# Math 40510, Algebraic Geometry

## Problem Set 1, due February 14, 2017

1. In this problem we explore polynomial rings.

   a) In class we defined the ring $k[x_1, \ldots, x_n]$ of polynomials in $n$ variables with coefficients in a field, $k$. We can similarly define $\mathbb{Z}_6[x_1, \ldots, x_n]$ to be the ring of polynomials in $n$ variables with coefficients in $\mathbb{Z}_6$. Prove by example that $\mathbb{Z}_6[x_1, \ldots, x_n]$ is not an integral domain.

   b) Now let $k$ be a field. Prove that if $f, g \in k[x_1, \ldots, x_n]$ then $\deg(fg) = \deg(f) + \deg(g)$.

   c) Prove that there are $\binom{d+2}{2}$ monomials of degree $d$ in the variables $x, y, z$. [Your proof should be from scratch, not by using a special case of some formula you find somewhere.]

2. In this problem we look at varieties in $\mathbb{R}^n$. (Part c) is only for $\mathbb{R}^2$.)

   a) Prove that a single point in $\mathbb{R}^n$ is an affine variety.

   b) Prove that the union of any finite number of points in $\mathbb{R}^n$ is an affine variety. [Hint: Use Lemma 2 of §2 of the book, and extend it to a finite union of varieties using induction.]

   c) In the next problem you'll show that a certain infinite union of points is not an affine variety. On the other hand, give an example of an infinite set of points in $\mathbb{R}^2$ whose union *is* an affine variety. Justify your answer.

3. Let
$$X = \{(m, m^3 + 1) \in \mathbb{R}^2 \mid m \in \mathbb{Z}\}.$$
   In this problem you'll show that $X$ is *not* an affine variety.

   a) Consider the following statement:

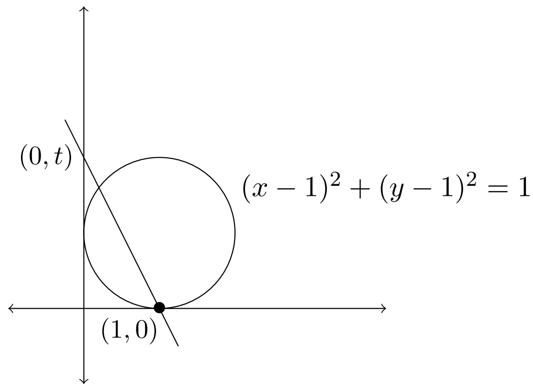
   *If $f(x,y)$ is a polynomial that vanishes at each point of $X$ then $f$ vanishes on the whole curve $x^3 - y + 1 = 0$.* $\quad (*)$

   Explain why proving $(*)$ will guarantee that $X$ is not an affine variety.

   b) Prove $(*)$.

4. In class we showed how to obtain the parametrization for the circle $x^2 + y^2 = 1$. Use the exact same idea (but slightly different algebra) to obtain the parametrization
$$x = \frac{(t-1)^2}{1+t^2}$$
$$y = \frac{2t^2}{1+t^2}.$$
   for the circle $(x-1)^2 + (y-1)^2 = 1$. Specifically:

   a) Verify that for any value of $t$ in this parametrization, we have $(x-1)^2 + (y-1)^2 = 1$.
   b) Derive the above parametrization. Show all your work. The following picture should help.




c) In particular, which point of the circle is missed by this parametrization?

5. Let $V$ be the parabola in $\mathbb{R}^2$ given by the equation $y = x^2$. Let $P = (a, a^2)$ be a point of $V$. (I don't mean that you should choose a specific value of $a$.)
   a) Find a polynomial $f$ so that $V = \mathbb{V}(f)$. [Hint: this is as easy as it looks. Don't look for anything tricky here.]
   b) Find a polynomial $\ell$ so that $\mathbb{V}(\ell)$ is the tangent line to $V$ at $P$.
   c) Prove directly that $\langle \ell, f\rangle$ is not a *radical* ideal. That is, find a polynomial $g$ such that some power of $g$ is in $\langle \ell, f\rangle$ but $g$ itself is not. Be sure to show all your work: prove that some power of $g$ is in this ideal (what power?), and prove that $g$ itself is not in the ideal. [Hint: look at vertical lines for one possible answer.]
   d) If $I = \langle \ell, f\rangle$, find $\mathbb{V}(I)$ and find $\mathbb{I}(\mathbb{V}(I))$. [Note that you can do this part even if you did not get part c). However, I would like you to justify your answer. No full credit if you find the right ideal but don't give a proof.]

6. In class we mentioned that if $k$ is a field then $k[x_1, \ldots, x_{n-1}][x_n] \cong k[x_1, \ldots, x_n]$. Give a proof of this fact. In particular, you should
   a) find a function $\phi : k[x_1, \ldots, x_{n-1}][x_n] \to k[x_1, \ldots, x_n]$ [Hint: don't try to do anything too fancy. For example, $(3x + y)z + (4xy + 5y^3)z^2$ is both an element of $k[x, y][z]$ and of $k[x, y, z]$];
   b) show that $\phi$ is a ring homomorphism,
   c) show that $\phi$ is injective,
   d) and show that $\phi$ is surjective.

   (Your proof of this whole problem should take very few lines. Just convince me that you understand what's going on.)

7. Consider the infinite family of polynomials $f_1, f_2, f_3, \ldots$ with
$$f_i = 3x^i + 5y^{i+7} - (i^2 + 3)x^{i-2}y \in \mathbb{R}[x, y] \quad \text{(where } i = 1, 2, 3, \ldots).$$
   Prove that there is some integer $N$ so that every $f_j$ with $j > N$ can be written as a linear combination of $f_1, f_2, \ldots, f_N$. [Hint: the form of the $f_i$ is a red herring. Also, I do *not* want to know specifically what $N$ is.]