

FAMILIES OF REDUCED ZERO-DIMENSIONAL SCHEMES

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ABSTRACT. A great deal of recent activity has centered on the question of whether, for a given Hilbert function, there can fail to be a unique minimum set of graded Betti numbers, and this is closely related to the question of whether the associated Hilbert scheme is irreducible or not. We give a broad class of Hilbert functions for which we show that there is no minimum, and hence that the associated Hilbert scheme is reducible. Furthermore, we show that the Weak Lefschetz Property holds for the general element of one component, while it fails for every element of another component.

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1. INTRODUCTION

In a fixed projective space \mathbb{P}^n , let Z be a zero-dimensional scheme. We denote by h_Z the Hilbert function of Z :

$$h_Z(t) = \dim(R/I_Z)_t$$

for all t . We also use the analogous notation $h_{R/I_Z}(t)$ or, if A is a graded ring (e.g. Artinian), $h_A(t)$.

Consider the possible zero-dimensional schemes in \mathbb{P}^n having a fixed Hilbert function, H . Recall [10] that the necessary and sufficient condition for the existence of H is that H be a differentiable O-sequence that stabilizes at some value d ; i.e. not only H but also its first difference ΔH has to satisfy Macaulay's growth condition [23] (see the next section for a review), and $\Delta H(t) = 0$ for all sufficiently large t . Conversely, for any such function there always exists a reduced zero-dimensional scheme with Hilbert function H .

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We first review some notation (mostly from [29]). We denote by $\mathbb{Hilb}^H(\mathbb{P}^n)$ the *pos-tulation Hilbert scheme associated to H* , i.e. the Hilbert scheme parameterizing zero-dimensional schemes with Hilbert function H . We define $\mathbb{Hred}^H(\mathbb{P}^n)$ to be the open subscheme of $\mathbb{Hilb}^H(\mathbb{P}^n)$ parameterizing the *reduced* zero-dimensional schemes with Hilbert function H .

If I is the ideal of a zero-dimensional scheme such that R/I has Hilbert function H , suppose that R/I has minimal free resolution

$$(1.1) \quad 0 \rightarrow \bigoplus_j R^{\beta_{n,j}}(-j) \rightarrow \cdots \rightarrow \bigoplus_j R^{\beta_{1,j}}(-j) \rightarrow R \rightarrow R/I \rightarrow 0,$$

with $\beta_{i,j} \in \mathbb{N}$. We denote by β^I the set of graded Betti numbers of R/I (or equivalently, the set of graded Betti numbers of I). We then write

$$\mathbb{B}_H := \{\beta^I : I \subset R \text{ and } h_{R/I} = H\},$$

and

$$\mathbb{B}'_H := \{\beta^I : I \subset R, h_{R/I} = H \text{ and } I \text{ is reduced}\}.$$

\mathbb{B}_H (resp. \mathbb{B}'_H) is a partially ordered set under the inequality \leq , where we write $\beta^I \leq \beta^{I'}$ if $\beta_{i,j} \leq \beta'_{i,j}$ (using the usual inequality for integers) for all i, j .

Two (related) questions may be asked, for a given Hilbert function H . First, is $\mathbb{Hred}^H(\mathbb{P}^n)$ reducible? Second, what can we say about \mathbb{B}_H and \mathbb{B}'_H ; and in particular, is there a unique minimum $\beta^I \in \mathbb{B}_H$ (resp. $\beta^I \in \mathbb{B}'_H$) under the partial ordering \leq ? One can also ask similar questions for Hilbert functions and graded Betti numbers of graded Artinian algebras.

The two questions are related by the following theorem of Ragusa and Zappalà:

Theorem 1.1 ([29]). *If \mathbb{B}_H (resp. \mathbb{B}'_H) has no unique minimum element then $\mathbb{Hilb}^H(\mathbb{P}^n)$ (resp. $\mathbb{Hred}^H(\mathbb{P}^n)$) is reducible.*

The point of this theorem is that the graded Betti numbers obey semicontinuity for ideals in an irreducible family (cf. [29], Lemma 1.2); indeed, it is well known that the number of minimal generators of a finitely generated module over a Noetherian ring obey semicontinuity, and from this fact we can extend to all the graded Betti numbers by standard methods. Here we do not need to restrict to zero-dimensional schemes. Furthermore, within such a family cancelation of Betti numbers can only come in consecutive terms in the resolution (cf. [28]). So we can make the following rephrasing of Theorem 1.1: Let I and I' be ideals of zero-dimensional schemes with the same Hilbert function, H , and assume that β^I and $\beta^{I'}$ are incomparable under the partial ordering. If no ideal J exists with β^J obtainable both from β^I and $\beta^{I'}$ by consecutive cancelation, then I and I' correspond to different components of $\mathbb{Hilb}^H(\mathbb{P}^n)$, which is thus reducible. If I and I' are both reduced then $\mathbb{Hred}^H(\mathbb{P}^n)$ is reducible. We will use this idea without comment in the following sections.

Definition 1.2. We say that the Betti diagrams of I and I' are *strongly incomparable* if the conditions of the previous paragraph are satisfied. \square

It is known that $\mathbb{Hilb}^H(\mathbb{P}^2)$ is smooth and irreducible [13] and that sometimes even $\mathbb{Hilb}^H(\mathbb{P}^n)$ is irreducible [14]. On the other hand, recently there has been a great deal

of activity in this area, giving examples of many different kinds to show that sometimes $\mathbb{H}\text{ilb}^H(\mathbb{P}^n)$ and $\mathbb{H}\text{red}^H(\mathbb{P}^n)$ (or related objects in the Artinian situation) are reducible, or that \mathbb{B}_H has no minimum element (cf. for instance [5], [6], [8], [9], [20], [29], [30], [31], [32]). Some of these examples are isolated, and some (e.g. [5], [30], [31], [33]) are infinite families. Other recent papers studying $\mathbb{H}\text{ilb}^H(\mathbb{P}^n)$ include [1], [4], [18], [19], [21], [22], [25]. While we present a new large class of Hilbert functions for which $\mathbb{H}\text{red}^H(\mathbb{P}^3)$ is reducible, the overlap with any of these previous results is minimal.

