Math 40510, Algebraic Geometry

Problem Set 3 Solutions, due April 25, 2018

1. For which varieties V in $\mathbb{C}^n$ is $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$ a field? [This should only take a few lines, but give a full explanation. In particular, be sure to mention which results and/or assumptions you are using.]

   Solution:
   $\mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$ is a field if and only if $\mathbb{I}(V)$ is a maximal ideal. We have seen that when k is algebraically closed, the maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ are exactly the ideals of single points,
   $$\mathfrak{m}_P = \langle x_1 - a_1, \ldots, x_n - a_n \rangle,$$
   where $P = (a_1, \ldots, a_n) \in \mathbb{C}^n$. So $\mathbb{C}[V]$ is a field if and only if V is a single point.

2. The following problems are in analogy with things we did in class about the rational normal curve.

   a) Let k be an infinite field. Let V be the variety in k^5 defined by
   $$\begin{aligned} x_1 &= a \\ x_2 &= a^2 \\ x_3 &= ab \\ x_4 &= b \\ x_5 &= b^2. \end{aligned}$$
   for any $a, b \in k$. That is, V is the image of the map $\phi : k^2 \to k^5$ defined by
   $$\phi((a, b)) = (a, a^2, ab, b, b^2).$$
   (For example, if $P = (2, 3) \in k^2$ then $\phi(P) = (2, 4, 6, 3, 9)$.) Prove that V is irreducible. [I would like a complete and careful proof, not a one-line proof quoting a result in the book. However, feel free to use without proof the fact that V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal in $k[x_1, \ldots, x_5]$.]

   Solution:
   We know that V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal, so we will prove that $\mathbb{I}(V)$ is prime. Suppose $f, g \in k[x_1, \ldots, x_5]$ such that $fg \in \mathbb{I}(V)$. We want to show that either $f \in \mathbb{I}(V)$ or $g \in \mathbb{I}(V)$.

   The fact that $fg \in \mathbb{I}(V)$ means that
   $$f(a, a^2, ab, b, b^2) \cdot g(a, a^2, ab, b, b^2) = 0 \quad \text{for all } a, b \in k.$$
   Thus the polynomial
   $$f(s, s^2, st, t, t^2) \cdot g(s, s^2, st, t, t^2) \in k[s, t]$$
   vanishes at every point of k^2. But this means that the polynomial $f(s, s^2, st, t, t^2) \cdot g(s, s^2, st, t, t^2)$ is the zero polynomial, since k is infinite. Since $k[s, t]$ is an integral domain, either $f(s, s^2, st, t, t^2) = 0$ (as polynomials) or $g(s, s^2, st, t, t^2) = 0$. Hence either
   $$f(a, a^2, ab, b, b^2) = 0 \text{ for all } (a, b) \in k^2 \quad \text{or} \quad g(a, a^2, ab, b, b^2) = 0 \text{ for all } (a, b) \in k^2.$$
   Since V consists of the set of all points (a, a^2, ab, b, b^2) for $(a, b) \in k^2$, this means that either $f(P) = 0$ for all $P \in V$ or else $g(P) = 0$ for all $P \in V$, i.e. either $f \in \mathbb{I}(V)$ or $g \in \mathbb{I}(V)$ as desired.

b) Is ϕ injective? If so, prove it. If not, explain why not.

Solution:
Yes. Suppose $(a, a^2, ab, b, b^2) = (c, c^2, cd, d, d^2)$ for some (a, b) and (c, d) in k^2. By comparing the first and fourth coordinates of each point we get $a = c$ and $b = d$) so $(a, b) = (c, d)$ and ϕ is injective.

c) Is ϕ surjective? If so, prove it. If not, explain why not.

Solution:
No. For example, $(1, 2, 3, 4, 5)$ can't be in the image since $2 \neq 1^2$ (among other reasons).

d) There are six linearly independent minimal generators for $\mathbb{I}(V)$, all of degree 2. Find them. You don't have to prove that they are linearly independent (i.e. that no non-trivial scalar linear combination is equal to zero), but if you give one that either has degree different from 2 or is linearly dependent on the others, you won't get credit for it.

