## Math 40510, Algebraic Geometry

Problem Set 2 Solutions, due March 21, 2018

1. Let $k$ be a field. You can use facts from CLO Chapter $4, \S 3$ for this problem.
a) Let $f, g, h \in k\left[x_{1}, \ldots, x_{n}\right]$. Prove that $\mathbb{V}(f, g h)=\mathbb{V}(f, g) \cup \mathbb{V}(f, h)$.

## Solution:

Let's prove the two inclusions.
Let $P \in \mathbb{V}(f, g h)$. So $f(P)=0$ and $(g h)(P)=0$. But the latter means

$$
0=(g h)(P)=g(P) h(P) .
$$

Since $k$ is a field, this means $f(P)=0$ and either $g(P)=0$ or $h(P)=0$. So $P \in \mathbb{V}(f, g) \cup \mathbb{V}(f, h)$.
Now let $P \in \mathbb{V}(f, g) \cup \mathbb{V}(f, h)$. So either $P \in \mathbb{V}(f, g)$ or $P \in \mathbb{V}(f, h)$ (or both). So in both cases we have $f(P)=0$, and either $g(P)=0$ or $h(P)=0$. This latter means in any case $(g h)(P)=0$. Thus $P \in \mathbb{V}(f, g h)$.
b) Let $V=\mathbb{V}\left(f_{1}, \ldots, f_{s}\right)$ and $W=\mathbb{V}\left(g_{1}, \ldots, g_{t}\right)$ be algebraic varieties in $k^{n}$ and let $h \in k\left[x_{1}, \ldots, x_{n}\right]$. Prove that

$$
(V \cup W) \cap \mathbb{V}(h)=\mathbb{V}\left(f_{1}, \ldots, f_{s}, h\right) \cup \mathbb{V}\left(g_{1}, \ldots, g_{t}, h\right) .
$$

## Solution:

The expression on the left is equal to $(V \cap \mathbb{V}(h)) \cup(W \cap \mathbb{V}(h))$. But we know

$$
V \cap \mathbb{V}(h)=\mathbb{V}\left(f_{1}, \ldots, f_{s}, h\right) \quad \text { and } \quad W \cap \mathbb{V}(h)=\mathbb{V}\left(g_{1}, \ldots, g_{t}, h\right)
$$

so we are done.
c) Now let $k=\mathbb{R}$. Find $\mathbb{V}(x y, x z, y z)$ in $\mathbb{R}^{3}$. (I.e. give a precise description of what this variety is from a geometric perspective.) [Hint: this was an example in class, but we didn't prove it. So this question is really asking for you to come up with the proof.]

## Solution:

We'll show that $\mathbb{V}(x y, x z, y z)$ is the union of the three coordinate axes in $\mathbb{R}^{3}$. Note that these axes are

$$
\begin{aligned}
& x \text {-axis }=\mathbb{V}(y, z) \\
& y \text {-axis }=\mathbb{V}(x, z) \\
& z \text {-axis }=\mathbb{V}(x, y)
\end{aligned}
$$

We'll prove the two inclusions. First let $P=(a, b, c) \in \mathbb{V}(x y, x z, y z)$. Since $x y$ vanishes at $P$, either $a=0$ or $b=0$.

- If $a=0$ then $x z$ automatically vanishes at $P$ too, but the fact that $y z$ vanishes at $P$ means that in addition either $b=0$ or $c=0$. So $P$ is of the form $(0,0, t)$ or $(0, t, 0)$, i.e. $P$ either lies on the $z$-axis or the $y$-axis.
- If $b=0$ then $y z$ automatically vanishes at $P$ too, but the fact that $x z$ vanishes at $P$ means that in addition either $a=0$ or $c=0$. So $P$ is of the form $(0,0, t)$ or $(t, 0,0)$, i.e. $P$ either lies on the $z$-axis or the $x$-axis.

So $P$ has to be on one of the three axes. We conclude that $\mathbb{V}(x y, x z, y z)$ is contained in the union of the three axes.

Now let $P$ be a point on one of the three axes.

- If $P$ lies on the $x$-axis then $P$ is of the form $(t, 0,0)$ so all three of $x y, x z, y z$ vanish at $P$.
- If $P$ lies on the $y$-axis then $P$ is of the form $(0, t, 0)$ so all three of $x y, x z, y z$ vanish at $P$.
- If $P$ lies on the $z$-axis then $P$ is of the form $(0,0, t)$ so all three of $x y, x z, y z$ vanish at $P$.

