

Throughput Capacities and Optimal Resource Allocation in Multiaccess Fading Channels

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Abstract

Mobile wireless environment would introduce special phenomena such as multipath fading which provide more challenges than wired networks. Resource allocation scheme can be implemented in the multi-access wireless systems to achieve better performance. This report presents the main result of [1] by Tse and Hanly, which characterize the throughput capacity region of multi-access fading channels, show that the boundary point of the region can be achieved by successive decoding and obtain the greedy algorithm to find the optimal rate and power allocation in fading state.

1. Introduction

The classical discrete memoryless multiple access channel with two transmitters, has their capacity region (R_1, R_2) satisfying

$$\begin{aligned} R_1 &< I(X_1; Y | X_2) \\ R_2 &< I(X_2; Y | X_1) \\ R_1 + R_2 &< I(X_1, X_2; Y) \end{aligned} \tag{1}$$

with fixed probability transitions $p(y | x_1, x_2)$ and for some independent input distribution $p(x_1)p(x_2)$.

In the case of the Gaussian multiple access channel, which can be modeled as:

$$Y_i = X_{1i} + X_{2i} + Z_i$$

The capacity region turns to be

$$\begin{aligned} R_1 &\leq \frac{1}{2} \log\left(1 + \frac{P_1}{\sigma^2}\right) \\ R_2 &\leq \frac{1}{2} \log\left(1 + \frac{P_2}{\sigma^2}\right) \\ R_1 + R_2 &\leq \frac{1}{2} \log\left(1 + \frac{P_1 + P_2}{\sigma^2}\right) \end{aligned} \tag{2}$$

where σ^2 is the variance of the white Gaussian noise Z_i , P_1, P_2 are the power constraint of X_1, X_2 . The upper bounds are achieved when $X_1 \sim N(0, P_1)$ and $X_2 \sim N(0, P_2)$.

The mobile wireless environment provides more challenges to reliable communication not found in wired networks by introducing the time-varying channel nature such as multipath fading, shadowing, and path losses. A general strategy to combat these detrimental effects is through the dynamic allocation of resources based on the states of the channels of the users. Such resources may include transmitter power, allocated bandwidth, and bit rates. Instead of finding a scheme with respect to specific multiple-access schemes, here the paper [1] addresses the problem at a more fundamental level as: **what are the information theoretically optimal resource allocation schemes and their achievable performance for multiple access?** The paper looks at the capacity region of the multi-access fading channel with Gaussian noise, when both the receiver and the transmitters can track the time-varying channel, considers the *throughput capacity* for the fading channel, in such case, the channel statistics are assumed to be fixed, and the codeword length can be chosen arbitrarily long to average over the fading of the channel, completes characterizations of this capacity region as well as the optimal resource-allocation schemes which attain the points on the boundary of this region are obtained. It is shown that every point on the boundary is achievable by successive decoding. Since the number of constraints defining the capacity region is exponential in the number of users, in order to obtain simple solutions of resource-allocation, a special *polymatroid* structure of the capacity region is exploited, and greedy optimization algorithm are derived to solve the problem. Besides, the paper [2] by Knopp and Humblet is a nice reference giving the same result on the information capacity and power control in single cell multiuser communications, which deals with the equal rate reward case of the resource allocation problem.

