Throughput Capacities and Optimal Resource Allocation in Multiaccess Fading Channels

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- Multiaccess Fadding Channel Model
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- Polymatroid Structure of Channel Capacity
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Preliminary:

The capacity region for classical discrete memoryless multiple access channel

(R_1, R_2) satisfying :

$$R_1 < I(X_1;Y \mid X_2)$$
$$R_2 < I(X_2;Y \mid X_1)$$
$$R_1 + R_2 < I(X_1, X_2;Y)$$

with fixed probability transitions $p(y \mid x_1, x_2)$

and for some independent input distribution $p(x_1)p(x_2)$

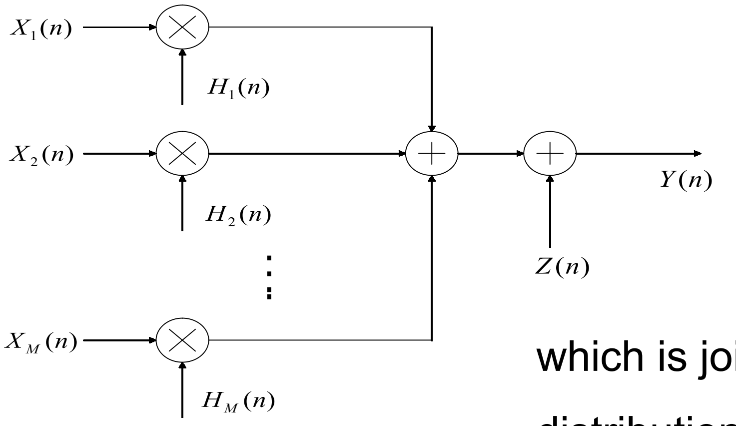
In the case of the Gaussian multiple access channel $Y_i = X_{1i} + X_{2i} + Z_i$

$$R_1 \le \frac{1}{2}\log(1 + \frac{P_1}{\sigma^2})$$
$$R_2 \le \frac{1}{2}\log(1 + \frac{P_2}{\sigma^2})$$
$$R_1 + R_2 \le \frac{1}{2}\log(1 + \frac{P_1 + P_2}{\sigma^2})$$

where $Z \sim N(0,\sigma^2)$, P_1, P_2 are the power constraint of X_1, X_2

The upper bounds are achieved when $X_1 \sim N(0, P_1)$, $X_2 \sim N(0, P_2)$

Multiaccess Fadding Channel Model:



$$Y(n)=\sum_{i=1}^{M}\sqrt{H_i(n)}X_i(n)+Z(n)$$

$H_i(n)$ is the fading process of the i th user, which is jointly stationary and ergodic and whose stationary distribution has continuous density and is bounded

I. Capacity region of the fixed Gaussian multiaccess channel

$$C_g(\mathbf{h},\mathbf{P})=\{\mathbf{R}:\mathbf{R}(S)\le\frac{1}{2}\log(1+\frac{\sum_{i\in S}h_iP_i}{\sigma^2})\qquad \textit{for every } S\subset\{1,\ldots,M\}\}$$

II. Capacity region of time-Varying channel with CSI on receiver side only

$$\{(R_1,\ldots,R_M):\mathbf{R}(S)\le\mathbf{E}_{\mathbf{H}}[\frac{1}{2}\log(1+\frac{\sum_{i\in S}H_iP_i}{\sigma^2})],\qquad \forall S\subset\{1,\ldots,M\}\}$$

Capacity Under Power Control: (CSI on all transceivers)

Def: A *power-control policy* PC: a mapping of $\mathbf{h} = (h_1, \ldots, h_M) \rightarrow R_+^M$

I. Capacity region under a power-control policy

$$C_f(PC) \equiv \{\mathbf{R} : \mathbf{R}(S) \leq \mathbf{E}_H[\frac{1}{2}\log(1 + \frac{1}{\sigma^2}\sum_{i \in S} H_i PC_i(\mathbf{H}))], \qquad \forall S \subset \{1, \ldots, M\}\}$$

