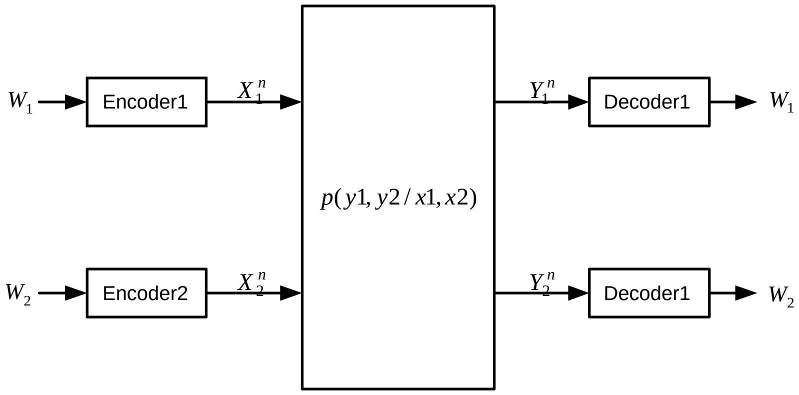

Figure 1: Model of interference channel

The error probabilities is defined as:

$$\lambda_{1,n} = \frac{1}{M_1 M_2} \sum_{w_1, w_2} p(d_1(Y_1) \neq w_1 / W_1 = w_1, W_2 = w_2) \tag{1}$$

$$\lambda_{2,n} = \frac{1}{M_1 M_2} \sum_{w_1, w_2} p(d_2(Y_2) \neq w_2 / W_1 = w_1, W_2 = w_2) \tag{2}$$

$$\lambda_n = max\{\lambda_{1,n}, \lambda_{2,n}\} \tag{3}$$

A rate pair(R_1, R_2) is said to be achievable for the interference channel if there exists a sequence of$((2^{nR_1}, 2^{nR_2}), n)$ codes with $\lambda_n \to 0$ as $n \to \infty$. The capacity region is defined as the closure of the set of all achievable rate pairs. As in the case of broadcast channel, the capacity region depends only on the marginal transition probability $p(y_1/x_1, x_2)$ and $p(y_2/x_1, x_2)$.

III. Strong Interference Case

Usually, we think interference is harmful and causes degradation to communication quality, and stronger interference is more harmful than weaker interference. So we try to suppress the interference to increase the communication quality. This is from the actual technological point of view. However, from theoretical point of view, it does not always look like that. Carleial, Gamal and Sato showed that strong interference can be less harmful than weak interference and enough strong interference can be even innocuous as no interference at all.

Definition 1 *: The discrete memeoryless (DM) interference channel is said to have very strong interference* [2] *if:*

$$I(X_1; Y_1/X_2) \leq I(X_1; Y_2) \tag{4}$$

$$I(X_2; Y_2/X_1) \leq I(X_2; Y_1) \tag{5}$$

for all $(X_1, X_2) \sim p(x_1)p(x_2)$.

Definition 2 *: The discrete memeoryless (DM) interference channel is said to have strong interference* [2] *if:*

$$I(X_1;Y_1/X_2) \leq I(X_1;Y_2/X_2) \tag{6}$$

$$I(X_2;Y_2/X_1) \leq I(X_2;Y_1/X_1) \tag{7}$$

for all $(X_1, X_2) \sim p(x_1)p(x_2)$.

Clearly, if the channel has very strong interference, it also has strong interference since the inequalities $I(X_1;Y_2/X_2) \geq I(X_1;Y_2)$ and $I(X_2;Y_1/X_1) \geq I(X_2;Y_1)$ hold for independent input distribution, but the converse is not necessarily true. The *very strong* and *strong* interference condition imply that the interference is strong enough such that the receiver can decode interference correctly and we can view the interference channel as two multiple access channels. With this idea [3] gave the capacity region for strong interference case:

Theorem 1 *:The capacity region of the DM interference channel with strong interference can be expressed as the union of the rate pairs* (R_1, R_2) *satisfying:*

$$0 \leq R_1 \leq I(X_1;Y_1/X_2,Q) \tag{8}$$

$$0 \leq R_2 \leq I(X_2;Y_2/X_1,Q) \tag{9}$$

$$R_1 + R_2 \leq min\{I(X_1,X_2;Y_1/Q), I(X_1,X_2;Y_2/Q)\} \tag{10}$$

for some $p(q)p(x_1/q)p(x_2/q)$*, where* Q *is time-sharing parameter of cardinality 4.*

We can note that this capacity region is the intersection of the capacity regions of the two multiple access channels $(X_1, X_2) \rightarrow Y_1$ and $(X_1, X_2) \rightarrow Y_2$.