2. Multiaccess Fading Channel Model

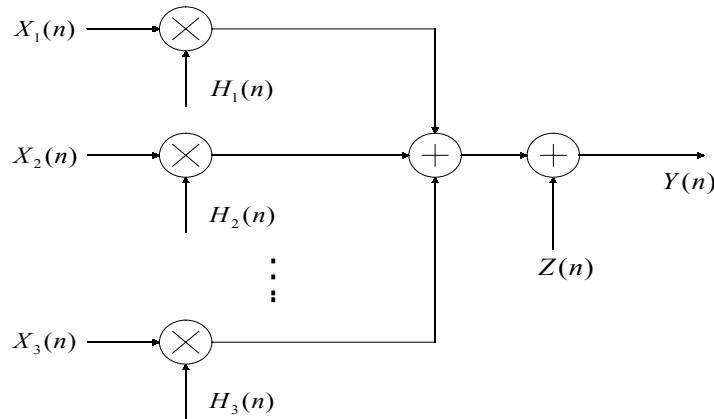


Figure 1. Discrete-time multiple-access Gaussian fading channel

Now let us consider the discrete-time multiple-access Gaussian channel

$$Y(n) = \sum_{i=1}^M \sqrt{H_i(n)} X_i(n) + Z(n) \quad (3)$$

where M is the number of users, $X_i(n)$ and $H_i(n)$ are the transmitted waveform and the fading process of the i th user, respectively, and $Z(n)$ is white Gaussian noise with variance σ^2 . We assume that the fading processes for all users are jointly stationary and ergodic, and the stationary distribution has continuous density and is bounded. User is also subject to an average transmitter power constraint of \bar{P}_i .

2.1. Capacity of fixed power scenario

A. Consider first the simple situation where the users' locations are fixed and the signal of user i is attenuated by a factor of h_i when received at the base station, i.e., $H_i(n) = h_i$ for all time n . In the case of the Gaussian multiaccess channel, this capacity region is

$$C_g(\mathbf{h}, \mathbf{P}) = \left\{ \mathbf{R} : \mathbf{R}(S) \leq \frac{1}{2} \log \left(1 + \frac{\sum_{i \in S} h_i P_i}{\sigma^2} \right) \quad \text{for every } S \subset \{1, \dots, M\} \right\} \quad (4)$$

where $\mathbf{h} = (h_1, \dots, h_M)$ and $\mathbf{P} = (P_1, \dots, P_M)$. Note that this region is characterized by $2^M - 1$ constraints, each corresponding to a nonempty subset of users. Moreover, it is known that the capacity region has precisely $M!$ vertices in the positive quadrant, each achievable by a successive decoding using one of the possible orderings.

B. We now turn to the case of interest where the channels are time-varying due to the motion of the users. When the receiver can perfectly track the channel but the transmitters have no such information, the capacity region is given by

$$\{(R_1, \dots, R_M) : \mathbf{R}(S) \leq \mathbf{E}_{\mathbf{H}} \left[\frac{1}{2} \log \left(1 + \frac{\sum_{i \in S} H_i P_i}{\sigma^2} \right) \right], \quad \forall S \subset \{1, \dots, M\}\} \quad (5)$$

where $\mathbf{H} = (H_1, \dots, H_M)$ is a random vector having the stationary distribution of the joint fading process. An intuitive understanding of this result can be obtained by viewing capacities in terms of time averages of mutual information, a rigorous proof can be found in [3].

2.2. Capacity under transmitters' power control

We will now focus on the scenario of the interest, where all the transmitters and the receiver know the current state of the channels of every user. Thus the codewords and the decoding scheme can both depend on the current state of the channels.

A *power-control policy* PC is a mapping from the fading state space to R_+^M . Given a joint fading state $\mathbf{h} = (h_1, \dots, h_M)$ for the users, $PC_i(\mathbf{h})$ can be interpreted as the transmitter power allocated to user i . For a given power control policy PC , consider the set of rates given by

$$C_f(PC) \equiv \left\{ \mathbf{R} : \mathbf{R}(S) \leq \mathbf{E}_{\mathbf{H}} \left[\frac{1}{2} \log \left(1 + \frac{1}{\sigma^2} \sum_{i \in S} H_i PC_i(\mathbf{H}) \right) \right], \quad \forall S \subset \{1, \dots, M\} \right\} \quad (6)$$

Theorem 2.1: The throughput capacity region for the multi-access fading Gaussian channel when all the transmitters as well as the receiver have side-information of the current state of the channel is given by

$$C(\bar{\mathbf{P}}) \equiv \bigcup_{PC \in F} C_f(PC)$$

where F is the set of all feasible power control policies satisfying the average power constraint

$$F \equiv \{PC : \mathbf{E}_{\mathbf{H}}[PC(\mathbf{H})] \leq \bar{P}_i \quad \forall i\} \quad (7)$$

The above theorem essentially says that the improvement in capacity due to the transmitters having knowledge of the channel state comes solely from the ability to allocate powers according to the channel state.