To describe the results in this paper, we first recall some notions.

Definition 1.3. An Artinian graded algebra, A , has the *Weak Lefschetz Property (WLP)* if, for a general linear form L , the induced multiplication $\times L : A_i \rightarrow A_{i+1}$ has maximal rank, for all i . A zero-dimensional scheme is said to have WLP if its general Artinian reduction does. A reduced zero-dimensional scheme Z has the *Uniform Position Property (UPP)* if all subsets of the same cardinality have the same Hilbert function.

In the expository paper [26], the author discussed the Weak Lefschetz Property (WLP) and the Uniform Position Property (UPP), describing both as open conditions. He showed in Example 3.4 that if H is the Hilbert function with first difference $(1, 3, 6, 9, 11, 11, 11)$ (corresponding to a set of 52 points in \mathbb{P}^3), then $\mathbb{H}\text{red}^H(\mathbb{P}^3)$ is reducible. Furthermore, in one component no element has WLP, while in another component the general element has WLP (and in fact also UPP).

The purpose of this paper is to put this example into a much more general framework, in the process giving a much larger class of Hilbert functions for which $\mathbb{H}\text{red}^H(\mathbb{P}^3)$ is reducible. In addition, one component has general element with WLP, and another has no element with WLP. We also recall (see above) that the Hilbert function, H , of a zero-dimensional scheme is a differentiable O-sequence. To specify H , it is equivalent to specify its first difference, ΔH . This is also known as the associated *h-vector*.

Our main result is the following (cf. Corollary 4.4):

Theorem *Let H be a Hilbert function in four variables, for which the first difference is of the form*

$$\Delta H = \{1, 3, b_2, b_3, \dots, b_{t-1}, d, \dots, d, b_{s+2}, b_{s+3}, \dots, b_r, 0\}.$$

Here t is simply the first degree for which the value is $= d$, and $s := \lfloor \frac{d-1}{2} \rfloor$. We assume that $(1, 3, b_2, \dots, b_{t-1}, d, \dots, d)$ is again a differentiable O-sequence, and that

$$\begin{cases} b_{s+2} \leq d - 2 & \text{if } d \text{ is even;} \\ b_{s+2} \leq d - 1 & \text{if } d \text{ is odd} \end{cases}$$

and $b_i \geq b_{i+1}$ for all $i \geq s + 2$. Assume furthermore that

$$t \leq s - 1 = \left\lfloor \frac{d - 3}{2} \right\rfloor.$$

Then $\mathbb{H}\text{red}^H(\mathbb{P}^3)$ is reducible. Furthermore, on one component no element has WLP, while on another component the general element has WLP.

To put this in a different context, recall the following result:

Theorem 1.4 ([17], Proposition 3.5). *Let $\underline{h} = (1, b_1, b_2, \dots, b_r)$ be a finite sequence of positive integers. Then \underline{h} is the Hilbert function of a graded Artinian k -algebra R/J having WLP if and only if \underline{h} is a unimodal O -sequence such that the positive part of the first difference is also an O -sequence.*

Notice that the conditions on ΔH in our result above are very similar to this necessary and sufficient condition for WLP. The only differences are the requirement that we have “the right number” of d ’s, and that $b_{s+2} \leq d-2$ (instead of $d-1$) when d is even. However, the condition on the right number of d ’s can be restrictive – see Example 4.6, where we also discuss how one might extend this result. Nevertheless, we have the nice fact that the first difference of the Hilbert function can begin with any differentiable O -sequence, and end with any non-increasing sequence that goes to zero. This degree of freedom is surprising: it is not difficult to construct different-looking sets of points with the same Hilbert function, but to show that they have strongly incomparable Betti diagrams is difficult in general.

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2. PRELIMINARIES

Let $R = k[x_0, \dots, x_n]$, where k is an algebraically closed field. We first recall some basic facts about Hilbert functions, especially those of arithmetically Cohen-Macaulay (ACM) curves in \mathbb{P}^3 .

Definition 2.1. Let $Z \subset \mathbb{P}^n$ be any closed subscheme with defining (saturated) ideal $I = I_Z$. We say that Z is *arithmetically Cohen-Macaulay* (ACM) if the coordinate ring R/I_Z is a Cohen-Macaulay ring. Note that if Z is a zero-dimensional scheme then it is automatically ACM. \square

Definition 2.2. The *i -binomial expansion* of the integer a ($i, a > 0$) is the unique expression

$$a = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j},$$

where $m_i > m_{i-1} > \dots > m_j \geq j \geq 1$. \square

Definition 2.3. If $a \in \mathbb{Z}$ ($a > 0$) has i -binomial expansion as in Definition 2.2, then we set

$$a^{(i)} = \binom{m_i + 1}{i + 1} + \binom{m_{i-1} + 1}{i} + \dots + \binom{m_j + 1}{j + 1}.$$

Note that this defines a collection of functions $^{(i)} : \mathbb{Z} \rightarrow \mathbb{Z}$. \square

For example, the 5-binomial expansion of 76 is

$$76 = \binom{8}{5} + \binom{6}{4} + \binom{4}{3} + \binom{2}{2},$$

so

$$76^{(5)} = \binom{9}{6} + \binom{7}{5} + \binom{5}{4} + \binom{3}{3} = 111.$$

Definition 2.4. A sequence of non-negative integers $\{a_i : i \geq 0\}$ is called an *O-sequence* if

$$a_0 = 1 \quad \text{and} \quad a_{i+1} \leq a_i^{\langle i \rangle},$$

for all i . An O-sequence is said to have *maximal growth from degree i to degree $i + 1$* if $a_{i+1} = a_i^{\langle i \rangle}$. \square

Theorem 2.5 ([23]). *The following are equivalent:*

- (i) $\{a_i : i \geq 0\}$ is an O-sequence;
- (ii) $\{a_i : i \geq 0\}$ is the Hilbert function of a standard graded k -algebra.