Thus $P \in \mathbb{V}(x y, x z, y z)$.
d) Let $h$ be a polynomial in $x, y, z$ of degree 1 (so $\mathbb{V}(h)$ is a plane in $\mathbb{R}^{3}$ - you can use this fact without further comment). Using geometric reasoning, what are all the possibilities for $\mathbb{V}(x y, x z, y z, h)$ ? For each of these possibilities, give a specific $h$ that achieves that outcome.
[For example, one possibility is that the plane contains two lines of $\mathbb{V}(x y, x z, y z)$, e.g. the $y$-axis and the $z$-axis. All I want from you is that one possibility for $\mathbb{V}(x y, x z, y z, h)$ is the union of two axes, coming for example when $h=x$. I don't want you to also give me $h=y$ and $h=z$ and I don't want you to do any algebraic manipulations with the ideal. This is mostly a geometric question.]

## Solution:

How can a plane intersect the three axes?

- A plane can meet the axes in three points (in fact "most" planes intersect the axes in three points), so for any such we have $\mathbb{V}(x y, x z, y z, h)$ consists of 3 points. An example is $h=x+y+z-1$.
- A plane can meet the axes in two points if it is parallel to one of the axes but not two. An example is $h=x+y+1$.
- A plane can meet the axes in one point. This could happen if it is parallel to one of the coordinate planes (e.g. $h=x+1$ ) or if it passes through the origin, not containing any of the axes. For example, $h=x+y+z$.
- If a plane contains only one axis, it also has to contain the origin. If it contained any other point on one of the other axes, it would thus contain two points (origin and one other) from that axis so it would contain the whole axis. So if it contains only one axis, $\mathbb{V}(x y, x z, y z, h)$ is the axis and nothing else. An example is $h=2 y+3 z$, which contains the $x$-axis.
- If a plane contains two axes, by reasoning as before we have $\mathbb{V}(x y, x z, y z, h)$ is the union of those two axes. An example is $h=x$, which contains the $y$ and $z$ axes.

2. In this problem we will work over the field of real numbers, $\mathbb{R}$.
a) Let $I=\left\langle f_{1}, \ldots, f_{s}\right\rangle$ be any ideal in $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$. Let $V=\mathbb{V}(I) \subset \mathbb{R}^{n}$ be the corresponding variety. Find a single polynomial $f$ such that $V=\mathbb{V}(f)$. Prove your answer.

## Solution:

We claim that $f=f_{1}^{2}+\cdots+f_{s}^{2}$ does the trick. First show $V \subseteq \mathbb{V}(f)$. If $P \in V$ then $f_{i}(P)=0$ for all $1 \leq i \leq s$, so $f_{i}^{2}(P)=0$ for all $1 \leq i \leq s$ and hence the sum $f(P)=0$ as well.
Conversely, we'll show that $V \supseteq \mathbb{V}(f)$. Let $P \in \mathbb{V}(f)$, so

$$
f(P)=\left(f_{1}^{2}+\cdots+f_{s}^{2}\right)(P)=f_{1}^{2}(P)+\cdots+f_{s}(P)=0
$$

But we are working over the real numbers, so each term of $f_{1}^{2}(P)+\cdots+f_{s}(P)$ is non-negative. Thus it can only equal zero if $f_{1}(P)=\cdots=f_{s}(P)=0$, i.e. if $P \in V$.
b) Let $I=\left\langle f_{1}, \ldots, f_{s}\right\rangle$ be any ideal in $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$. Suppose that $\mathbb{V}(I)=\emptyset$. Show that there is at least one element of $I$ that has no zero in $\mathbb{R}^{n}$. Justify your answer. (Notice that $\mathbb{R}$ is not algebraically closed, so you can't use the Nullstellensatz.)

Solution: Let $f=f_{1}^{2}+\cdots+f_{s}^{2}$, which is certainly in $I=\left\langle f_{1}, \ldots, f_{s}\right\rangle$. From part a) we know that

$$
\emptyset=\mathbb{V}(I)=\mathbb{V}\left(\left\langle f_{1}, \ldots, f_{s}\right\rangle\right)=\mathbb{V}(f),
$$

so $f$ has no zeros.
3. Let $V$ and $W$ be varieties in $\mathbb{C}^{n}$ such that $V \cap W=\emptyset$. Prove that there exist $f \in \mathbb{I}(V)$ and $g \in \mathbb{I}(W)$ such that $f+g=1$.

## Solution:

Let $J=\mathbb{I}(V)+\mathbb{I}(W)$. We first claim that $\mathbb{V}(J)=\emptyset$. If $P \in \mathbb{V}(J)$ then in particular every element of $\mathbb{I}(V)$ vanishes at $P$ and every element of $\mathbb{I}(W)$ vanishes at $P$. Thus $P \in V$ and $P \in W$, i.e. $P \in V \cap W$. This is impossible since $V \cap W=\emptyset$.