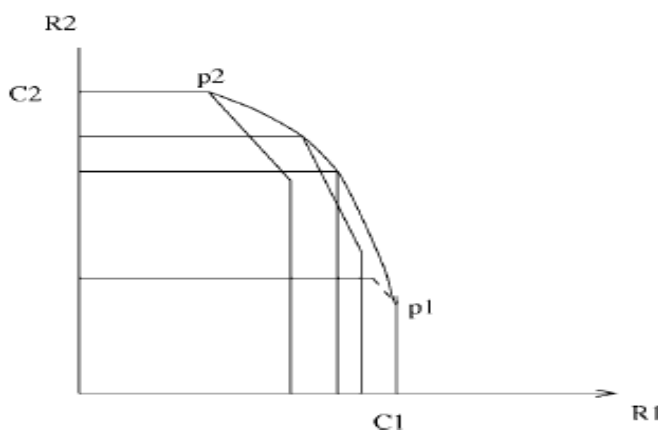


Figure 2. A two-user throughput capacity region as a union of capacity regions, each corresponding to a feasible power control PC . Note each of these regions is a pentagon. The boundary surface of $C(\bar{\mathbf{P}})$ is the curved part.

3. Explicit Characterization of the Capacity Region

A. Polymatroid Structure

Lemma 3.4: Consider a discrete memoryless multiaccess channel with transition matrix $p(y | x_1, \dots, x_M)$, for any independent distribution $p(x_1) \dots p(x_M)$ on the inputs, the polyhedron $\{\mathbf{R} \in R_+^M : \mathbf{R}(S) \leq I[Y; \mathbf{X}(S) | \mathbf{X}(S^c)] \forall S \subset E\}$ is a polymatroid.

Corollary 3.5: The capacity region $C_g(\mathbf{h}, \mathbf{P})$ of a memoryless Gaussian multiaccess channel is a polymatroid.

Lemma 3.6: Let PC be any power control policy. Then $C_f(PC)$ defined in (6) is a polymatroid.

Definition 3.7: A rate allocation policy R is a mapping from the set of joint fading states to R_+^M ; for each fading state \mathbf{h} , $R_i(\mathbf{h})$ can be interpreted as the rate allocated to user i while the users are in state \mathbf{h} .

Lemma 3.8: For any power control policy PC

$$C_f(PC) = \{\mathbf{E}_H[R(\mathbf{H})]: R \text{ is a rate allocation policy s.t. } \forall \mathbf{h} R(\mathbf{h}) \in C_g(\mathbf{h}, PC(\mathbf{h}))\} \quad (8)$$

Furthermore, for any permutation π on E , $\mathbf{v}(\pi) = \mathbf{E}_H[\mathbf{v}_H(\pi)]$

$\mathbf{v}(\pi)$ is the vertex of $C_f(PC)$ corresponding to the permutation π , and,

$\mathbf{v}_h(\pi)$ is the vertex of $C_g(\mathbf{h}, PC(\mathbf{h}))$ corresponding to permutation π for each state \mathbf{h} .

The above structural result shows that the region $C_f(PC)$ can be written as a weighted sum of the capacity regions of parallel time-invariant Gaussian channels $C_g(\mathbf{h}, PC(\mathbf{h}))$.

B. A Lagrangian Characterization of the Capacity Region

Definition 3.9: The boundary surface of $C(\bar{\mathbf{P}})$ is the set of those rates such that no component can be increased with the other components remaining fixed, while remaining in $C(\bar{\mathbf{P}})$.