II. Throughput capacity region under power control

Theorem 2.1: $$C(\overline{\mathbf{P}}) \equiv \bigcup_{PC \in F} C_f(PC)$$

where $$F \equiv \{PC : \mathbf{E}_\mathbf{H}[PC(\mathbf{H})] \leq \overline{P_i} \qquad \forall i\}$$

- achievability: for any PC , look the channel as unit power channel with $h_i PC_i(\mathbf{h})$, $C_f(PC)$ Is achievable and $\bigcup_{PC \in F} C_f(PC) \subset C(\overline{\mathbf{P}})$
- converse: By Fano’s inequality.

Polymatroid Structure and Greedy Algorithm

I. Polymatroid Structure

Def: $E = \{1, ..., M\}$ and $f : 2^E \to R_+$ be a set function. The polyhedron $B(f) \equiv \{(x_1, ..., x_M) : \mathbf{x}(S) \leq f(S)\ \forall S \subset E, x_i \geq 0 \forall i\}$ is a polymatroid if the set function f satisfies

1) $f(\phi) = 0$ (normalized).

2) $f(S) \leq f(T)$ if $S \subset T$ (nondecreasing).

3) $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ (submodular)

II. Greedy Algorithm

optimization problem $\max \mathbf{l} \cdot \mathbf{x}$ subject to $\mathbf{x} \in B(f)$

- **Initialization**: Set $x_i = 0$ for all i. Set $k = 1$
- **Step** k : Increase the value of $x_{\pi^*(k)}$ until a constraint becomes tight. Goto Step $k+1$
- After M steps, the optimal solution is reached.

Polymatroid Structure of Channel Capacity

I. Classical discrete memoryless multiaccess channel

$\{\mathbf{R} \in R_{+}^{M} : \mathbf{R}(S) \le I[Y; \mathbf{X}(S) \mid \mathbf{X}(S^{\subset})] \forall S \subset E\}$ **is a polymatroid.**

II. Memoryless Gaussian multiaccess channel

$C_g(\mathbf{h}, \mathbf{P})$ **is a polymatroid.**

III. A power control policy PC, $C_f(PC)$ **is a polymatroid**

Def: A rate allocation policy R is a mapping: $\mathbf{h} \rightarrow R_{+}^{M}$

for each fading state $\mathbf{h}$, $R_i(\mathbf{h})$ can be interpreted as the rate allocated to user i

IV. For any power control policy PC

$$C_f(PC) = \{\mathbf{E_H}[R(\mathbf{H})] : R \textit{ is a rate allocation policy s.t.} \forall \mathbf{h}\, R(\mathbf{h}) \in C_g(\mathbf{h}, PC(\mathbf{h}))\}$$

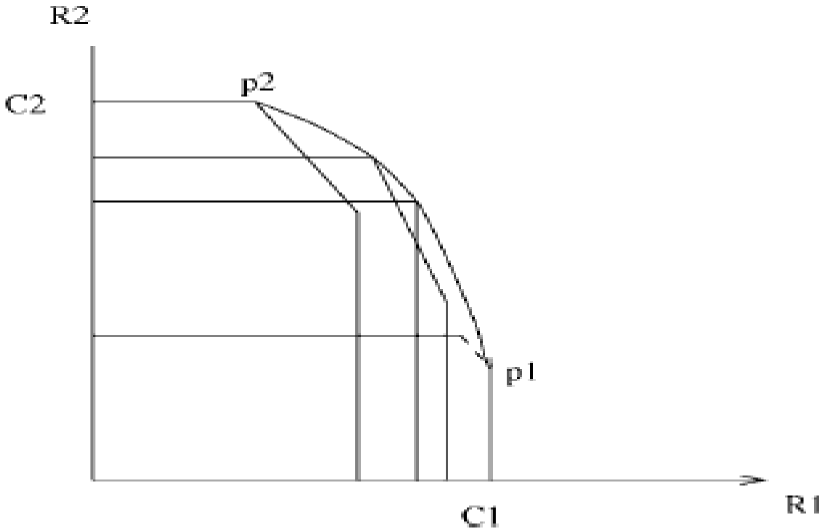
Furthermore, $\mathbf{v}(\pi) = \mathbf{E_H}[\mathbf{v_H}(\pi)]$, for any permutation π on E ,