Definition 2.6. Given a sequence of non-negative integers $\underline{a} = \{a_i : i \geq 0\}$, the *first difference* of this sequence is the sequence $\Delta \underline{a} := \{b_i\}$ defined by $b_i = a_i - a_{i-1}$ for all i . (We make the convention that $a_{-1} = 0$, so $b_0 = a_0 = 1$.) We say that \underline{a} is a *differentiable O-sequence* if $\Delta \underline{a}$ is again an O-sequence. By taking successive first differences, we inductively define the *k -th difference* $\Delta^k \underline{a}$. \square

Recall that if $H = \{h_i : i \geq 0\}$ is the Hilbert function of an arithmetically Cohen-Macaulay subscheme of \mathbb{P}^n of dimension r (so the coordinate ring R/I has Krull dimension $r + 1$) then the sequences $\Delta^i H$, $1 \leq i \leq r + 1$, are all O-sequences. We say that H is $(r + 1)$ -times differentiable.

In particular, let H be the Hilbert function of a non-degenerate ACM curve, C , of degree d in \mathbb{P}^3 . Then ΔH is the Hilbert function of a proper hyperplane section of C , which is a zero-dimensional scheme of degree d in the hyperplane \mathbb{P}^2 .

In his paper [15], Gotzmann showed a number of very strong connections between the growth of the Hilbert function, the Hilbert polynomial and the regularity. Applying his results, Bigatti, Geramita and Migliore showed the following, which we will use later. Note that by “maximal growth in degree d ” we mean maximal growth from degree d to degree $d + 1$, as defined above. Also, we say that a coherent sheaf \mathcal{F} on \mathbb{P}^n is *s -regular* if $H^q(\mathcal{F}(s - q)) = 0$ for all $q > 0$. The least integer r for which \mathcal{F} is r -regular is called the *regularity* of \mathcal{F} . Finally, by $I_{\leq d}$ we mean the homogeneous ideal generated by the components of I of degrees $\leq d$.

Proposition 2.7 ([3]). *Let I be a saturated ideal in $R = k[x_0, \dots, x_n]$ and suppose that L is a linear form which is not a zero divisor on $A = R/I$. Denote by J the ideal $I/LI \cong (I, L)/(L)$ in the polynomial ring $S = R/L$ and suppose that the Hilbert function of S/J has maximal growth in degree d .*

Let $\bar{I} = \langle I_{\leq d} \rangle$ and let $\bar{\mathcal{I}}$ be the sheafification of \bar{I} . Then $\bar{\mathcal{I}}$ is d -regular.

Another result that we will need is the following. Let V be any reduced subscheme of \mathbb{P}^n (not necessarily ACM) with Hilbert function $H = (1, n + 1, h_2, h_3, \dots)$. Let e be any positive integer, and define a new sequence $\{e_i : i \geq 0\}$ by $e_i = \min\{h_i, e\}$. It is shown in [10] that $\{e_i\}$ is the Hilbert function of some reduced set of points on V . The sequence $\{e_i\}$ is called *the truncation of H at e* .

A useful way of presenting the graded Betti numbers occurring in the minimal free resolution of a graded module is by the so-called *Betti diagram*, introduced in the computer algebra program `macaulay` [2], a program which was used to generate many examples that

contributed to the theorems in this paper. Specifically, for a standard graded algebra R/I we have the diagram

$$\begin{array}{cccccc} 0 & : & 1 & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} \\ 1 & : & - & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} \\ 2 & : & - & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} \\ 3 & : & - & \beta_{1,4} & \beta_{2,5} & \beta_{3,6} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \end{array}$$

Note that the minimal free resolution has redundant, or “ghost” terms if and only if two non-zero entries are adjacent diagonal (with slope 1); for instance, if $\beta_{2,5}$ and $\beta_{3,5}$ are both non-zero, then there is a copy of $R(-5)$ in both the second and third free modules in the resolution.

3. BASIC CONSTRUCTION

For most of the remainder of this paper we will work in the ring $R = k[x_0, x_1, x_2, x_3]$ but we will also need the ring $S = k[x_1, x_2, x_3]$. The main results of this paper are all built on the construction that we will describe in this section. First, there are certain invariants that we will need.

Notation 3.1. Consider a twice differentiable O -sequence, $H = \{h_i : i \geq 0\}$ and assume that $h_1 = 4$ and that ΔH levels off at the value d . So H is the Hilbert function of an ACM curve (see for instance [10]), C , in \mathbb{P}^3 and ΔH is the Hilbert function of its general hyperplane section.

Let $\Delta H = \{b_i : i \geq 0\}$ and let $\Delta^2 H = \{c_i : i \geq 0\}$. Then we define the integer t by

$$\Delta H = (1, 3, b_2, \dots, b_{t-1}, b_t = d, d, \dots)$$

where $b_2 < b_3 < \dots < b_{t-1} < b_t = d$, and we note that

$$\Delta^2 H = (1, 2, c_2, \dots, c_{t-1}, c_t, 0)$$

with $c_t > 0$ and $\sum_{i=0}^t c_i = d$. The integer t has been given many different names in the literature. Most importantly, we have the connection to the regularity of C : $t = \text{reg}(I_C) - 1$, where

$$\text{reg}(I_C) = \min\{s \mid h^i(\mathbb{P}^3, \mathcal{I}_C(s-i)) = 0 \text{ for all } i \geq 1\}$$

is the regularity of the ideal sheaf of C , as defined above. The finite sequence $\Delta^2 H$ is also known as the h -vector of C .