Lemma 3.10: The boundary surface of $C(\bar{\mathbf{P}})$ is the closure of all points \mathbf{R}^* such that \mathbf{R}^* is a solution to the optimization problem

$$\max_{\mathbf{R}} \mathbf{u} \cdot \mathbf{R} \quad \text{subject to} \quad \mathbf{R} \in C(\bar{\mathbf{P}}) \quad (9)$$

for some positive $\mathbf{u} \in R_+^M$. For a given \mathbf{u} , \mathbf{R}^* is a solution to the above problem if and only if there exists a $\mathbf{l} \in R_+^M$, rate allocation policy $R(\cdot)$, and power control policy $PC(\cdot)$ such that for every joint fading state \mathbf{h} , $(R(\mathbf{h}), PC(\mathbf{h}))$ is a solution to the optimization problem

$$\max_{(\mathbf{r}, \mathbf{p})} \mathbf{u} \cdot \mathbf{r} - \mathbf{l} \cdot \mathbf{p} \quad \text{subject to} \quad \mathbf{r} \in C_g(\mathbf{h}, \mathbf{p})$$

$$\text{and } \mathbf{E}_H[R_i(\mathbf{H})] = R_i^*, \quad \mathbf{E}_H[PC_i(\mathbf{H})] = \bar{P}_i, \quad i = 1, \dots, M \quad (10)$$

where \bar{P}_i is the constraint on the average power of user i .

One can interpret \mathbf{u} as a vector of rate rewards, prioritizing the users, which will determine the order of the successive decoding; the vector \mathbf{l} can be interpreted as a set of power prices, for a given \mathbf{u} , \mathbf{l} is chosen such that the average power constraints are satisfied.

C. Optimal Power and Rate Allocation for a Fading State \mathbf{h}

Consider the problem of determining $(R^*(\mathbf{h}), PC^*(\mathbf{h}))$ for each fading state \mathbf{h} .

Theorem 3.14: Consider the problem

$$\max_{(\mathbf{x}, \mathbf{y})} \mathbf{u} \cdot \mathbf{x} - \mathbf{l} \cdot \mathbf{y} \quad \text{subject to} \quad \mathbf{x}(S) \leq g(\mathbf{y}(S)) \quad \forall S \subset E$$

where g is a monotonically increasing concave function.

Define the marginal utility functions

$$u_i(z) \equiv u_i g'(z) - l_i, \quad i = 1, \dots, M$$

$$u^*(z) \equiv [\max_i u_i(z)]^+$$

(Here, $x^+ \equiv \max(x, 0)$).

Then the solution to the above problem is given by $\int_0^\infty u^*(z) dz$ and an optimizing point $(\mathbf{x}^*, \mathbf{y}^*)$ to achieve this can be found by a greedy algorithm.

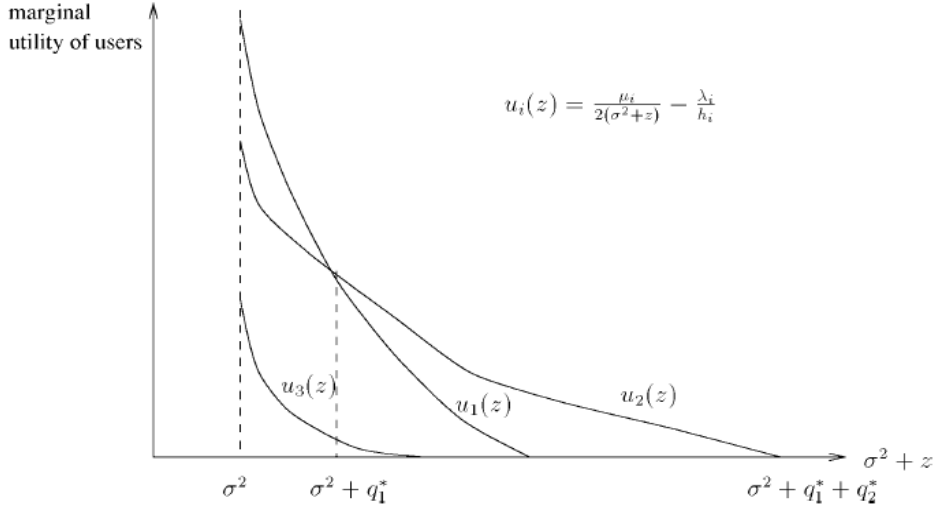


Figure 3. A three-user example illustrating the greedy power allocation. The x-axis represents the received interference level; y-axis the marginal utility of each user at the interference levels. At each interference level, the user who is selected to transmit is the one with the highest marginal utility. Here, user 1 gets decoded after user 2, and user 3 gets no power at all. The optimal received powers for user 1 and user 2 are q_1^* and q_2^* , respectively.