Where $\mathbf{v}(\pi) \leftrightarrow C_f(PC)$,

and $\mathbf{v_h}(\pi) \leftrightarrow C_g(\mathbf{h}, PC(\mathbf{h}))$,

for each state $\mathbf{h}$,corresponding to same permutation π

A Lagrangian Characterization of the Capacity Region



Capacity Region: $C_f(PC)$ and $C(\overline{\mathbf{P}})$

Def: The *boundary surface of* $C(\overline{\mathbf{P}})$: is the set of those rates such that no component can be increased with the other components remaining fixed, while remaining in $C(\overline{\mathbf{P}})$

Lemma 3.10: I. The boundary surface of $C(\overline{\mathbf{P}})$ is the closure of all points $\mathbf{R}^*$ such that $\mathbf{R}^*$ is a solution to the optimization problem:

$$\max_{\mathbf{R}} \mathbf{u}\cdot\mathbf{R} \quad \textit{subject to} \quad \mathbf{R}\in C(\overline{\mathbf{P}}) \qquad \text{for some} \quad \mathbf{u}\in R_+^M$$

II. For a given $\mathbf{u}$, $\mathbf{R}^*$ is a solution to the above problem if and only if there exists a $\mathbf{l}\in R_+^M$, rate allocation policy $R(\cdot)$ and power control policy $PC(\cdot)$ such that for every joint fading-state $\mathbf{h}$, $(R(\mathbf{h}), PC(\mathbf{h}))$ is a solution to the optimization problem:

$$\max_{(\mathbf{r},\mathbf{P})} \mathbf{u}\cdot\mathbf{r}-\mathbf{l}\cdot\mathbf{p} \quad \textit{subject to} \;\; \mathbf{r}\in C_g(\mathbf{h},\mathbf{p})$$

and $\mathbf{E}_{\mathbf{H}}[R_i(\mathbf{H})]=R_i^*, \quad \mathbf{E}_{\mathbf{H}}[PC_i(\mathbf{H})]=\overline{P}_i, \qquad i=1,\ldots,M$