Now we set

$$s := \left\lfloor \frac{d-1}{2} \right\rfloor$$

and we let CI be the Hilbert function of a complete intersection of type $(2, s)$ in four variables. That is,

$$\Delta CI = (1, 3, 5, \dots, 2s-3, 2s-1, 2s) \quad \text{and} \quad \Delta^2 CI = (1, 2, 2, \dots, 2, 2, 1)$$

and note that $\Delta^2 CI$ ends in degree s .

We make the following key assumptions:

$$(3.1) \quad d > 3 \quad \text{and} \quad t \leq s - 1 = \left\lfloor \frac{d-3}{2} \right\rfloor.$$

We will assume that d is even. The case d odd is entirely analogous and will be left to the reader (but we make comments from time to time about this case). Note that if d is even then we have $2s = d - 2$.

Claim 1: $b_2 = 6$.

Suppose otherwise. Recall that H is twice differentiable, i.e. $\Delta^2 H$ is an O-sequence. We have:

- $b_2 \geq 3$ since ΔH is differentiable;
- if $b_2 = 3$ then $d = 3$ (again using differentiability), contradicting our assumption;
- if $b_2 = 4$ then $\Delta^2 H$ is the sequence

$$1, 2, 1, \dots$$

and by Macaulay's growth condition each subsequent entry must be 1 until degree t , so $d = t + 2$, which clearly violates (3.1);

- if $b_2 = 5$ then $\Delta^2 H$ is the sequence

$$1, 2, 2, c_3, \dots,$$

where each c_i is ≤ 2 and $c_{t+1} = 0$. Hence $d \leq 2t + 1$, so combining with (3.1), we get

$$\frac{d-1}{2} \leq t \leq \left\lfloor \frac{d-3}{2} \right\rfloor,$$

a contradiction.

Therefore we have finished Claim 1.

As noted, CI is the Hilbert function of a complete intersection in R of type $(2, \frac{d-2}{2})$, $\Delta^2 CI$ ends in degree $\frac{d-2}{2} = s$, and $2s = d - 2$. Consider the following table, where the last line is obtained by taking the difference $\Delta H - \Delta CI$.

(3.2)	deg	0	1	2	3	4	...	$t-1$	t	$t+1$...	$s-2$	$s-1$	s	$s+1$...
	ΔH	1	3	6	b_3	b_4	...	b_{t-1}	d	d	...	d	d	d	d	...
	ΔCI	1	3	5	7	9	...	$2t-1$	$2t+1$	$2t+3$...	$2s-3$	$2s-1$	$2s$	$2s$...
				1	e_1	e_2	...	e_{t-3}	e_{t-2}	e_{t-1}	...	5	3	2	2	...

Since $t \leq s - 1$, we verify that the sequence $\{e_i : i \geq 0\}$ ends with the subsequence $(\dots, 3, 2, 2, 2, \dots)$, although the columns between t and $s - 1$ could be redundant. (When d is odd, the sequence $\{e_i : i \geq 0\}$ ends with $(\dots, 2, 1, 1, 1, \dots)$.)

Claim 2: $\{e_i : i \geq 0\}$ is an O-sequence.

Note that $e_1 \leq 3$. We want to show that for any i , the growth from e_i to e_{i+1} does not violate Macaulay's theorem.

Case 1: $i \geq t - 2$.

Then $i + 2 \geq t$ so $b_{i+2} = d$ and it follows that $e_i \geq e_{i+1}$ (and in fact $e_i > e_{i+1}$ for $t - 2 \leq i \leq s - 3$). Hence e_i is an O-sequence in this range.

Case 2: $i \leq t - 3$.

Note that under this assumption we have

$$(3.3) \quad b_{i+2} = e_i + 2(i + 2) + 1 = e_i + 2i + 5$$

and

$$(3.4) \quad b_{i+2} = \binom{i+4}{2} \text{ if and only if } e_i = \binom{i+2}{2}.$$

Now note that

$$(3.5) \quad b_{i+3} - b_{i+2} = e_{i+1} - e_i + 2.$$

What is the i -binomial expansion of e_i ? We have

$$e_i = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

for some $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. From (3.4) it is not hard to see that $e_i \leq \binom{i+2}{2}$, with equality as given there, and this represents the i -binomial expansion of e_i in that case since $\binom{i+2}{2} = \binom{i+2}{i}$. So without loss of generality we may assume that $e_i < \binom{i+2}{2}$. Again using the fact that $\binom{a}{b} = \binom{a}{a-b}$ for $a > b$, and taking into account $e_i < \binom{i+2}{2}$, we see that without loss of generality we may write

$$\begin{aligned} e_i &= \left[\binom{i+1}{1} + \binom{i}{1} + \cdots + \binom{k}{1} \right] + \left[\binom{k-1}{0} + \cdots + \binom{k-l}{0} \right] \\ &= [(i+1) + i + \cdots + k] + l \end{aligned}$$

for some k, l . Then from (3.3) we get

$$b_{i+2} = [(i+3) + (i+2) + (i+1) + i + \cdots + k] + l.$$

Similarly we obtain

$$e_{i+1} = [(i+2) + \cdots + k'] + l'$$

$$b_{i+3} = [(i+4) + (i+3) + (i+2) + \cdots + k'] + l'.$$

Now, we know that ΔH is an O-sequence, so the growth from b_{i+2} to b_{i+3} satisfies Macaulay's bound. That is,

$$\begin{aligned} (i+4) + (i+3) + \cdots + k' + l' &\leq b_{i+2}^{\langle i+2 \rangle} \\ &= [(i+4) + (i+3) + \cdots + (k+1)] + l. \end{aligned}$$

Then

$$\begin{aligned} e_{i+1} &= b_{i+3} - (2i+7) \\ &\leq b_{i+2}^{\langle i+2 \rangle} - (2i+7) \\ &= e_i^{\langle i \rangle} \end{aligned}$$

proving that $\{e_i : i \geq 0\}$ is an O-sequence as claimed. This complete the proof of Claim 2.