Solution:

$$\begin{array}{ll}
x_1x_4 - x_3 & \text{since } (a)(b) - ab = 0 \text{ for all } a, b \in k; \\
x_1^2 - x_2 & \text{since } (a)^2 - a^2 = 0 \text{ for all } a, b \in k; \\
x_4^2 - x_5 & \text{since } (b)^2 - b^2 = 0 \text{ for all } a, b \in k; \\
x_1x_3 - x_2x_4 & \text{since } (a)(ab) - (a^2)(b) = 0 \text{ for all } a, b \in k; \\
x_3^2 - x_2x_5 & \text{since } (ab)^2 - (a^2)(b^2) = 0 \text{ for all } a, b \in k; \\
x_3x_4 - x_1x_5 & \text{since } (ab)(b) - (a)(b^2) = 0 \text{ for all } a, b \in k.
\end{array}$$

3. Consider the ideal $\langle x^2 + 1 \rangle$. For this problem, remember what you know about $\mathbb{F}[x]$, where $\mathbb{F}$ is a field.

We will use the following facts in this problem. Since $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a principal ideal domain. Furthermore, again since $\mathbb{F}$ is a field, a polynomial f of degree two factors (into a product of linear polynomials) if and only if it has a root. Finally, if $f, g \in \mathbb{F}[x]$ both have degree ≥ 1 and $\deg(g) < \deg(f)$ then $f = gh$ for some $h \in \mathbb{F}[x]$ if and only if $\langle f \rangle \subsetneq \langle g \rangle$.

a) Prove that $\langle x^2 + 1 \rangle$ is a maximal ideal in $\mathbb{R}[x]$.

Solution:
Since $x^2 + 1$ has no root in $\mathbb{R}$, it does not factor into a product of linear polynomials. Suppose that $\langle x^2 + 1 \rangle$ were not maximal, so there is some ideal I with $\langle x^2 + 1 \rangle \subsetneq I \subsetneq \mathbb{R}[x]$. Since $\mathbb{R}[x]$ is a principal ideal domain, $I = \langle g \rangle$ for some $g \in \mathbb{R}[x]$. Since $I \subsetneq \mathbb{R}[x]$, g must have degree > 0. Since $\langle x^2 + 1 \rangle \subset \langle g \rangle$, g divides f. But f has degree two and does not have any linear factors, so it does not have any factors at all (of positive degree). This contradicts the fact that g divides f. Thus I does not exist and $\langle x^2 + 1 \rangle$ is a maximal ideal.

b) Prove that $\langle x^2 + 1 \rangle$ is not a maximal ideal in $\mathbb{C}[x]$.

Solution:
$x^2 + 1 = (x + i)(x - i)$ in $\mathbb{C}[x]$ so $\langle x^2 + 1 \rangle \subsetneq \langle x + i \rangle$ and $\langle x^2 + 1 \rangle$ is not maximal.

c) Find three other fields $\mathbb{F}$ where $\langle x^2 + 1 \rangle$ is not a maximal ideal in $\mathbb{F}[x]$ and explain your answer.

Solution:
From what we said above, we just want three fields where $x^2 + 1$ factors.

- In $\mathbb{Z}_2$, $x^2 + 1 = (x + 1)(x + 1)$ so $\langle x^2 + 1 \rangle \subsetneq \langle x + 1 \rangle$.

- In $\mathbb{Z}_5$, $x^2+1=(x+2)(x+3)$ so $\langle x^2+1\rangle \subsetneq \langle x+2\rangle$.
- In $\mathbb{Z}_{17}$, $x^2+1=(x+4)(x+13)$ so $\langle x^2+1\rangle \subsetneq \langle x+4\rangle$.

4. If $I \subset \mathbb{R}[x_1,\ldots,x_n]$ is a maximal ideal, show that either $\mathbb{V}(I) \subset \mathbb{R}^n$ is empty or $\mathbb{V}(I)$ is a single point in $\mathbb{R}^n$.

Solution:
We've seen that $\mathbb{V}(I)$ can be empty in $\mathbb{R}^n$ (e.g. when $n=1$, take $I=\langle x^2+1\rangle$) and $\mathbb{V}(I)$ can be a single point (e.g. when $n=2$, take $I=\langle x,y\rangle$, which we showed in class is maximal).