where $\overline{P_i}$ is the constraint on the average power of user i

$\mathbf{u}$ ----- rate rewards

$\mathbf{l}$ ----- power prices

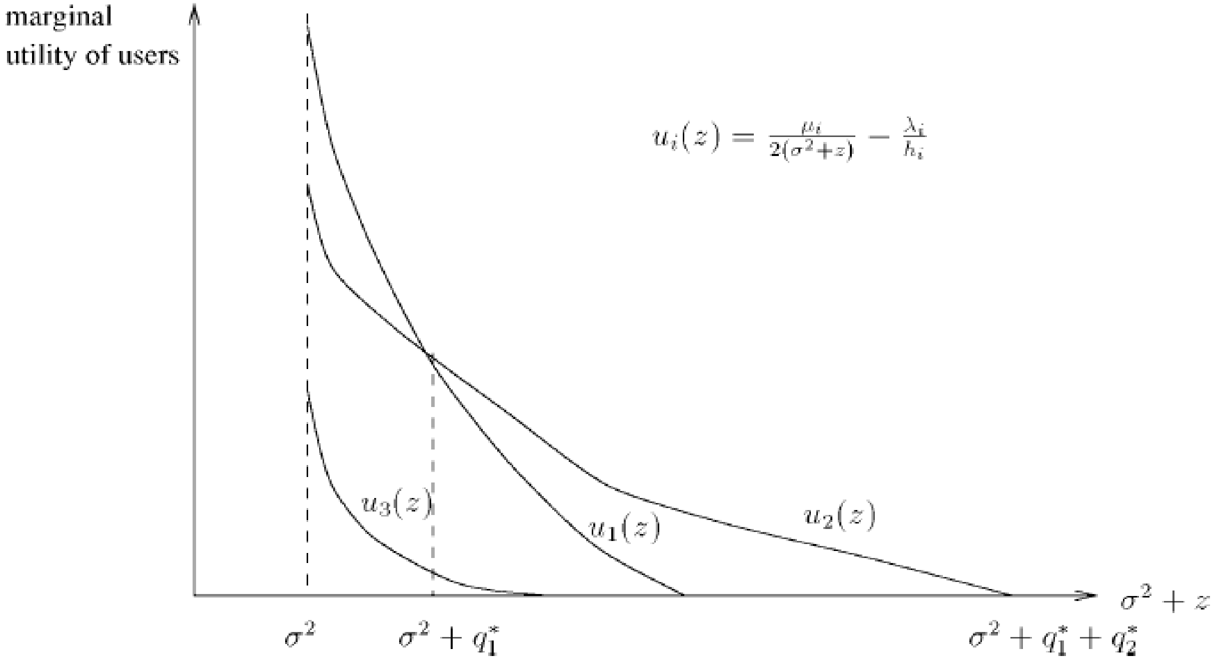
$\mathbf{h}$ ----- joint fading state

$\overline{\mathbf{P}}$ ----- average power constraints

$$(R(\mathbf{h}), PC(\mathbf{h}))$$

Optimal Power and Rate Allocation for a Fading State $\mathbf{h}$

Theorem 3.14: generic problem	Time-invariant Gaussian channel
$\max_{(\mathbf{x},\mathbf{y})} \mathbf{u}\cdot\mathbf{x}-\mathbf{l}\cdot\mathbf{y} \quad \textit{subject to } \mathbf{x}(S)\le g(\mathbf{y}(S))$	$\max \sum_i u_i r_i - \sum_i \frac{l_i}{h_i} q_i \quad \textit{subject to } \mathbf{r}(S)\le g(\mathbf{q}(S))$
g : monotonically increasing concave function	$g(z)\equiv\frac{1}{2}\log(1+\frac{z}{\sigma^2})$
Define the marginal utility functions	
$u_i(z)\equiv u_i g'(z)-l_i, \quad i=1,\dots,M$	$u_i(z)\equiv \frac{u_i}{2(\sigma^2+z)}-\frac{l_i}{h_i}$
$u^*(z)\equiv[\max_i u_i(z)]^+$	$u^*(z)\equiv[\max_i u_i(z)]^+$
The solution to the problem is:	
$\int_0^\infty u^*(z)dz$	$\int_0^\infty u^*(z)dz$



A three-user example illustrating the greedy power allocation

With probability 1, the optimal power and rate allocation is unique and is explicitly given by

$$R_i^*(\mathbf{h}) = \int_{A_i} \frac{1}{2(\sigma^2 + z)} dz \qquad PC_i^*(\mathbf{h}) = |A_i|$$

where $A_i \equiv \{z \in [0,\infty) : u_i(z) > u_j(z) \ \forall j \neq i \ \ and \ \ u_i(z) > 0\}$

Boundary of the Capacity Region $C(\overline{\mathbf{P}})$

I. *Lemma 3.15*: (Uniqueness) for $\mathbf{u} \in R_+^M$, there is a unique $\mathbf{R}^*$ on the boundary which maximizes $\mathbf{u} \cdot \mathbf{R}$, and there is a unique Lagrangian power price $\mathbf{l}$ such that the optimal power allocation satisfies the average power constraints