Now we have that all three sequences in (3.2) are infinite O-sequences. Let J be the lex-segment ideal in S with Hilbert function $\{e_i : i \geq 0\}$. Note that since J is lex-segment and $e_i = 2$ for all i sufficiently large, we have that $J_i = \langle x_1, x_2 \rangle_i$ for all i sufficiently large.

Let Y be the reduced subscheme of \mathbb{P}^3 obtained by "lifting" the monomial ideal J (cf. for instance [27]). Y is reduced, the reduction modulo x_0 of I_Y is J , and the top

dimensional part, \bar{Y} , of Y is a reduced curve of degree 2. Note that the Betti diagram for I_Y is the same as the Betti diagram for J . The first difference of the Hilbert function of Y is precisely $\{e_i : i \geq 0\}$.

The next claim is not needed for the rest of the proof, but is an interesting observation.

Claim 3: \bar{Y} is ACM, and in fact it is a plane curve consisting of two lines meeting in a point.

(Note that when d is odd, \bar{Y} is just a line and there is nothing to prove here. But when d is even, \bar{Y} could in principle be two skew lines. We have to prove that regardless of the sequence $\{e_i : i \geq 0\}$, the curve \bar{Y} obtained in this way consists of two lines meeting in a point.)

The reduction of $I_{\bar{Y}}$ modulo x_0 is the ideal of a zero-dimensional scheme of degree 2 (not necessarily saturated, a priori, if \bar{Y} is not ACM), and this reduction contains J . But in large degree J agrees with the ideal $\langle x_1, x_2 \rangle_i$, which is the saturated ideal of a zero-dimensional scheme of degree 2. Therefore the reduction of $I_{\bar{Y}}$ modulo x_0 agrees with $\langle x_1, x_2 \rangle$ in all sufficiently large degrees, and hence its saturation is $\langle x_1, x_2 \rangle$. But this ideal is supported at a point, so the curve \bar{Y} meets the hyperplane defined by x_0 in one point (up to multiplicity). Hence \bar{Y} is the union of two lines meeting at this point, and we have proved Claim 3.

We now choose polynomials $F \in (I_Y)_s$ and $Q \in R$ such that Q is a quadric and (F, Q) is a regular sequence. Let V be the complete intersection defined by (F, Q) . Note that $h_V = CI$ and that $\deg V = d - 2$. We consider the ideal

$$I = Q \cdot I_Y + (F).$$

The ideal I is formed by Basic Double Linkage [12]. (Note, however, that since Y is not unmixed, it does not participate in any linkage, so strictly speaking this is a misnomer. However, this is the standard name for this construction even in this more general setting.) The following can easily be deduced using [12].

- (1) I is a saturated ideal defining a curve, X , of degree d .
- (2) The Hilbert function of X is H . (This comes directly from (3.2), using the methods of [12].)
- (3) X is reduced, and as sets, we have $X = Y \cup V$. Since Q can be chosen freely, V avoids any 0-dimensional components of Y .
- (4) F can be chosen to be a union of planes, as can Q , in which case X is a union of lines and reduced points.
- (5) There is an exact sequence

$$0 \rightarrow R(-s-2) \rightarrow I_Y(-2) \oplus R(-s) \rightarrow I \rightarrow 0.$$

Now we can say something about the Betti diagram of I . Suppose that I_Y has minimal free resolution

$$0 \rightarrow \mathbb{F}_3 \rightarrow \mathbb{F}_2 \rightarrow \mathbb{F}_1 \rightarrow I_Y \rightarrow 0.$$

Notice that because of the values of $\{e_i : i \geq 0\}$ in degrees $\geq s-3$ (recall the shift in (3.2)), we obtain from [7] that S/J (the reduction of R/I_Y by x_0) has a non-zero socle element

Again, the case d odd is entirely analogous, and is left to the reader. (This time we use a complete intersection of type $(2, \frac{d-1}{2})$.)

4. MAIN RESULTS

In this section we make minor modifications on the construction of the last section in order to get our results.

Theorem 4.1. *Let $H = \{h_i : i \geq 0\}$ be a twice differentiable O -sequence over R , with invariants d, t, s as in Notation 3.1; in particular, we continue to assume that*

$$t \leq s - 1 = \left\lfloor \frac{d - 3}{2} \right\rfloor.$$

Let \bar{H} be the truncation at the value h_{s+1} , so \bar{H} is the Hilbert function of some set of h_{s+1} points in \mathbb{P}^3 . Then $\mathbb{B}'_{\bar{H}}$ does not have a unique smallest element, and the postulation scheme $\mathbb{H}\text{red}^{\bar{H}}(\mathbb{P}^3)$ is reducible. Furthermore, on one component, \mathcal{H}_1 , the general point corresponds to a set of points with WLP, while on another component, \mathcal{H}_2 , no point corresponds to a set of points with WLP.

Proof. We continue to assume that d is even, leaving the case d odd for the reader (but occasionally remarking on the case of d odd). Notice that the first difference of \bar{H} is

$$\Delta \bar{H} = (1, 3, 6, b_3, \dots, b_{t-1}, d, \dots, d, 0)$$

where the last d occurs in degree $s+1$.