All that's left is to show that $\mathbb{V}(I)$ can't contain more than one point. Indeed, suppose $P,Q\in\mathbb{V}(I)$ with $P\neq Q$. Then

$$\mathbb{R}[x_1,\ldots,x_n] \supsetneq \mathbb{I}(P) \supsetneq \mathbb{I}(P\cup Q) \supseteq \mathbb{I}(\mathbb{V}(I)) \supseteq I$$

so I is not a maximal ideal (since $\mathbb{I}(P)$ is properly between I and $\mathbb{R}[x_1,\ldots,x_n]$).

5. In class we said that if f is homogeneous in $k[x_0,x_1,\ldots,x_n]$ then $\mathbb{V}(f)$ is well-defined in $\mathbb{P}^n_k$. (You can use this fact in this problem.) We didn't talk much about the converse.

In all parts of this problem we let $f_1,f_2\in k[x_0,\ldots,x_n]$ be homogeneous, not necessarily of the same degree. Let $V=\mathbb{V}(f_1,f_2)$. Let $f=f_1+f_2$. Assume that k is an infinite field.

a) (Still no assumption on the degrees of f_1 and f_2.) Prove that the vanishing of f at any point $P=[a_0,a_1,\ldots,a_n]$ of V is well-defined. Remember that $P=[ta_0,ta_1,\ldots,ta_n]$ for any $t\in k$, $t\neq 0$, so this is asking you to show that if $P\in V$ then $f(ta_0,ta_1,\ldots,ta_n)=0$ for all $t\in k$, and no matter whether f_1 and f_2 have the same degree or not.

Solution:
We are assuming that $P\in V$, so $f_1(P)=0$ and $f_2(P)=0$ are well-defined. That is, $f_1(ta_0,ta_1,\ldots,ta_n)=0$ for all $t\in k$, and $f_2(ta_0,ta_1,\ldots,ta_n)=0$ for all $t\in k$. Then

$$\begin{aligned} f(P) &= f(ta_0,ta_1,\ldots,ta_n) \\ &= f_1(ta_0,ta_1,\ldots,ta_n)+f_2(ta_0,ta_1,\ldots,ta_n) \\ &= 0+0 \\ &= 0 \end{aligned}$$

for all $t\in k$.

b) Assume that $\deg(f_1)=\deg(f_2)$. Give an example to show that there may be points of $\mathbb{P}^n_k$ not in V where $f(P)=0$ is well-defined.

Solution:
Let's take $n=2$. Let $f_1=x$ and $f_2=y$. Let $P=[1,-1,1]\in\mathbb{P}^2_k$. Then neither f_1 nor f_2 vanish at P, so $P\notin V$, but clearly f_1+f_2 does vanish at P.

c) Now assume that $\deg(f_1)\neq\deg(f_2)$. Prove that if $P\notin V$ then f does not vanish at P. In other words, what I'm asking you to show is that there is some value of t for which $f(ta_0,ta_1,\ldots,ta_n)\neq 0$. [Hint: we know that the vanishing of f at P *is* well-defined if f_1 and f_2 have the same degree, so somewhere you should use the fact that they have different degrees.]

Solution:
Assume $\deg(f_1) = d_1 < d_2 = \deg(f_2)$. We have assumed that $P \notin V$, which means that it is not true that both $f_1(ta_0, ta_1, \ldots, ta_n) = 0$ for all $t \in k$ and also $f_2(ta_0, ta_1, \ldots, ta_n) = 0$ for all $t \in k$. Since both f_1 and f_2 are homogeneous, this means that we can assume that either $f_1(a_0, \ldots, a_n)$ or $f_2(a_0, \ldots, a_n)$ is a non-zero scalar. (Notice that we are not allowing the $(n+1)$-tuple to be multiplied by scalars here – we are fixing $a_0, \ldots, a_n$.)
Then

$$\begin{aligned} f(ta_0, ta_1, \ldots, ta_n) &= f_1(ta_0, ta_1, \ldots, ta_n) + f_2(ta_0, ta_1, \ldots, ta_n) \\ &= t^{d_1} f_1(a_0, a_1, \ldots, a_n) + t^{d_2} f_2(a_0, a_1, \ldots, a_n) \\ &= t^{d_1}[f_1(a_0, \ldots, a_n) + t^{d_2 - d_1} f_2(a_0, a_1, \ldots, a_n)] \end{aligned}$$

Since $d_1 < d_2$, the exponent of t in front of f_2 is non-zero. We have seen that $f_1(a_0, \ldots, a_n)$ and $f_2(a_0, \ldots, a_n)$ are just scalars, possibly zero but not both zero.