II. *Theorem 3.16*: For independent users' fading processes, the boundary of $C(\overline{\mathbf{P}})$ is the closure of the parametrically defined surface

$$\{\mathbf{R}^*(\mathbf{u}) : \mathbf{u} \in R_+^M, \sum_i u_i = 1\}$$

where

$$R_i^*(\mathbf{u}) = \int_0^\infty \frac{1}{2(\sigma^2 + z)} \{\int_{\frac{2l_i(\sigma^2+z)}{u_i}}^\infty \prod_{k \neq i} F_k\left(\frac{2l_k h(\sigma^2 + z)}{2l_i(\sigma^2 + z) + (u_k - u_i)h}\right) f_i(h)dh\} dz$$

and where the vector $\mathbf{l}$ is the unique solution of the equations

$$\int_0^\infty \{\int_{\frac{2l_i(\sigma^2+z)}{u_i}}^\infty \frac{1}{h} \times \prod_{k \neq i} F_k\left(\frac{2l_k h(\sigma^2 + z)}{2l_i(\sigma^2 + z) + (u_k - u_i)h}\right) f_i(h)dh\} dz = \overline{P_i}$$

F_i and f_i are the cdf and pdf of the stationary fading distribution of user i

Cases for throughput capacity and power allocation in fading channels

I. *Single-User Channel*

$$R^* = \int_0^\infty \frac{1}{2(\sigma^2+z)} \{\int_{\frac{2\lambda(\sigma^2+z)}{u}}^\infty f(h)dh\}dz = \int_0^\infty \frac{1}{2}\log(1+\frac{h}{\sigma^2}(\frac{u}{2l}-\frac{\sigma^2}{h})^+) f(h)dh$$

$\frac{u}{2l}$ satisfies the power constraint $\int_0^\infty (\frac{u}{2l}-\frac{\sigma^2}{h})^+ f(h)dh = \overline{P}$

Note:

- Time Water-Filling Solution: Power allocation over a set of parallel single-user channels, one for each fading level h
- More power is used when the channel is good and little or even no power when it is bad.

II. *Maximum Sum-Rate Point* $(u_1 = \ldots = u_M = 1)$

Utility Functions: $$u_i(z) = \frac{1}{2(\sigma^2 + z)} - \frac{l_i}{h_i}$$

Power Control Strategy: $$PC_i^*(\mathbf{h}, \mathbf{l}) = \begin{cases} \left(\frac{1}{2l_i} - \frac{\sigma^2}{h_i}\right)^+, & \text{if } h_i > \frac{l_i}{l_j} h_j \ \text{ for all } j \\ 0, & \text{else.} \end{cases}$$

Optimal Rates : $$R_i^* = \int_0^\infty \frac{1}{2} \log\left(1 + \frac{h}{\sigma^2}\left(\frac{1}{2\lambda_i} - \frac{\sigma^2}{h}\right)^+\right) \times \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h)\, dh$$

Power Prices Constant: $$\int_0^\infty \left(\frac{1}{2l_i} - \frac{\sigma^2}{h}\right)^+ \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h)\, dh = \overline{P}_i,$$

Note:

- Here the optimal rates are in the sense of sum of all users' rates.
- The optimal (TDMA) strategy allows at most one user to transmit at any given fading state, this lucky user has the best channel and largest available power.
- As for the two-user example, the boundary point corresponding to $u_1 = u_2$ Is the corner point of a rectangular $C_f(PC)$

III. *Multiple Classes of Users*

If the fading processes of the users have very different statistics, in order to equalize users' rates, unequal rate awards can be assigned to users.

Two classes users example:

Class 1: *far* users at the cell boundary, all assigned rate award u_1

Class 2: *near* users close to the base station, all assigned rate award u_2

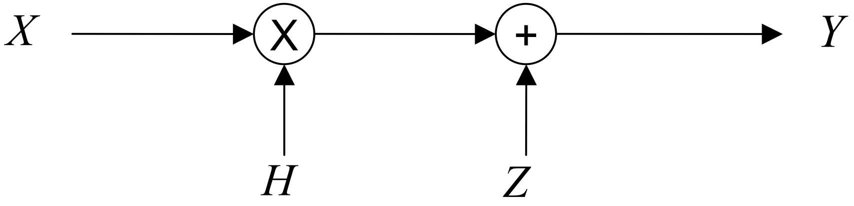
$$u_1 > u_2$$

Note:

- At each fading state, only the strongest user in each class transmits.
- The two strongest users are decoded by successive cancellation, with nearby user decoded first.