Consider the sequence $\{e'_i : i \geq 0\}$ defined by

$$e'_i = \begin{cases} e_i & \text{if } i \leq s - 1 \\ 0 & \text{if } i \geq s. \end{cases}$$

Note that $\{e'_i\}$ ends with the subsequence $(\dots, 3, 2, 2, 0)$. Let J' be the lexsegment ideal in S with Hilbert function $\{e'_i : i \geq 0\}$. Note that J' has two minimal generators in degree s . Let Y' be the reduced (but now zero-dimensional) subscheme of \mathbb{P}^3 obtained by “lifting” the monomial ideal J' . Note that J' agrees with the ideal J described above in degrees $\leq s - 1$, so the same is true of $I_{Y'}$ and I_Y , respectively. The first difference of the Hilbert function of Y' is $\{e'_i : i \geq 0\}$. Since the ideal I_Y has no generator in degree s , $I_{Y'}$ must have exactly two minimal generators in degree s . (When d is odd, $I_{Y'}$ has one minimal generator in degree s .)

Now we choose $F \in (I_{Y'})_s$ and $Q \in R_2$ as above, but we choose F to be a minimal generator of $I_{Y'}$. We form the ideal $I' = QI_{Y'} + (F)$. This is again a saturated ideal, defining a reduced subscheme, X' , of \mathbb{P}^3 which consists of the disjoint union of the curve defined by (F, Q) (which has degree $d - 2$) and Y' . Let H' be the Hilbert function of X' . We make the following Hilbert function calculation using (3.2):

(4.1)	deg	0	1	2	3	4	...	$t-1$	t	$t+1$...	$s-1$	s	$s+1$	$s+2$	$s+3$...
	e_{i-2}			1	e_1	e_2	...	e_{t-3}	e_{t-2}	e_{t-1}	...	3	2	2	0	0	
	ΔCI	1	3	5	7	9	...	$2t-1$	$2t+1$	$2t+3$...	$2s-1$	$2s$	$2s$	$2s$	$2s$...
	$\Delta H'$	1	3	6	b_3	b_4	...	b_{t-1}	d	d	...	d	d	d	$d-2$	$d-2$...

Note that $I' = I_{X'}$ agrees with I_X in degrees $\leq s + 1$, and its Hilbert function is H up to and including degree $s + 1$. Note also that $I_{X'}$ has only one minimal generator in degree $s + 2$ (thanks to Remark 3.2), rather than the two that one might guess by looking at (4.1).

Thanks to [10], there exists a set of points, Z' , on X' whose Hilbert function is the truncation of $h_{X'}$ at level h_{s+1} , i.e. whose Hilbert function is \bar{H} . The table (4.1) shows that $I_{Z'}$ has $d - 2$ more generators in degree $s + 2$ than does $I_{X'}$, so in fact $I_{Z'}$ has $d - 1$ minimal generators in degree $s + 2$.

Thanks to the Betti diagram (3.6) and Remark 3.2, we have the following Betti diagram for $R/I_{Z'}$:

$$(4.2) \quad \begin{array}{cccccc} 0 & : & 1 & * & * & * \\ 1 & : & - & * & * & * \\ & & \vdots & & & \\ s-2 & : & - & * & * & * \\ s-1 & : & - & * & * & 1 \\ s & : & - & - & - & - \\ s+1 & : & - & d-1 & * & * \end{array}$$

Now we let C be an ACM curve with Hilbert function H . Since $\text{reg}(I_C) = t + 1$, R/I_C has Betti diagram

$$(4.3) \quad \begin{array}{cccccc} 0 & : & 1 & * & * & - \\ 1 & : & - & * & * & - \\ & & \vdots & & & \\ t-1 & : & - & * & * & - \\ t & : & - & * & * & - \\ t+1 & : & - & - & - & - \end{array}$$

Let Z be a finite subset of C whose Hilbert function is the truncation of H at level h_{s+1} , i.e. $h_Z = \bar{H}$; note that as before I_Z agrees with I_C in degrees $\leq s + 1$. Since $t \leq s - 1$, the Betti diagram of R/I_Z is

$$(4.4) \quad \begin{array}{cccccc} 0 & : & 1 & * & * & - \\ 1 & : & - & * & * & - \\ & & \vdots & & & \\ t-1 & : & - & * & * & - \\ t & : & - & * & * & - \\ t+1 & : & - & - & - & - \\ & & \vdots & & & \\ s-1 & : & - & - & - & - \\ s & : & - & - & - & - \\ s+1 & : & - & d & * & * \end{array}$$

Comparing the Betti diagrams (4.2) and (4.4), one sees that $\beta'_{3,s+2} > \beta_{3,s+2}$ and $\beta_{1,s+2} > \beta'_{1,s+2}$. In fact it is clear that the diagrams are strongly incomparable (in the sense of Definition 1.2), since there can be no element smaller than both. The non-existence

of a unique smallest element and the reducibility of the postulation scheme then follow immediately. Z corresponds to a point in \mathcal{H}_1 and Z' corresponds to a point in \mathcal{H}_2 .

The assertion about WLP comes from the observation (from the Betti diagram) that the Artinian reduction of $R/I_{Z'}$ has a socle element in degree $s - 1$. But the value of the Hilbert function of this Artinian reduction in both degrees $s - 1$ and s is d , so the map induced by a general linear form can be neither injective nor surjective (since it has a one-dimensional kernel). Hence this Artinian reduction does not have WLP. On the other hand, the Artinian reduction of R/I_Z does have WLP – it follows from the Cohen-Macaulay property of R/I_C and the fact that R/I_C agrees with R/I_Z in all degrees $\leq s+1$.