Specializing this result to the case of the time-invariant Gaussian channel gives exactly the proposed solution to the optimization problem (10) discussed earlier. The function g is taken to be

$$g(z) \equiv \frac{1}{2} \log\left(1 + \frac{z}{\sigma^2}\right) \quad \text{then} \quad u_i(z) \equiv \frac{u_i}{2(\sigma^2 + z)} - \frac{l_i}{h_i} \quad (11)$$

In terms of the received powers $\mathbf{q} = (h_1 p_1, \dots, h_M p_M)$, the optimization problem can be rewritten as

$$\max \sum_i u_i r_i - \sum_i \frac{l_i}{h_i} q_i \quad \text{subject to} \quad \mathbf{r}(S) \leq g(\mathbf{q}(S)) \quad \forall S \subset E \quad (12)$$

Moreover, it shows that the optimal solution is achieved by successive decoding among the actual users. Any such solution can be represented by a permutation π and a set of intervals $[z_i, z_{i+1}]$, $i = 1, \dots, M$ of the real line such that $z_1 = 0$, $z_{i+1} - z_i$ is the received power of user $\pi(i)$, and users are decoded in the order given by $\pi(M), \pi(M-1), \dots, \pi(1)$.

With probability 1, the optimal power and rate allocation is unique and is explicitly given by

$$R_i^*(\mathbf{h}) = \int_{A_i} \frac{1}{2(\sigma^2 + z)} dz \quad PC_i^*(\mathbf{h}) = |A_i|$$

where $A_i \equiv \{z \in [0, \infty) : u_i(z) > u_j(z) \quad \forall j \neq i \text{ and } u_i(z) > 0\}$ (13)

D. Boundary of the Capacity Region $C(\bar{\mathbf{P}})$

Lemma 3.15: Let \mathbf{u} be a given positive rate reward vector. Then there is a unique \mathbf{R}^* on the boundary which maximizes $\mathbf{u} \cdot \mathbf{R}$, and there is a unique Lagrangian power price \mathbf{l} such that the optimal power allocation solving (10) satisfies the average power constraints.

For any such positive \mathbf{u} , the above lemma implies that we can define a parameterization $\mathbf{R}^*(\mathbf{u})$ which is the unique rate vector on the boundary which maximizes $\mathbf{u} \cdot \mathbf{R}$. Its value can be obtained using the greedy rate and power allocation solution, with \mathbf{l} chosen such that the average power constraints are satisfied. For the given \mathbf{u} and \mathbf{l} , let $R^*(\mathbf{h}, \mathbf{u}, \mathbf{l})$ and $PC^*(\mathbf{h}, \mathbf{u}, \mathbf{l})$ be the optimal solution to the problem (10). Thus we have

Theorem 3.16: Assume that the fading processes of users are independent of each other. The boundary of $C(\bar{\mathbf{P}})$ is the closure of the parametrically defined surface

$$\{\mathbf{R}^*(\mathbf{u}) : \mathbf{u} \in R_+^M, \sum_i u_i = 1\}$$

where for $i = 1, \dots, M$

$$R_i^*(\mathbf{u}) = \int_0^\infty \frac{1}{2(\sigma^2 + z)} \left\{ \int_{\frac{2l_i(\sigma^2+z)}{u_i}}^\infty \prod_{k \neq i} F_k \left(\frac{2l_k h(\sigma^2 + z)}{2l_i(\sigma^2 + z) + (u_k - u_i)h} \right) f_i(h) dh \right\} dz \quad (14)$$

where the vector \mathbf{l} is the unique solution of the equations

$$\int_0^\infty \left\{ \int_{\frac{2l_i(\sigma^2+z)}{u_i}}^\infty \frac{1}{h} \times \prod_{k \neq i} F_k \left(\frac{2l_k h(\sigma^2 + z)}{2l_i(\sigma^2 + z) + (u_k - u_i)h} \right) f_i(h) dh \right\} dz = \bar{P}_i \quad (15)$$

$i = 1, \dots, M$. Moreover, every point can be attained by successive decoding.