But now $\mathbb{C}$ is algebraically closed, so the Weak Nullstellensatz holds. This means

$$
J=\mathbb{I}(V)+\mathbb{I}(W)=\langle 1\rangle,
$$

so the desired result holds.
4. Let $I \subset k\left[x_{1}, \ldots, x_{n}\right]$ be an ideal. Let $\sqrt{I}$ be its radical. Show that there is a positive integer $p$ such that for every $f \in \sqrt{I}, f^{p} \in I$. (The thing to stress is that the choice of $p$ does not depend on what $f$ you choose; rather, $p$ depends only on what $\sqrt{I}$ is.) [Hint: $\sqrt{I}$ is an ideal in a Noetherian ring. You can also review our proof in class that $\sqrt{I}$ is an ideal.]

## Solution:

Since $k\left[x_{1}, \ldots, x_{n}\right]$ is Noetherian, $\sqrt{I}$ is finitely generated. Say

$$
\sqrt{I}=\left\langle f_{1}, \ldots, f_{s}\right\rangle
$$

In particular, each $f_{i}$ is in $\sqrt{I}$. Define $m_{1}, \ldots, m_{s}$ so that $f_{i}^{m_{i}} \in I$ for each $i$. Let $p=m_{1}+\cdots+m_{s}$.
Let $f \in \sqrt{I}$, so we can write $f=g_{1} f_{1}+\cdots+g_{s} f_{s}$, where $g_{i} \in k\left[x_{1}, \ldots, x_{n}\right]$. Then

$$
f^{p}=\left(g_{1} f_{1}+\cdots+g_{s} f_{s}\right)^{p}
$$

Each term in the expansion of $f^{p}$ is of the form

$$
B f_{1}^{i_{1}} f_{2}^{i_{2}} \cdots f_{s}^{i_{s}}
$$

where $B$ is some (ugly) polynomial and $i_{1}+i_{2}+\cdots+i_{s}=p=m_{1}+\cdots+m_{s}$. As in class, we claim that for at least one subscript $k$ we have $i_{k} \geq m_{k}$. This is a sort of pigeon-hole principle - if $i_{k}$ is always less than $m_{k}$, it is impossible for $i_{1}+i_{2}+\cdots+i_{s}=p=m_{1}+\cdots+m_{s}$. But if $i_{k} \geq m_{k}$ then $f_{k}^{i_{k}} \in I$. So every such term in the expansion of $f^{p}$ is in $I$, hence $f^{p} \in I$.
5. Let $I$ and $J$ be ideals in $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ such that

$$
I+J=\langle 1\rangle=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] .
$$

a) Prove that the varieties $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint.

## Solution:

We have seen that

$$
\mathbb{V}(I) \cap \mathbb{V}(J)=\mathbb{V}(I+J)
$$

Hence under our conditions, $\mathbb{V}(I) \cap \mathbb{V}(J)=\emptyset$, i.e. $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint.
b) Prove that $I J=I \cap J$. [Don't forget the assumption at the beginning of this problem!!!!]

## Solution:

It is always true that $I J \subseteq I \cap J$ so we only have to prove the reverse inclusion.
Let

$$
I=\left\langle f_{1}, \ldots, f_{s}\right\rangle \quad \text { and } \quad J=\left\langle g_{1}, \ldots, g_{t}\right\rangle .
$$

From our assumption we have that for some $f \in I$ and $g \in J, 1=f+g$. Say $f=a_{1} f_{1}+\cdots+a_{s} f_{s}$ and $g=b_{1} g_{1}+\cdots+b_{t} g_{t}$. So

$$
1=\left(a_{1} f_{1}+\cdots+a_{s} f_{s}\right)+\left(b_{1} g_{1}+\cdots+b_{t} g_{t}\right) .
$$

Now let $h \in I \cap J$. We want to show that $h \in I J$. Multiplying both sides in the above quality by $h$ we get

$$
h=\left(a_{1} f_{1} h+\cdots+a_{s} f_{s} h\right)+\left(b_{1} h g_{1}+\cdots+b_{t} h g_{t}\right) .
$$

The fact that each $f_{i} \in I$ and $h \in J$ means that every term in the first set of parentheses is in $I J$. The fact that $h \in I$ and every $g_{i} \in J$ means that every term in the second set of parentheses is in $I J$. Thus $h \in I J$.
6. Let $X$ be a topological space (not necessarily with the Zariski topology). Let $A$ be a subset of $X$ with the following property:

For each $P \notin A$ there exists a closed set $V_{P}$ that contains $A$ but does not contain $P$.
Prove that $A$ must be closed, making sure to justify each step.

## Solution:

We claim that

$$
A=\bigcap_{P \notin A} V_{P}
$$

where $V_{P}$ is the closed set containing $A$ but not containing $P$, as stated in the assumption, and $P$ ranges over all points in $X$ that are not in $A$. This will prove the result, since arbitrary intersections of closed sets are closed.