Again, the case d odd is almost identical and is left to the reader. \square

Corollary 4.2. *Under the assumptions of Theorem 4.1, if H is of decreasing type (i.e. $\Delta^2 H$ is strictly decreasing once it starts to decrease, so H is the Hilbert function of an irreducible ACM curve in \mathbb{P}^3) then the general element of \mathcal{H}_1 also satisfies UPP.*

Proof. The fact that having $\Delta^2 H$ be of decreasing type implies that H is the Hilbert function of an irreducible ACM curve in \mathbb{P}^3 is due to several authors ([16], [34], [24]), and we refer to [11] for a history and summary of this problem. It is clear from the proof of Theorem 4.1. We simply choose C to be an irreducible ACM curve and Z to be a general set of points on C of the right cardinality. We do not know if the general element of \mathcal{H}_2 has UPP, although we believe that it does not. \square

Example 4.3. Let

$$\Delta H = (1, 3, 6, 10, 14, 16, 17, 17, \dots).$$

Then $t = 6$, $d = 17$ and $s = 8$, so Theorem 4.1 considers the Hilbert function

$$\Delta \bar{H} = (1, 3, 6, 10, 14, 16, 17, 17, 17, 0).$$

That is,

$$\bar{H} = (1, 4, 10, 20, 34, 50, 67, 84, 101, 118, 118, \dots).$$

The corresponding postulation scheme $\text{Hilb}^{\bar{H}}(\mathbb{P}^3)$, parameterizing sets of 118 points in \mathbb{P}^3 with Hilbert function \bar{H} , is reducible. \square

Corollary 4.4. *Let $H = \{h_i : i \geq 0\}$ be a twice differentiable O -sequence, with invariants d, t, s as in Notation 3.1; in particular, we assume that*

$$t \leq s - 1 = \left\lfloor \frac{d - 3}{2} \right\rfloor.$$

Write $\Delta H = \{1, 3, b_2, b_3, \dots, b_{t-1}, d, d, \dots\}$. Let \bar{H} be the Hilbert function whose first difference is

$$\Delta \bar{H} = \{1, 3, b_2, b_3, \dots, b_{t-1}, d, \dots, d, b_{s+2}, b_{s+3}, \dots, b_r, 0\}$$

where

$$\begin{cases} b_{s+2} \leq d - 2 & \text{if } d \text{ is even} \\ b_{s+2} \leq d - 1 & \text{if } d \text{ is odd} \end{cases}$$

and $b_i \geq b_{i+1}$ for all $i \geq s + 2$. Then all the conclusions of Theorem 4.1 continue to hold for \bar{H} .

Proof. As before, we prove the case where d is even, leaving to the reader the case of d odd. Our approach here is similar to that taken in [17], Proposition 3.5.

We return to the monomial ideal J' and the reduced subscheme X' obtained at the beginning of the proof of Theorem 4.1. By the mechanism of liftings of monomial ideals (cf. [27]), we may assume that $F \in (I_{Y'})_s$ is a union of planes. Clearly Q can also be chosen to be a (general) union of planes. Hence X' is the disjoint union of $2s = d - 2$ lines (defined by the complete intersection of F and Q) and a finite set of points (Y') whose regularity is s . We saw that the Hilbert function of X' has first difference

$$(4.5) \quad (1, 3, 6, b_3, \dots, b_{t-1}, d, \dots, d, d-2, d-2, \dots)$$

with the last d occurring in degree $s + 1$.

Note that the union, A , of $2s = d - 2$ lines is a complete intersection of type $(2, s)$; in particular it is ACM. Its Hilbert function has first difference

$$(1, 3, 5, \dots, d-5, d-3, d-2, d-2, \dots)$$

where the first $d - 2$ occurs in degree s . It is easy to check (e.g. by considering liftings) that we may order the lines $\lambda_1, \lambda_2, \dots, \lambda_{d-2}$ such that for each i , $A_i := \lambda_1 \cup \dots \cup \lambda_i$ is ACM. Notice that the Hilbert function of A_i has first difference which reaches value i in degree $< s$, except for A_{d-2} , which reaches value $d - 2$ in degree exactly s .

For any i , the base locus of the linear system $|(I_{X'})_i|$ includes all the lines λ_j . Furthermore, in degrees $\geq s + 2$, the reduction of $R/I_{X'}$ modulo x_0 agrees with the reduction of $R/(F, Q)$ modulo x_0 .

We will proceed inductively. Consider the Hilbert function

$$1, 3, b_2, \dots, b_{t-1}, d, \dots, d, b_{s+2}, b_{s+2}, \dots$$

This can be obtained by the ideal

$$I_{X_{s+2}} := I_{Z'} \cap I_{\lambda_1} \cap \dots \cap I_{\lambda_{b_{s+2}}}$$

i.e. by taking the ideal of $X_{s+2} := Z' \cup A_{b_{s+2}}$, where Z' is the reduced zero-dimensional scheme obtained in Theorem 4.1. Indeed, since $\lambda_1, \dots, \lambda_{b_{s+2}}$ are contained in the base locus of $I_{Z'}$ in degrees $\leq s + 1$, they impose no additional conditions in those degrees. And in degrees $\geq s + 2$ the reduction modulo x_0 sees only the b_{s+2} lines. We choose a finite set of points Z_{s+2} whose Hilbert function is the truncation, i.e. whose Hilbert function has first difference

$$(4.6) \quad 1, 3, b_2, \dots, b_{t-1}, d, \dots, d, b_{s+2}, 0.$$

For the next step we proceed as we have done here. We want to produce a finite set of points with Hilbert function having first difference

$$(4.7) \quad 1, 3, b_2, \dots, b_{t-1}, d, \dots, d, b_{s+2}, b_{s+3}, 0.$$

To achieve this, we note that we have a finite set of points Z_{s+2} with the indicated Hilbert function (4.6), and we consider $X_{s+3} := Z_{s+2} \cup A_{b_{s+3}}$; that is, we have

$$Z_{s+3} = Z_{s+2} \cup \lambda_1 \cup \dots \cup \lambda_{b_{s+3}}.$$

Arguing as before, we have that X_{s+2} has Hilbert function with first difference

$$1, 3, b_2, \dots, b_{t-1}, d, \dots, d, b_{s+2}, b_{s+3}, b_{s+3}, \dots$$

(repeating b_{s+3}). Again we choose a finite set of points, Z_{s+3} whose Hilbert function is the truncation given by (4.7).

