Multiuser Capacity in Block Fading Channel

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1 Introduction and Model

We use a block-fading model, with coherence interval T where M independent users simultaneously transmit to a single receiver equipped with N antennas in a flat-fading environment, where each user has sole access to one of M transmit antennas, and where nobody has any channel state information. The fading is described by an $M \times N$ complex-valued propagation matrix H, which remains constant for T symbol periods, after which it jumps to a new independent value for another T symbol, and so on. During a coherence interval, the M users collectively transmit a $T \times M$ complex matrix S, whose columns, representing different users, are statistically independent, and the receiver records a $T \times N$ complex matrix X. The basic equation describing this channel is as follows:

$$X = \sqrt{\frac{\rho}{M}} SH + W$$

where W is a $T \times N$ vector of additive receiver noise, whose components are independent, zero-mean complex Gaussian with unit variance ($\mathcal{CN}(0,1)$). The components of H are assumed independent and zero mean with unit variance. The independence of the columns of S is necessary if the users are to act with no cooperation among themselves. We enforce an expected power constraints

$$\operatorname{tr} E\{SS^*\} = TM$$

This constraint, when combined with the normalization $1/\sqrt{M}$ implies that the total transmitted power remains constant as the number of users changes, and the ρ represents the expected SNR at each receiver antenna.

We wish to choose the joint probability density of the components of S, subject to the independence of its M columns, and subject to the power constraint to maximize the mutual information with no CSI available to anyone (receiver and transmitters).

$$I(X;S) := E\{\log(\frac{p(X|S)}{p(X)})\}$$

This maximization yields the total throughout or capacity.

It is convenient to assume Rayleigh fading, where the independent components of H are distributed as $\mathcal{CN}(0,1)$, although this assumption can be relaxed for some of the asymptotic results. For Rayleigh fading, the conditional density takes the form

$$p(X|S) = \frac{\exp(-X^* \Lambda X)}{\pi^{TN} \det^N \Lambda} \tag{1.1}$$

where $\Lambda = I_T + (\rho/M)SS^*$ and I_T denote the $T \times T$ identity matrix.

2 Known Results about Single User Channel

2.1 coherent channel

If the receiver somehow knew the random propagation coefficients, the capacity would be greater than for the case of interest where the receiver does not know propagation coefficients. Telatar [3] computes the perfect-knowledge capacity for the case $T = 1$. It is straightforward to extend his analysis to $T > 1$.

The perfect-knowledge fading link is completely described by the conditional probability density, $p(X, H|S) = p(X|H, S)p(H)$. The perfect-knowledge capacity is obtained by maximizing the mutual information between (X, H) and S with respect to $p(S)$. The mutual information is

$$I(X, H; S) = E \log \frac{p(X, H|S)}{p(X, H)} = E \log \frac{p(X|H, S)}{p(X|H)} = E\{E\{\log(\frac{p(X|H, S)}{p(X|H)})|H\}\}.$$

The inner expectation, conditioned on H, is simply the mutual information for the classical additive Gaussian noise case and is maximized by making the components of S independent $\mathcal{CN}(0,1)$. The resulting perfect-knowledge capacity is

$$C = E \log \det[I_N + \frac{\rho}{M} H^* H].$$

The asymptotical capacity can be described as follows:

$$\lim_{M \to \infty} C = N \log(1+\rho).$$

Using the identity $\det[I_N + \frac{\rho}{M} H^* H] = \det[I_M + \frac{\rho}{M} H H^*]$, we can derive the following limiting result:

$$\lim_{M \to \infty} C = M \log(1 + \frac{\rho N}{M}).$$

2.2 non-coherent channel

The following lemma introduce Cholesky factorization:

Lemma 2.1. *For a positive definite matrix $A \in \mathcal{C}^{T \times T}$, we have the following factorization:*

$$A = LL^*,$$

where L is lower triangle matrix.