If $f_2(a_0, \ldots, a_n) = 0$ then we are assuming that $f_1(a_0, \ldots, a_n) \neq 0$ so just take $t = 1$.

If $f_1(a_0, \ldots, a_n) = 0$ then $f_2(a_0, \ldots, a_n) \neq 0$ so again we can take $t = 1$.

If neither $f_1(a_0, \ldots, a_n) = 0$ nor $f_2(a_0, \ldots, a_n) = 0$ then because k is infinite, the polynomial

$$[f_1(a_0, \ldots, a_n) + t^{d_2 - d_1} f_2(a_0, a_1, \ldots, a_n)]$$

cannot be zero for all $t \in k$, so choose any t for which this is non-zero.

6. One of the really cool things about projective space is the notion of **duality**. Let's limit ourselves to $\mathbb{P}^2_{\mathbb{R}}$, the real projective plane. (We will understand that we are working over $\mathbb{R}$ and not bother writing the subscript $\mathbb{R}$ each time.)

   Recall that a line ℓ in $\mathbb{P}^2$ is the vanishing locus of a homogeneous linear polynomial, i.e. $\ell = \mathbb{V}(ax + by + cz)$ for some choice of $a, b, c \in \mathbb{R}$ not all zero.

   a) Show that $ax + by + cz = 0$ defines the same line as $3x + 4y + 5z = 0$ if and only if there exists some $t \in \mathbb{R}$ such that $a = 3t$, $b = 4t$ and $c = 5t$. (Of course 3, 4, 5 is just an example.) [Hint: $\Leftarrow$ is almost immediate. For $\Rightarrow$, you can use the fact that in $\mathbb{P}^2$, either two lines meet at a single point or they are the same line. It may help to take the linear algebra point of view.]

   Solution:
   $\Leftarrow$:
   If we know in advance that $a = 3t$, $b = 4t$ and $c = 5t$ then

   $$ax + by + cz = 0 \quad \Leftrightarrow \quad (3t)x + (4t)y + (5t)z = 0 \quad \Leftrightarrow \quad 3x + 4y + 5z = 0$$

   so they define the same line.

   $\Rightarrow$:
   Consider the lines $\mathbb{V}(ax + by + cz)$ and $\mathbb{V}(3x + 4y + 5z)$ in $\mathbb{P}^2$. Either they meet in a single point or they are the same line. To find out which, we solve a system of homogeneous linear equations

   $$\begin{array}{ccccccc} 3x & + & 4y & + & 5z & = & 0 \\ ax & + & by & + & cz & = & 0. \end{array}$$

Each equation represents a plane through the origin in $\mathbb{R}^3$. The lines in $\mathbb{P}^2$ meet in a single point if and only if the solution space of these two equations is a 1-dimensional subspace of $\mathbb{R}^3$ (i.e. a line through the origin in $\mathbb{R}^3$, i.e. a point of $\mathbb{P}^2$). Looking at the coefficient matrix

$$\begin{bmatrix} 3 & 4 & 5 \\ a & b & c \end{bmatrix}$$

we know that the solution space is 1-dimensional if and only if the rank of this matrix is 2, if and only if neither row is a multiple of the other. So the lines are the same in $\mathbb{P}^2$ if and only if the solution space is 2-dimensional, if and only if $a = 3t$, $b = 4t$ and $c = 5t$ for some non-zero t as claimed.

b) Based on a), show that the **set** of lines in $\mathbb{P}^2$ itself can be viewed as a projective plane, which we will denote by $(\mathbb{P}^2)^\vee$.