Appendix:

The channel capacity for the receiver tracking the channel only.



$$C = \max_{p(x)} I(X;Y,H) = \max_{p(x)} [I(X;H) + I(X;Y \mid H)]$$

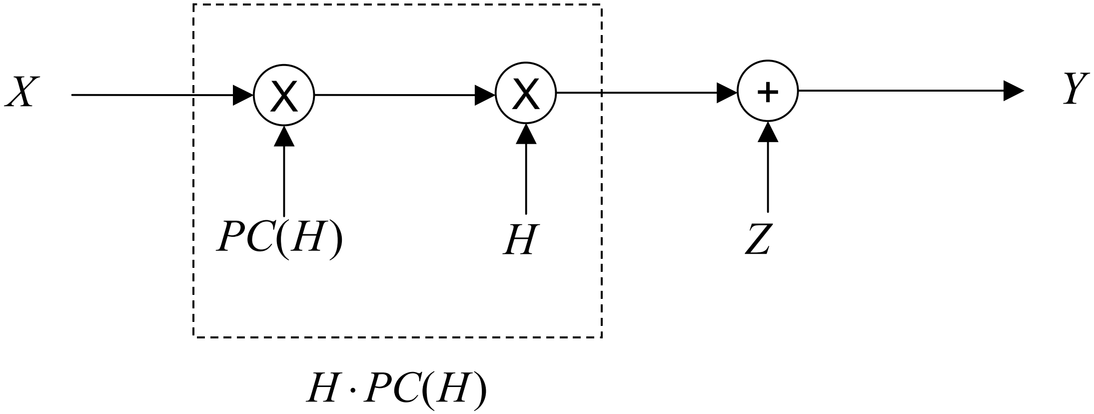
($I(X;H) = 0$ --- the fading variable and the channel input are independent)

$$= \max_{p(x)} E[I(X;Y \mid H = h)]$$

$$\{(R_1,\dots,R_M) : \mathbf{R}(S) \le \mathbf{E}_{\mathbf{H}}[\frac{1}{2}\log(1 + \frac{\sum_{i\in S} H_i P_i}{\sigma^2})], \qquad \forall S \subset \{1,\dots,M\}\}$$

Appendix:

Capacity region under a power-control policy



$$C_f(PC) \equiv \{\mathbf{R} : \mathbf{R}(S) \le \mathbf{E}_H[\frac{1}{2}\log(1+\frac{1}{\sigma^2}\sum_{i\in S} H_i PC_i(\mathbf{H}))], \qquad \forall S \subset \{1,\ldots,M\}\}$$

Appendix:

$B(f) \equiv \{(x_1,\ldots,x_M) : \mathbf{x}(S) \le f(S)\ \forall S \subset E, x_i \ge 0\, \forall i\}$ is a polymatroid

π is a permutation on the set E, define the vector $\mathbf{v}(\pi) \in R^M$ by

$$v_{\pi(1)}(\pi) = f(\pi(1))$$

$$v_{\pi(i)}(\pi) = f(\{\pi(1),\ldots,\pi(i)\}) - f(\{\pi(1),\ldots,\pi(i-1)\}) \qquad i = 2,\ldots,M.$$

Then $\mathbf{v}(\pi)$ is a vertex of $B(f)$ for every permutation π

Conversely, suppose f is a set function and $B(f)$ is the polyhedron defined as above. Then if $\mathbf{v}(\pi) \in B(f)$ for every permutation π, then $B(f)$ is a polymatroid.