Theorem 2.1. [1] *For any coherence interval T and any number of receiver antennas, the capacity obtained with $M > T$ transmitter antennas is the same as the capacity obtained with $M = T$ transmitter antennas.*

Proof. Suppose that a particular joint probability density of the elements of SS^* achieve capacity with $M > T$ antennas. We can perform the Cholesky factorization $SS^* = LL^*$, where L is a $T \times T$ lower triangular matrix. Using T transmitter antennas, with a signal matrix that has the same joint probability density as the joint probability density of L, we may therefore also achieve the same probability density on SS^*, thus we get the same channel conditional probability 1.1. Moreover if S satisfies power constraints, then so does L. □

Definition 2.1. *A $T \times T$ unitary matrix Φ is called isotropically distributed if its probability density is unchanged when pre-multiplied by a deterministic unitary matrix:*

$$p(\Phi) = p(\Theta\Phi), \forall\Theta : \Theta^*\Theta = I$$

Lemma 2.2. *Let $A \in \mathcal{C}_r^{m\times n}$, then there exist unitary matrices $U \in \mathcal{C}^{m\times m}$ and $V \in \mathcal{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} \sum_r & 0 \\ 0 & 0 \end{bmatrix} V,$$

where $\sum_r = diag(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Theorem 2.2. [1] *The signal matrix that achieves capacity can be written as $S = \Phi V$, where Φ is an $T \times T$ isotropically distributed unitary matrix, and V is an independent $T \times M$ real, nonnegative, diagonal matrix. Furthermore we can choose the joint density of the diagonal elements of V of be unchanged by rearrangements of its arguments.*

Proof. Combine the singular value decomposition $S = U\Phi V$ and channel conditional density function 1.1, one can prove the same capacity can be achieved by $S' = \Phi V$. Suppose S' has probability density function $p(S')$ and generate mutual information I_0. Let Θ be an isotropically distributed unitary matrix that is statistically independent of Φ and V, and define a new signal matrix $S'' = \Theta S$, generating mutual information I_1. One can check conditioned on Θ, the mutual information generated by S'' equals I_0. The concavity of mutual information as a function as a function of $p(S)$ and Jensen's inequality imply that $I_1 \geq I_0$. The same argument can be applied to the distribution of diagonal matrix.

□

Theorem 2.3. [5]

$$\lim_{\rho\to\infty} \frac{C(M, N, T; \rho)}{\log \rho} = TM^*(1 - \frac{M^*}{T}),$$

where $M^* = \min\{M, N, \lfloor \frac{T}{2} \rfloor\}$.

3 Capacity of Multiuser Channel

From [3], we conclude that if perfect CSI were available to the receiver, the components of S would be independent $\mathcal{CN}(0,1)$, and the capacity would increase monotonically with $\min\{M,N\}$. If perfect CSI were available to the receiver, this single-user capacity could be achieved by M independent users. The absence of CSI drastically changes the problem, and that the total M-user capacity is generally less that single-user/M-antenna capacity.

Conjecture 3.1. [2] *For multiuser antenna channel, the total capacity for any $M > T$ is equal to to the total capacity for $M \leq T$.*

Supporting case 1: T=1

According to Theorem 2.1, for the special case where $T = 1$, a single user having M antennas can attain the same performance with a single antenna, so the total capacity for M users is equal to the capacity for one user.

Supporting case 2: large T, M

By chain rule,

$$I(X;S) = I(X;S,H) - I(X;H|S),$$

we have

$$I(X;S) \leq I(X;S,H).$$

Again by chain rule, the following inequality holds

$$I(X;S,H) = I(X;S|H) + I(X;H) \geq I(X;S|H).$$

The combination of the above inequalities gives

$$I(X;S|H) - I(X;H|S) \leq I(X;S) \leq I(X;S,H).$$

Thus, we have

$$I(X;S) \leq I(X,H;S) = I(H;S) + I(X;S|H) = I(X;S|H) \leq I(X;S_G|H)$$

$$= TE\{\log\det(I_N + \frac{\rho}{M}H^*H)\} \leq T\log\det|I_N + \frac{\rho}{M}E\{H^*H\}| = TN\log(1+\rho),$$

where S_G denote zero-mean complex Gaussian, with independent components having unit variance.