Sketch of Proof:

Achievability: It is immediate. We encode W_1 and W_2 with rate pair (R_1, R_2) within this region. With the strong interference condition, receiver1(receiver2) is able to decode interference from sender2 (sender1) since $R_2 \leq I(X_2;Y_1/X_1)$ $(R_1 \leq I(X_1;Y_2/X_1))$, strips it out, and then decodes the intended signal from the residual. received signal.

Converse: The first two inequalities are trivial bounds. By symmetry, it suffices to show that $R_1 + R_2 \leq I(X_1, X_2; Y_2/Q)$. From Fano's inequality, we have:

$$H(W_1/Y_1) \leq nR_1\lambda_{1,n} + h(\lambda_{1,n}) = n\epsilon_n \tag{11}$$

$$H(W_2/Y_2) \leq nR_2\lambda_{2,n} + h(\lambda_{2,n}) = n\epsilon_n \tag{12}$$

Then consider

$$\begin{aligned}
n(R_1+R_2) &= H(W_1) + H(W_2) \\
&= I(W_1;Y_1^n) + I(W_2;Y_2^n) + H(W_1/Y_1^n) + H(W_2/Y_2^n) \\
&\leq I(W_1;Y_1^n) + I(W_2;Y_2^n) + 2n\epsilon_n \\
&\leq I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \\
&\leq I(X_1^n;Y_1^n/X_2^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \\
&\leq^* I(X_1^n;Y_2^n/X_2^n) + I(X_2^n;Y_2^n) + 2n\epsilon_n \\
&= I(X_1^n,X_2^n;Y_2^n) + 2n\epsilon_n \\
&\leq nI(X_1,X_2;Y_2) + 2n\epsilon_n
\end{aligned} \tag{13}$$

where step $(*)$ follows by the lemma in [3].

□

With the similar argument and very strong interference condition, we can get the capacity region for interference channel with very strong interference easily:

Theorem 2 *:The capacity region of the DM interference channel with very strong interference can be expressed as the union of the rate pairs* (R_1, R_2) *satisfying:*

$$0 \leq R_1 \leq I(X_1; Y_1/X_2, Q) \tag{14}$$

$$0 \leq R_2 \leq I(X_2; Y_2/X_1, Q) \tag{15}$$

for some $p(q)p(x_1/q)p(x_2/q)$*, where* Q *is time-sharing parameter of cardinality 4.*

IV. Inner Bound for General Case

The capacity region of general interference case is not known yet. What we know now are only some bounds to the capacity region. The best known achievable region for the general interference channel is due to Han and Kobayashi [4]. As shown in Fig.2., Their idea is: Split each sender's message into two independent parts, one private part(U_j with rate $R_{jj}, j = 1, 2$) which is only decodable by the intended reciever, and one public part (V_j with rate $R_{0j}, j = 1, 2$) which can be decoded by both receivers. The achievable region is given by the following theorem:

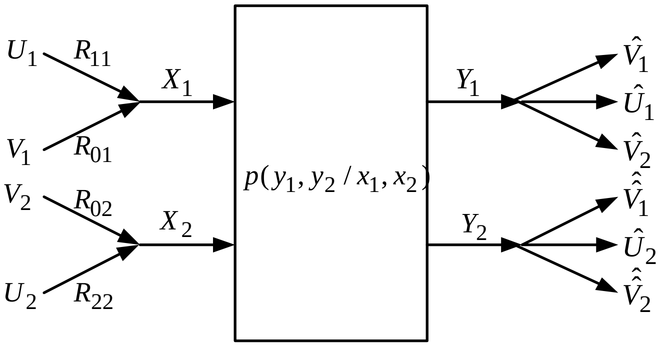

Figure 2: Model of interference channel: Split message into two parts

Theorem 3 *:For DM interference channel* $(\mathcal{X}_1, \mathcal{X}_2, p(y_1, y_2/x_1, x_2), \mathcal{Y}_1, \mathcal{Y}_2)$*, any rate pair* (R_1, R_2) *is achievable if* R_1 *and* R_2 *can be rewritten as* $R_1 = R_{01} + R_{11}$ *and* $R_2 = R_{02} + R_{22}$ *such that:*

$$R_{01} \leq min\{I(V_1; Y_1/U_1, V_2, Q), I(V_1; Y_2/U_2, V_2, Q)\} \tag{16}$$

$$R_{02} \leq min\{I(V_2; Y_1/U_1, V_1, Q), I(V_2; Y_2/U_2, V_1, Q)\} \tag{17}$$

$$
\begin{aligned}
R_{11} &\leq I(U_1;Y_1/V_1,V_2,Q) && (18)\\
R_{22} &\leq I(U_2;Y_2/V_1,V_2,Q) && (19)\\
R_{01}+R_{02} &\leq min\{I(V_1,V_2;Y_1/U_1,Q),I(V_1,V_2;Y_2/U_2,Q)\} && (20)\\
R_{01}+R_{11} &\leq I(V_1,U_1;Y_1/V_2,Q) && (21)\\
R_{02}+R_{11} &\leq I(V_2,U_1;Y_1/V_1,Q) && (22)\\
R_{01}+R_{22} &\leq I(V_1,U_2;Y_2/V_2,Q) && (23)\\
R_{02}+R_{22} &\leq I(V_2,U_2;Y_2/V_1,Q) && (24)\\
R_{01}+R_{02}+R_{11} &\leq I(V_1,V_2,U_1;Y_1/Q) && (25)\\
R_{01}+R_{02}+R_{22} &\leq I(V_1,V_2,U_2;Y_2/Q) && (26)
\end{aligned}
$$

for some distribution $p(q)p(u_1/q)p(v_1/q)p(u_2/q)p(v_2/q)$

The proof is very long and complicated. So I don't present it here. The technique used is a generalization of superposition coding to the multivariable case. However, it can understand more easily if we think there are two multiple access channels $(U_1,V_1,V_2)\to Y_1$ and $(U_2,V_1,V_2)\to Y_2$ roughly and use the results of multiple access channel here.
Other bounds are also demonstrated to the capacity region of interference channel. Carleial [6] gave two outer bounds for capacity region of DM interference channels and Gaussian interference channels. The bound for DM case coincides with the capacity region in special cases. In 2001, Kramer demonstrated two outer bounds in [7]: the first bound is to let a genie give one of the receivers just enough information to decode both messages, the other bound is derived by degraded interference channel. These can be refered as further reading.

V. Gaussian Interference Channel

In this section, some results of Gaussian Interference channel are presents. As shown in Fig.3. it is the continuous alphabet Gaussian interference channel. Its inputs X_1 and X_2 and its

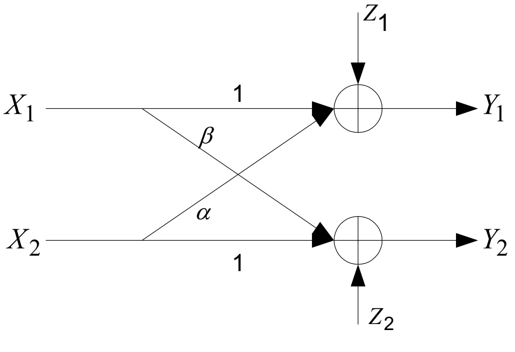

Figure 3: Gaussian Interference Channel

outputs Y_1 and Y_2 are real numbers related by

$$
\begin{aligned}
Y_1 &= X_1+\alpha X_2+Z_2 && (27)\\
Y_2 &= X_2+\beta X_1+Z_1 && (28)
\end{aligned}
$$

where α and β are given interference coefficients and are constants, and Z_1 and Z_2 are independent Gaussian noise random variables with zero mean and variance N_1 and N_2 respectively. We assume average power constraint P_1 over X_1 and P_2 over X_2.
As in the general interference channel case, the capacity region of Gaussian interference channel is also not known in general. In [1], Carleial showed the capacity region of Gaussian interference channel is rectangle with very strong interference. The capacity region of Gaussian interference channel with very strong interference is given by the following theorem:

Theorem 4 *The capacity region for Gaussian interference channel is the set of* (R_1, R_2) *such that:*

$$\begin{aligned} R_1 &\leq \frac{1}{2}log_2(1 + P_1/N_1) = C_1 && (29) \\ R_2 &\leq \frac{1}{2}log_2(1 + P_2/N_2) = C_2 && (30) \end{aligned}$$

provided

$$\begin{aligned} \alpha^2 &\geq (P_1 + N_1)/N_2 && (31) \\ \beta^2 &\geq (P_2 + N_2)/N_1 && (32) \end{aligned}$$

which is the very strong interference condition.

We can see that, with very strong interference, Gaussian interference channel just looks like two independent Gaussian channel without interference.
Sketch of Proof:
Achievability: At each sender, we use random coding technology. At sender1, we indepently generate 2^{nR_1} codewords x_1^n which are n-sequence of independet zaro-mean Gaussian random wariable with variance P_1. Similarly, we generate x_2^n for transmitter 2.At each receiver, we consider there are two decoding steps. At reciever 1, the interfering signal αX_2 is estimated in the presence of desired signal X_1 and noise. X_1 and Z_1 are independent Gaussian variables, and their sum is also a Gaussian varible with zaro mean and $P_1 + N_1$ variance. So in the first step, interfering signal can be decoded with arbitrarily small error probability if:

$$R_2 < \frac{1}{2}log_2(1 + \frac{\alpha^2 P_2}{P_1 + N_1}) \tag{33}$$

We subtrate it after haveing determined αX_2, and then desired signal X_1 can be otianed with noise since $R_1 < C_1$. The similar two-step decoding is performed by receiver 2 reliably i:

$$R_1 < \frac{1}{2}log_2(1 + \frac{\beta^2 P_1}{P_2 + N_2}) \tag{34}$$

Also we note the condition in inequalities(33) and (34) are weaker than condition those of the Theorem. So the rate (R_1, R_2) is achievable.
Converse: It is immediate since C_1 and C_2 are obvious upper bounds.

□

The Gaussian interference channel is said to have strong interference if both of

$$\alpha^2 \geq N_1/N_2 \qquad \text{and} \qquad \beta^2 \geq N_2/N_1 \tag{35}$$

are simultaneously satisfied. Sato[2] demonstrated the capacity region of Gaussian interference channel with strong interference is:

Theorem 5 *The capacity region for the Gaussian interference channel with strong interference is the set of rate pair* (R_1, R_2) *such that*

$$\begin{aligned} R_1 &\leq \frac{1}{2} log_2(1 + P_1/N_1) \qquad (36) \\ R_2 &\leq \frac{1}{2} log_2(1 + P_2/N_2) \qquad (37) \\ R_1 + R_2 &\leq min\{\frac{1}{2} log_2(1 + \frac{P_1 + \alpha^2 P_2}{N_1}), \frac{1}{2} log_2(1 + \frac{P_2 + \beta^2 P_1}{N_2})\} \qquad (38) \end{aligned}$$

The region is just the intersection of the two capacity region of the two multiple access gaussian channels.
Except for demonstrating the inner bound for DM interference channel case, Han [4] also showed that the result is applicable with obvious modifications to the Gaussian interference channel. The inner bound to capacity region of general Gaussian interference channel is given by the following theorem:

Theorem 6 *For the Gaussian interference channel, any rate pair* (R_1, R_2) *is achievable, if* R_1 *and* R_2 *can be written as* $R_1 = R_{01} + R_{11}$ *and* $R_2 = R_{02} + R_{22}$*, and they satisfy:*

$$\begin{aligned} R_{01} &\leq min\{\mathcal{C}(\frac{\bar{\lambda}_1 P_1}{N_1 + \alpha^2 \lambda_2 P_2}), \mathcal{C}(\frac{\beta^2 \bar{\lambda}_1 P_1}{N_2 + \beta^2 \lambda_1 P_1})\} \qquad (39) \\ R_{02} &\leq min\{\mathcal{C}(\frac{\bar{\lambda}_2 P_2}{N_2 + \beta^2 \lambda_1 P_1}), \mathcal{C}(\frac{\alpha^2 \bar{\lambda}_2 P_2}{N_1 + \alpha^2 \lambda_2 P_2})\} \qquad (40) \\ R_{11} &\leq \mathcal{C}(\frac{\lambda_1 P_1}{N_1 + \alpha^2 \lambda_2 P_2}) \qquad (41) \\ R_{22} &\leq \mathcal{C}(\frac{\lambda_2 P_2}{N_2 + \beta^2 \lambda_1 P_1}) \qquad (42) \\ R_{01} + R_{02} &\leq min\{\mathcal{C}(\frac{\bar{\lambda}_1 P_1 + \alpha^2 \bar{\lambda}_2 P_2}{N_1 + \alpha^2 \lambda_2 P_2}), \mathcal{C}(\frac{\bar{\lambda}_2 P_2 + \beta^2 \bar{\lambda}_1 P_1}{N_2 + \beta^2 \lambda_1 P_1})\} \qquad (43) \\ R_{01} + R_{11} &\leq \mathcal{C}(\frac{P_1}{N_1 + \alpha^2 \lambda_2 P_2}) \qquad (44) \\ R_{02} + R_{11} &\leq \mathcal{C}(\frac{\lambda_1 P_1 + \alpha^2 \bar{\lambda}_2 P_2}{N_1 + \alpha^2 \lambda_2 P_2}) \qquad (45) \\ R_{01} + R_{22} &\leq \mathcal{C}(\frac{\lambda_2 P_2 + \beta^2 \bar{\lambda}_1 P_1}{N_2 + \beta^2 \lambda_1 P_1}) \qquad (46) \\ R_{02} + R_{22} &\leq \mathcal{C}(\frac{P_2}{N_2 + \beta^2 \lambda_1 P_1}) \qquad (47) \\ R_{01} + R_{02} + R_{11} &\leq \mathcal{C}(\frac{P_1 + \alpha^2 \bar{\lambda}_2 P_2}{N_1 + \alpha^2 \lambda_2 P_2}) \qquad (48) \\ R_{01} + R_{02} + R_{22} &\leq \mathcal{C}(\frac{P_2 + \beta^2 \bar{\lambda}_1 P_1}{N_2 + \beta^2 \lambda_1 P_1}) \qquad (49) \end{aligned}$$

for some $0 \leq \lambda_j \leq 1, \bar{\lambda} = 1 - \lambda_j, j = 1, 2$. $\mathcal{C}(x)$ *is defined as* $\mathcal{C}(x) = \frac{1}{2} log_2(1 + x)$

Costa [5] showed that the following "corner points" lie on the boundary of the the capacity region of the general Gaussian interference channel:

$$(R_1, R_2) = \left(\mathcal{C}(\frac{P_1}{N_1}), min\left\{\mathcal{C}(\frac{\alpha^2 P_2}{P_1+N_1}), \mathcal{C}(\frac{P_2}{\beta^2 P_1+N_2})\right\}\right) \tag{50}$$

$$(R_1, R_2) = \left(min\left\{\mathcal{C}(\frac{\beta^2 P_1}{P_2+N_2}), \mathcal{C}(\frac{P_1}{\alpha^2 P_2+N_1}), \mathcal{C}(\frac{P_2}{N_2}), \right\}\right) \tag{51}$$

The achievability of these points follows from the previous theorem, the receiver first decodes the other sender's message , subtracts it off and then decodes his message. The converse uses the entropy power inequality and it is quite complicated.

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