Math 40510, Algebraic Geometry

Problem Set 2, due March 21, 2018

1. Let k be a field. You can use facts from CLO Chapter 4, §3 for this problem.

   a) Let $f, g, h \in k[x_1, \ldots, x_n]$. Prove that $\mathbb{V}(f, gh) = \mathbb{V}(f, g) \cup \mathbb{V}(f, h)$.

   b) Let $V = \mathbb{V}(f_1, \ldots, f_s)$ and $W = \mathbb{V}(g_1, \ldots, g_t)$ be algebraic varieties in k^n and let $h \in k[x_1, \ldots, x_n]$. Prove that
   $$(V \cup W) \cap \mathbb{V}(h) = \mathbb{V}(f_1, \ldots, f_s, h) \cup \mathbb{V}(g_1, \ldots, g_t, h).$$

   c) Now let $k = \mathbb{R}$. Find $\mathbb{V}(xy, xz, yz)$ in $\mathbb{R}^3$. (I.e. give a precise description of what this variety is from a geometric perspective.) [Hint: this was an example in class, but we didn't prove it. So this question is really asking for you to come up with the proof.]

   d) Let h be a polynomial in x, y, z of degree 1 (so $\mathbb{V}(h)$ is a plane in $\mathbb{R}^3$ – you can use this fact without further comment). Using geometric reasoning, what are all the possibilities for $\mathbb{V}(xy, xz, yz, h)$? For each of these possibilities, give a specific h that achieves that outcome.

   [For example, one possibility is that the plane contains two lines of $\mathbb{V}(xy, xz, yz)$, e.g. the y-axis and the z-axis. All I want from you is that one possibility for $\mathbb{V}(xy, xz, yz, h)$ is the union of two axes, coming *for example* when $h = x$. I don't want you to *also* give me $h = y$ and $h = z$ and I don't want you to do any algebraic manipulations with the ideal. This is mostly a geometric question.]

2. In this problem we will work over the field of real numbers, $\mathbb{R}$.

   a) Let $I = \langle f_1, \ldots, f_s \rangle$ be any ideal in $\mathbb{R}[x_1, \ldots, x_n]$. Let $V = \mathbb{V}(I) \subset \mathbb{R}^n$ be the corresponding variety. Find a single polynomial f such that $V = \mathbb{V}(f)$. Prove your answer.

   b) Let $I = \langle f_1, \ldots, f_s \rangle$ be any ideal in $\mathbb{R}[x_1, \ldots, x_n]$. Suppose that $\mathbb{V}(I) = \emptyset$. Show that there is at least one element of I that has no zero in $\mathbb{R}^n$. Justify your answer. (Notice that $\mathbb{R}$ is not algebraically closed, so you can't use the Nullstellensatz.)

3. Let V and W be varieties in $\mathbb{C}^n$ such that $V \cap W = \emptyset$. Prove that there exist $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f + g = 1$.

4. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Let $\sqrt{I}$ be its radical. Show that there is a positive integer p such that for every $f \in \sqrt{I}$, $f^p \in I$. (The thing to stress is that the choice of p does not depend on what f you choose. p depends only on what $\sqrt{I}$ is.) [Hint: $\sqrt{I}$ is an ideal in a Noetherian ring. You can also review our proof in class that $\sqrt{I}$ is an ideal.]

5. Let I and J be ideals in $\mathbb{C}[x_1, \ldots, x_n]$ such that
   $$I + J = \langle 1 \rangle = \mathbb{C}[x_1, \ldots, x_n].$$

   a) Prove that the varieties $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint.

   b) Prove that $IJ = I \cap J$. [Don't forget the assumption at the beginning of this problem!!!]

6. Let X be a topological space (*not* necessarily with the Zariski topology). Let A be a subset of X with the following property:

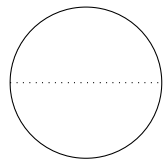
   For each $P \notin A$ there exists a closed set V_P that contains A but does not contain P.

Prove that A must be closed, making sure to justify each step.

7. Find the Zariski closure for each of the following sets in $\mathbb{R}^2$, and explain your answer. Some of them may already be closed. Your explanations do not have to be rigorous proofs, but they should be convincing!

   a) The unit circle.

   b) $A \cup B$, where A is the unit circle and B is the set of points in $\mathbb{R}^2$ of the form $(x, 0)$ where $-1 \leq x \leq 1$ is a rational number.



   c) The sine curve in $\mathbb{R}^2$, i.e. $\{(x, y) \mid y = \sin x\}$.

8. Let k be a field and let $R = k[x_1, \ldots, x_n]$. Let I, J and K be ideals in R.

   a) If I is radical, prove that $I : J$ must also be radical.

   b) Give an example to show that if $I : J$ is radical, it is not necessarily true that I is radical.

   c) If V and W are varieties in k^n, prove that $\mathbb{I}(V) : \mathbb{I}(W)$ is a radical ideal. [Hint: "c" comes after "a" in the alphabet.]

   d) Prove that $J \subseteq I$ if and only if $I : J = R$.

   e) If $J \subseteq K$, prove that $I : K \subseteq I : J$.

   f) Assume that I is radical. Prove that $I : \sqrt{J} = I : J$.