Solution:

$$\{ \text{ Lines in } \mathbb{P}^2 \ \} = \{ \ \mathbb{V}(ax + by + cz) \ \} = \{ \ [a, b, c] \ \}$$

where the latter is the set of triples of real numbers, not all zero, up to scalar multiples, i.e. the latter is a projective plane.

$(\mathbb{P}^2)^\vee$ is called the **dual projective plane**. So what we have so far is that a point $P = [a, b, c]$ in $(\mathbb{P}^2)^\vee$ corresponds to the line $\ell_P = \mathbb{V}(ax + by + cz)$ in $\mathbb{P}^2$. You can use this for the next two parts even if you didn't get a) and/or b). Furthermore, even if you don't get c) you can use the statement of c) to do d) and e).

c) Let P_1, P_2, P_3 be points of $(\mathbb{P}^2)^\vee$ and let $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ be the lines in $\mathbb{P}^2$ that they correspond to. Show that P_1, P_2, P_3 all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ all pass through a common point. [Hint: if you look at the equation $ax + by + cz = 0$, you can think of a, b, c as given and x, y, z as the variables, OR you can think of x, y, z as given and a, b, c as the variables!]

Solution:
Say $P_i = [a_i, b_i, c_i]$ for $i = 1, 2, 3$. Then the P_i all lie on a line in $(\mathbb{P}^2)^\vee$ if and only if there are some **constants** $p, q, r \in \mathbb{R}$ such that $[a_1, b_1, c_1], [a_2, b_2, c_2]$ and $[a_3, b_3, c_3]$ are all solutions to the equation

$$pa + qb + rc = 0$$

in the variables a, b, c. That is, we have

$$\begin{array}{ccccccc} a_1p & + & b_1q & + & c_1r & = & 0 \\ a_2p & + & b_2q & + & c_2r & = & 0 \\ a_3p & + & b_3q & + & c_3r & = & 0 \end{array}$$

But this means that $[p, q, r]$ is a common solution of the equations

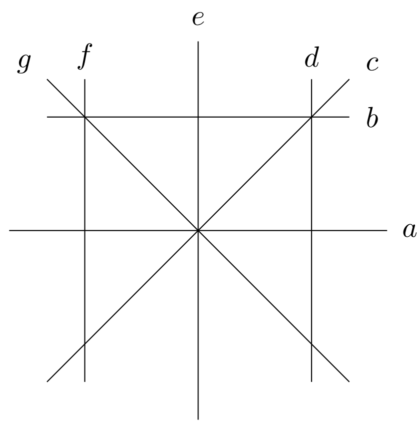
$$\begin{array}{ccccccc} a_1x & + & b_1y & + & c_1z & = & 0 \\ a_2x & + & b_2y & + & c_2z & = & 0 \\ a_3x & + & b_3y & + & c_3z & = & 0 \end{array}$$

i.e. $[p, q, r]$ is common to the lines $\mathbb{V}(a_1x + b_1y + c_1z), \mathbb{V}(a_2x + b_2y + c_2z), \mathbb{V}(a_3x + b_3y + c_3z)$, i.e. to the lines $\ell_{P_1}, \ell_{P_2}, \ell_{P_3}$ as desired.

d) Using c), if you take a **line** in $(\mathbb{P}^2)^\vee$, what does the collection of all the points on this line correspond to back in $\mathbb{P}^2$? Explain your answer carefully.

Solution:
The points on this line are all on the same line (obviously), so the corresponding lines in $\mathbb{P}^2$ all pass through the same common point, by c). This collection of lines through a common point is called a **pencil** of lines.

e) The following is a set of lines in $\mathbb{P}^2_{\mathbb{R}}$, labelled a to g.



Sketch the set of points in $(\mathbb{P}^2)^\vee$ dual to these lines, and label them A to G **corresponding to the similarly named lines**. Make sure that your sketch reflects when three or more of the points are on a line. [Hint: in addition to the obvious places where three or more lines meet, the three vertical lines meet at infinity!! Part c) is crucial in this problem.]

Solution:
We have to make sure that A, C, E, G are collinear, B, C, D are collinear, B, F, G are collinear and D, E, F are collinear. Here is one possible sketch. The blue lines are just to emphasize which points are collinear.