The mutual information $I(X;H|S)$ corresponds to the fictitious case of a user sending the signal H through a random propagation matrix S, where S is known at the receiver. Therefore, the mutual information is maximized by making the elements of H complex Gaussian, which gives

$$I(X;H|S) \leq NE\{\log\det(I_M + \frac{\rho}{M}S^*S)\} \leq N\log\det(I_M + \frac{\rho}{M}E\{S^*S\})$$

$$= N\log\det(I_M + \frac{\rho T}{M}I_M) = MN\log(1+\frac{\rho T}{M}).$$

We fix N and simultaneously let T and M become big while maintaining a constant ratio $\beta = T/M$. This implies that $H^*H/M \to I_N$ almost surely. Thus we have

$$I(X;S) \geq I(X;S|H) - I(X;H|S) = TE\{\log\det(I_N + \frac{\rho}{M}H^*H)\} - I(X;H|S)$$

$$\geq TN\log(1+\rho) - T\frac{N}{\beta}\log(1+\rho\beta)$$

The combination of the above derivation gives

$$N\log(1+\rho) - \frac{N}{\rho}\log(1+\rho\beta) \leq \frac{I(X;S_G)}{T} \leq N\log(1+\rho).$$

Finally we let ratio grow big, $\beta = T/M \to \infty$, which yields the asymptotically result

$$\lim_{\beta\to\infty}\frac{I(X;S)}{T} = N\log(1+\rho).$$

The expression $N\log(1+\rho)$ is equal to the capacity for a single user having an unlimited number of transmit antennas, where the receiver has perfect knowledge of the propagation matrix. We have shown that this same capacity can collectively be attained by M independent users, where no CSI is available to anyone, in the limit as T and M become large, with $M \ll T$. And the capacity could not be increased by having $M > T$ users.

Supporting case 3: large ρ

Consider the capacity of non-coherent channel described by Theorem 2.3. When $N = 1$, one can show that having a single operating transmitter yields the optimal possible asymptotical result with respect to ρ.

4 Vector Multiple-Access Channels and Water-Filling [4]

In the multiple-antenna model, a baseband model for a synchronous multiple-access antenna array channel is

$$\mathbf{y}(n) = d(N) \sum_{i=1}^{K} x_i(n) \tilde{h}_i^s(n) \tilde{\mathbf{h}}_i^f(n) + \mathbf{w}(n).$$

Here n denotes the time of channel use, $x_i(n)$ is the transmitted symbol of user i at time n, and $\mathbf{y}(n)$ is an N-dimensional vector of received symbols at the N antenna elements of the array at the receiver. The vector $\tilde{h}_i^s(n)\tilde{\mathbf{h}}_i^f(n)$ represents the channel from the ith user to the antenna array at time n. The scalars $\tilde{h}_i^s(n)$ captures the slowly varying component of the fading channel and the vector $\tilde{\mathbf{h}}_i^f(n)$ is the fast varying component of the fading channel. We will assume that $\{\tilde{h}_i^s(n)\}_n$ and $\{\tilde{\mathbf{h}}_i^f(n)\}_n$ are independent complex stationary and ergodic processes and are perfectly known.

The sum capacity of the multiple antenna MAC can be described as follows:

$$C_{opt} = \sup_{\mathcal{P}} C_{sum}(\mathcal{P})$$

$$= \sup_{\mathcal{P}} \frac{1}{2N} \left[\log\det(I + d^2(N) N \sum_{i=1}^{K} \sigma^{-2} h_i^s \tilde{\mathbf{s}}_i \tilde{\mathbf{s}}_i^* \mathcal{P}_i(h_1^s, \cdots, h_K^2, S) \right].$$

The power allocation policy $\mathcal{P}$ depends on both the slow-fading components $h_1^s, \cdots, h_K^s$ and the fast varying components $S = \frac{1}{\sqrt{N}}[\mathbf{h}_1^f, \cdots, \mathbf{h}_K^f]$. The calculation can be converted to an optimization problem with constraints. By analyzing the Kuhn-Tucker point, one can have the optimum solution which is called water-filling policy. The main result in [4] is that a water-filling power policy that depends only on the slow fading component of the fading channel is asymptotically optimal(the asymptotic is in the number of users and the correspondingly large number of antennas at the receiver).

References

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