After a finite number of steps, we are finished. Let us denote by W' this reduced zero-dimensional scheme with the desired Hilbert function.

For the scheme W (analogous to Z in Theorem 4.1) having the same Hilbert function as W' but strongly incomparable Betti diagram, we first note that a curve C can be constructed with Hilbert function having first difference $\{1, 3, b_2, \dots, b_{t-1}, d, d, \dots\}$ and such that C is ACM, and as before, it is a union of lines $\lambda_1 \cup \lambda_2 \cup \dots \cup \lambda_d$ such that $\lambda_1 \cup \dots \cup \lambda_i$ is ACM for all i . We then proceed exactly as we did above, considering truncations and slowly removing lines, to produce a reduced zero-dimensional scheme with the desired Hilbert function.

Now we have to check that these zero-dimensional schemes W and W' have strongly incomparable Betti diagrams. Note that the Betti diagram for R/I_C is the same as that in (4.3), by construction.

Recall that in Theorem 4.1, we produced zero-dimensional schemes Z and Z' with Hilbert function having first difference

$$1, 3, b_2, \dots, b_{t-1}, d, \dots, d, 0$$

where the last d occurs in degree $s + 1$. We saw that the Betti diagram for $R/I_{Z'}$ was

$$\begin{array}{rcccc} 0 & : & 1 & * & * & * \\ 1 & : & - & * & * & * \\ & & \vdots & & & \\ s-2 & : & - & * & * & * \\ s-1 & : & - & * & * & 1 \\ s & : & - & - & - & - \\ s+1 & : & - & d-1 & * & * \end{array}$$

while the Betti diagram for R/I_Z was

$$\begin{array}{rcccc} 0 & : & 1 & * & * & - \\ 1 & : & - & * & * & - \\ & & \vdots & & & \\ t-1 & : & - & * & * & - \\ t & : & - & * & * & - \\ t+1 & : & - & - & - & - \\ & & \vdots & & & \\ s & : & - & - & - & - \\ s+1 & : & - & d & * & * \end{array}$$

Now instead we have constructed reduced zero-dimensional schemes W and W' with Hilbert function having first difference

$$1, 3, b_2, \dots, b_{t-1}, d, \dots, d, b_{s+2}, \dots$$

In particular, $I_{Z'}$ (resp. I_Z) agrees with $I_{W'}$ (resp. I_W) in all degrees $\leq s + 1$. It follows that the Betti diagram for $R/I_{Z'}$ (resp. R/I_Z) agrees with the Betti diagram for $R/I_{W'}$ (resp. R/I_W) in all rows above the one labelled s . The fact that the Hilbert function

Here $t = 10$ and $s = 14$. Note that the bottom row is indeed an O-sequence. Theorem 4.1 then draws a conclusion about the Hilbert function with first difference

$$(1, 3, 6, 10, 15, 19, 23, 26, 27, 28, 29, 29, 29, 29, 0).$$

But if we tried to reduce the number of 29's by choosing, for instance, a complete intersection of type (4, 7), we obtain the diagram

deg	0	1	2	3	4	5	6	7	8	9	10	11	...
ΔH	1	3	6	10	15	19	23	26	27	28	29	29	...
ΔCI	1	3	6	10	14	18	22	25	27	28	28	28	...
					1	1	1	1	0	1	1	1	...

and the bottom line is not an O-sequence.

However, in (4.8) if we instead use a complete intersection of type (3, 9) then we obtain

deg	0	1	2	3	4	5	6	7	8	9	10	11	12	...
ΔH	1	3	6	10	15	19	23	26	27	28	29	29	29	...
ΔCI	1	3	6	9	12	15	18	21	24	26	27	27	27	...
				1	3	4	5	5	3	2	2	2	2	...

One might think that the method of this paper would allow us to start with the O-sequence (1, 3, 4, 5, 5, 3, 2, 2, 0) and proceed as before to obtain a result about the Hilbert function with first difference (1, 3, 6, 10, 15, 19, 23, 26, 27, 28, 29, 0). But recall that the complete intersection needs to include a minimal generator for $I_{Y'}$, and the O-sequence (1, 3, 4, 5, 5, 3, 2, 2, 0) does not allow a minimal generator of degree 9. So in fact we have to start with the O-sequence (1, 3, 4, 5, 5, 3, 2, 2, 2, 0) and then we can indeed draw a conclusion about the Hilbert function with first difference (1, 3, 6, 10, 15, 19, 23, 26, 27, 28, 29, 29, 0). This is not covered by Theorem 4.1.

Another obstruction to extending this technique is that if CI is the Hilbert function of a complete intersection of type (a, b) , then in order for the sequence $\{e_i\}$ to begin with a 1 we need $a < b$, and also we need that a be strictly smaller than the initial degree of H (or some small variant which we need not make explicit here). If we impose this condition, and also the condition that $a + b \geq t + 3$ and that ab is smaller than d but “not too much smaller” (so that we have maximal growth of the first difference of the Hilbert function) then very likely the technique could be extended, as illustrated above. But the gain would only be to shorten the string of d 's in the middle of the h -vector. Given the complicated nature of these numerical conditions (we have not even addressed what we would need to ensure that $\{e_i : i \geq 0\}$ is an O-sequence), and the fact that already our result gives a very large new class of Hilbert functions for which the postulation scheme is reducible, it does not seem worthwhile to pursue this. \square

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