Math 40510, Algebraic Geometry

Problem Set 3, due April 25, 2018

1. For which varieties V in $\mathbb{C}^n$ is $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$ a field? [This should only take a few lines, but give a full explanation. In particular, be sure to mention which results and/or assumptions you are using.]

2. The following problems are in analogy with things we did in class about the rational normal curve.

   a) Let k be an infinite field. Let V be the variety in k^5 defined by
   $$\begin{aligned} x_1 &= a \\ x_2 &= a^2 \\ x_3 &= ab \\ x_4 &= b \\ x_5 &= b^2. \end{aligned}$$
   for any $a, b \in k$. That is, V is the image of the map $\phi : k^2 \to k^5$ defined by
   $$\phi((a, b)) = (a, a^2, ab, b, b^2).$$
   (For example, if $P = (2, 3) \in k^2$ then $\phi(P) = (2, 4, 6, 3, 9)$.) Prove that V is irreducible. [I would like a complete and careful proof, not a one-line proof quoting a result in the book. However, feel free to use without proof the fact that V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal in $k[x_1, \ldots, x_5]$.]

   b) Is ϕ injective? If so, prove it. If not, explain why not.

   c) Is ϕ surjective? If so, prove it. If not, explain why not.

   d) There are six linearly independent minimal generators for $\mathbb{I}(V)$, all of degree 2. Find them. You don't have to prove that they are linearly independent (i.e. that no non-trivial scalar linear combination is equal to zero), but if you give one that either has degree different from 2 or is linearly dependent on the others, you won't get credit for it.

3. Consider the ideal $\langle x^2 + 1 \rangle$. For this problem, remember what you know about $\mathbb{F}[x]$, where $\mathbb{F}$ is a field.

   a) Prove that $\langle x^2 + 1 \rangle$ is a maximal ideal in $\mathbb{R}[x]$.

   b) Prove that $\langle x^2 + 1 \rangle$ is not a maximal ideal in $\mathbb{C}[x]$.

   c) Find three other fields $\mathbb{F}$ where $\langle x^2 + 1 \rangle$ is not a maximal ideal in $\mathbb{F}[x]$ and explain your answer.

4. If $I \subset \mathbb{R}[x_1, \ldots, x_n]$ is a maximal ideal, show that either $\mathbb{V}(I) \subset \mathbb{R}^n$ is empty or $\mathbb{V}(I)$ is a single point in $\mathbb{R}^n$.

5. In class we said that if f is homogeneous in $k[x_0, x_1, \ldots, x_n]$ then $\mathbb{V}(f)$ is well-defined in $\mathbb{P}^n_k$. (You can use this fact in this problem.) We didn't talk much about the converse.

   In all parts of this problem we let $f_1, f_2 \in k[x_0, \ldots, x_n]$ be homogeneous, not necessarily of the same degree. Let $V = \mathbb{V}(f_1, f_2)$. Let $f = f_1 + f_2$. Assume that k is an infinite field.

   a) (Still no assumption on the degrees of f_1 and f_2.) Prove that the vanishing of f at any point $P = [a_0, a_1, \ldots, a_n]$ of V is well-defined. Remember that $P = [ta_0, ta_1, \ldots, ta_n]$ for any $t \in k$, $t \neq 0$, so this is asking you to show that if $P \in V$ then $f(ta_0, ta_1, \ldots, ta_n) = 0$ for all $t \in k$, and no matter whether f_1 and f_2 have the same degree or not.

   b) Assume that $\deg(f_1) = \deg(f_2)$. Give an example to show that there may be points of $\mathbb{P}^n_k$ not in V where $f(P) = 0$ is well-defined. [Editor's note: let's rephrase that. Show that there may exist points $P \in \mathbb{P}^n_k$ such that $P \notin V$ but nevertheless it is well-defined to say that $f(P) = 0$, where $f = f_1 + f_2$. Your example should tell me what f_1 and f_2 are, what V is, and give at least one P that does the trick.]

   c) Now assume that $\deg(f_1) \neq \deg(f_2)$. Prove that if $P \notin V$ then f does not vanish at P. In other words, what I'm asking you to show is that there is some value of t for which $f(ta_0, ta_1, \ldots, ta_n) \neq 0$. [Hint: we know that the vanishing of f at P *is* well-defined if f_1 and f_2 have the same degree, so somewhere you should use the fact that they have different degrees.]

6. One of the really cool things about projective space is the notion of **duality**. Let's limit ourselves to $\mathbb{P}^2_{\mathbb{R}}$, the real projective plane. (We will understand that we are working over $\mathbb{R}$ and not bother writing the subscript $\mathbb{R}$ each time.)

   Recall that a line ℓ in $\mathbb{P}^2$ is the vanishing locus of a homogeneous linear polynomial, i.e. $\ell = \mathbb{V}(ax + by + cz)$ for some choice of $a, b, c \in \mathbb{R}$ not all zero.

   a) Show that $ax + by + cz = 0$ defines the same line as $3x + 4y + 5z = 0$ if and only if there exists some $t \in \mathbb{R}$ such that $a = 3t$, $b = 4t$ and $c = 5t$. (Of course 3, 4, 5 is just an example.) [Hint: $\Leftarrow$ is almost immediate. For $\Rightarrow$, you can use the fact that in $\mathbb{P}^2$, either two lines meet at a single point or they are the same line. It may help to take the linear algebra point of view.]

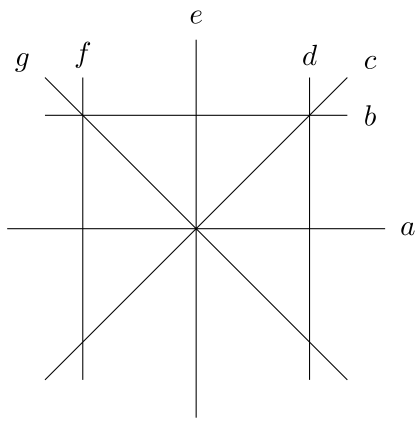
   b) Based on a), show that the **set** of lines in $\mathbb{P}^2$ itself can be viewed as a projective plane, which we will denote by $(\mathbb{P}^2)^\vee$.

   $(\mathbb{P}^2)^\vee$ is called the **dual projective plane**. So what we have so far is that a point $P = [a, b, c]$ in $(\mathbb{P}^2)^\vee$ corresponds to the line $\ell_P = \mathbb{V}(ax + by + cz)$ in $\mathbb{P}^2$. You can use this for the next two parts even if you didn't get a) and/or b). Furthermore, even if you don't get c) you can use the statement of c) to do d) and e).

   c) Let P_1, P_2, P_3 be points of $(\mathbb{P}^2)^\vee$ and let $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ be the lines in $\mathbb{P}^2$ that they correspond to. Show that P_1, P_2, P_3 all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ all pass through a common point. [Hint: if you look at the equation $ax + by + cz = 0$, you can think of a, b, c as given and x, y, z as the variables, OR you can think of x, y, z as given and a, b, c as the variables!]

   d) Using c), if you take a **line** in $(\mathbb{P}^2)^\vee$, what does the collection of all the points on this line correspond to back in $\mathbb{P}^2$? Explain your answer carefully.

e) The following is a set of lines in $\mathbb{P}^2_{\mathbb{R}}$, labelled a to g.



Sketch the set of points in $(\mathbb{P}^2)^\vee$ dual to these lines, and label them A to G **corresponding to the similarly named lines**. Make sure that your sketch reflects when three or more of the points are on a line. [Hint: in addition to the obvious places where three or more lines meet, the three vertical lines meet at infinity!! Part c) is crucial in this problem.]