E. Cases for throughput capacity and power allocation in fading channels

1) Single-User Channel:

It is the simple case of *Theorem 3.16* with $M = 1$:

$$\begin{aligned} R^* &= \int_0^\infty \frac{1}{2(\sigma^2 + z)} \left\{ \int_{\frac{2l(\sigma^2+z)}{u}}^\infty f(h) dh \right\} dz \\ &= \int_0^\infty \frac{1}{2} \log \left(1 + \frac{h}{\sigma^2} \left(\frac{u}{2l} - \frac{\sigma^2}{h} \right)^+ \right) f(h) dh \end{aligned}$$

by reversing the order of integration. Using (15), the constant $\frac{u}{2l}$ is shown to satisfy the power

constraint
$$\int_0^\infty \left(\frac{u}{2l} - \frac{\sigma^2}{h} \right)^+ f(h) dh = \bar{P}$$

This is just the classic water-filling solution to the problem of power allocation over a set of parallel single-user channels, one for each fading level h . The strategy has the characteristic that more power is used when the channel is good and little or even no power when it is bad.

2) Maximum Sum-Rate Point:

If we set $u_1 = \dots = u_M = 1$, we get the point on the boundary of the capacity region that maximizes the sum of the rates of the individual users. For this case, the marginal utility functions $u_i(z)$'s are given by

$$u_i(z) = \frac{1}{2(\sigma^2 + z)} - \frac{l_i}{h_i}$$

We note that for a given fading state \mathbf{h} , the marginal utility function of the user with the smallest $\frac{\lambda_i}{h_i}$ dominates all the others for all z . This means that in the optimal strategy, at most one user is allowed to transmit at any given fading state. The optimal power control strategy PC^* can be readily calculated to be

$$PC_i^*(\mathbf{h}, \mathbf{l}) = \begin{cases} \left(\frac{1}{2l_i} - \frac{\sigma^2}{h_i}\right)^+, & \text{if } h_i > \frac{l_i}{l_j} h_j \text{ for all } j \\ 0, & \text{else.} \end{cases}$$

The optimal rates are given by

$$R_i^* = \int_0^\infty \frac{1}{2} \log\left(1 + \frac{h}{\sigma^2} \left(\frac{1}{2\lambda_i} - \frac{\sigma^2}{h}\right)^+\right) \times \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h) dh, \quad i = 1, \dots, M$$

where the constants λ_i 's satisfy

$$\int_0^\infty \left(\frac{1}{2\lambda_i} - \frac{\sigma^2}{h}\right)^+ \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h) dh = \bar{P}_i, \quad i = 1, \dots, M$$

This solution was just the same as which by Knopp and Humblet [2]. Note that this power control gives rise to a time-division multiple-access strategy.

3) Multiple Classes of Users:

While the above strategy maximizes the total throughput of the system, it can be unfair if the fading processes of the users have very different statistics. Generally, one way of remedying this situation is to assign unequal rate rewards to users. Let us consider an example where there are two classes of users. Users in the same class have the same fading statistics and power constraints; the class 1 can represent users at the cell boundary, while the class 2 consists of users close to the base station. To maintain equal rates for everyone, we can assign rate rewards u_1 to all users in class 1 (far), and u_2 to users in class 2 (near), with $u_1 > u_2$. By symmetry, the power prices of users in the same class are the same. We observe that at any fading state, the marginal utility function of the user with the best channel within each class dominates those of other users in the same class. Thus the optimal strategy has the form that at each fading state, only the strongest user in each class transmits, and the two users are decoded by successive cancellation, with the nearby user (class 2) decoded first. This gives an advantage to the user far away. Adjusting the rate rewards can be thought of as a way to maintain fairer allocation of resources to the users.

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