Consider the polyhedron $\{\mathbf{R} \in R_+^M : \mathbf{R}(S) \le I[Y;\mathbf{X}(S) \mid \mathbf{X}(S^{\subset})]\forall S \subset E\}$

$$f(S) = I[Y;\mathbf{X}(S) \mid \mathbf{X}(S^{\subset})]$$

π is a permutation on the set E, define the vector $\mathbf{R}(\pi) \in R^M$ by

$$R_{\pi(1)}(\pi) = I[Y;X_{\pi(1)} \mid \mathbf{X}(\{\pi(2),\ldots,\pi(M)\})]$$

$$R_{\pi(i)}(\pi) = I[Y;X_{\pi(i)} \mid \mathbf{X}(\{\pi(i+1),\ldots,\pi(M)\})], \qquad i = 1,\ldots,M-1 \qquad \text{(Chain rule)}$$

$$R_{\pi(M)}(\pi) = I[Y;X_{\pi(M)}]$$

Obviously, $\mathbf{R}(\pi)$ lies in $\mathbf{R}(S)$, so the polyhedron region is a ploymatroid

Appendix:

A Lagrangian multiplier:

Optimization problem: $$\max_{\mathbf{R}} \mathbf{u}\cdot\mathbf{R} \quad \textit{subject to} \quad \mathbf{R}\in C(\overline{\mathbf{P}})$$

$$\Downarrow$$

$$\max_{\mathbf{R}\in C(\mathbf{P})} \mathbf{u}\cdot\mathbf{R} \quad \textit{subject to} \;\; \mathbf{P}=\overline{\mathbf{P}}$$

Lagrangian multipliers $$\Downarrow$$

$$\textit{there exists an } \mathbf{l}, \quad \max_{\mathbf{R}\in C(\mathbf{P})} \mathbf{u}\cdot\mathbf{R}-\mathbf{l}\cdot(\mathbf{P}-\overline{\mathbf{P}})$$

$$\Downarrow$$

$$\textit{there exists an } \mathbf{l}, \quad \max_{\mathbf{R}\in C(\mathbf{P})} \mathbf{u}\cdot\mathbf{R}-\mathbf{l}\cdot\mathbf{P}$$

Appendix:

The classic water-filling solution for the single-user case

For each fading state h, the optimization problem:

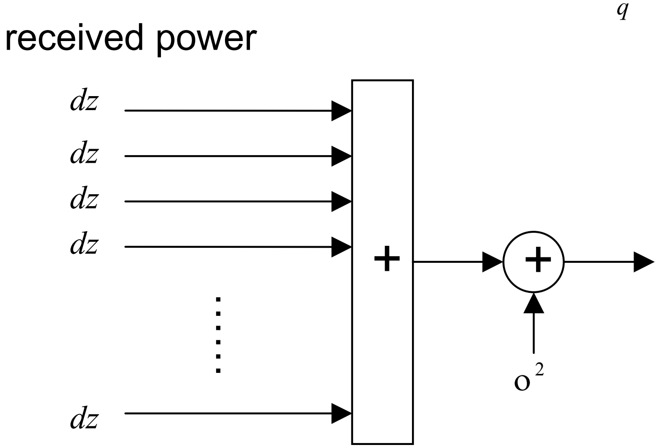
$$\max_{r,q}[r-\frac{l}{h}q] \quad \textit{subject to} \quad r\le\frac{1}{2}\log(1+\frac{q}{\sigma^2})$$

$$\Downarrow$$

$$\max_{q}[\frac{1}{2}\log(1+\frac{q}{\sigma^2})-\frac{l}{h}q]$$

$$\Downarrow \quad \longleftarrow \quad \frac{1}{2}\log(1+\frac{q}{\sigma^2})=\int_0^q\frac{1}{2(\sigma^2+z)}dz$$

$$\max_{q}\int_0^q[\frac{1}{2(\sigma^2+z)}-\frac{l}{h}]dz$$



marginal utility function $u(z)\equiv\frac{1}{2(\sigma^2+z)}-\frac{l}{h}$

Optimization problem turns out to be :

Adding more virtual users until $u(z)\cdot dz<0$