So we just have to prove the claim. The inclusion $\subseteq$ is clear since each $V_{P}$ contains $A$. For the reverse inclusion, let

$$
Q \in \bigcap_{P \notin A} V_{P}
$$

We want to show that $Q \in A$. But suppose that $Q \notin A$ (looking for a contradiction). Then associated to $Q$ is a closed set $V_{Q}$ containing $A$ but not containing $Q$. This participates in the intersection. But since $Q \notin V_{Q}$, we get that

$$
Q \notin \bigcap_{P \notin A} V_{P}
$$

(since we are exhibiting at least one $V_{P}$ that doesn't contain $Q$, namely $V_{Q}$ ), giving a contradiction.
7. Find the Zariski closure for each of the following sets in $\mathbb{R}^{2}$, and explain your answer. Some of them may already be closed. Your explanations do not have to be rigorous proofs, but they should be convincing!
a) The unit circle.

## Solution:

This is a variety, namely $\mathbb{V}\left(x^{2}+y^{2}-1\right)$, so it is already closed. Hence it is equal to its Zariski closure.
b) $A \cup B$, where $A$ is the unit circle and $B$ is the set of points in $\mathbb{R}^{2}$ of the form $(x, 0)$ where $x$ is a rational number.


## Solution:

We claim that any polynomial vanishing at the rational points between -1 and 1 has to vanish along the whole $x$-axis. Indeed, this is because such a polynomial has to be either zero along the whole line or else have only finitely many zeros. So the Zariski closure, i.e. the smallest closed set containing $A \cup B$, is the union of $A$ (which is closed thanks to the previous problem) and the $x$-axis, remembering that the union of two closed sets is closed.
c) The sine curve in $\mathbb{R}^{2}$, i.e. $\{(x, y) \mid y=\sin x\}$.

## Solution:

We claim that the Zariski closure of the sine curve is the whole plane. Let $C$ be this curve.
First consider any horizontal line, $y=c$, where $-1 \leq c \leq 1$. Such a line meets $C$ in infinitely many points, so the Zariski closure of $C$ has to contain the whole line. But there are infinitely many such lines, so the Zariski closure has to contain the whole horizontal strip $-1 \leq y \leq 1$ (and arbitrary $x$ ). But now any vertical line meets this strip in infinitely many points, so the Zariski closure has to contain all vertical lines as well. The union of all the vertical lines is the whole plane, so we are done.
8. Let $k$ be a field and let $R=k\left[x_{1}, \ldots, x_{n}\right]$. Let $I, J$ and $K$ be ideals in $R$.
a) If $I$ is radical, prove that $I: J$ must also be radical.

## Solution:

Let $f \in R$ be a polynomial such that $f^{r} \in I: J$ for some $r \geq 1$. We want to show that $f \in I: J$. That is, we want to show that $f g \in I$ for all $g \in J$.

By hypothesis we know that

$$
f^{r} \cdot g \in I \quad \text { for all } \quad g \in J
$$

This implies that $f^{r} g^{r}=(f g)^{r} \in I$ for all $g \in J$. But $I$ is radical, so $f g \in I$ for all $g \in J$, as desired.
b) Give an example to show that if $I: J$ is radical, it is not necessarily true that $I$ is radical.

## Solution:

$I=\left\langle x^{2}\right\rangle, J=\langle x\rangle, I: J=\langle x\rangle$.
c) If $V$ and $W$ are varieties in $k^{n}$, prove that $\mathbb{I}(V): \mathbb{I}(W)$ is a radical ideal. [Hint: "c" comes after "a" in the alphabet.]

## Solution:

We know that if $V$ is a variety then $\mathbb{I}(V)$ is a radical ideal. So this follows from a).
d) Prove that $J \subseteq I$ if and only if $I: J=R$.

## Solution:

First we prove $\Longrightarrow$. Assume that $J \subseteq I$. Let $f \in R$ be any polynomial. We want to show that $f g \in I$ for all $g \in J$. But if $g \in J \subset I$ then $g \in I$, so automatically $f g \in I$.

Now we prove $\Longleftarrow$. Assume $I: J=R$. We want to show that $J \subseteq I$. Let $g \in J$. We want to show that $g \in I$. Since $I: J=R$, we have $1 \in I: J$. Hence $g=1 \cdot g \in I$.
e) If $J \subseteq K$, prove that $I: K \subseteq I: J$.

## Solution:

Let $f \in I: K$. Then $f g \in I$ for all $g \in K$. Since $J \subseteq K$, in particular $f g \in I$ for all $g \in J$, so $f \in I: J$.
f) Assume that $I$ is radical. Prove that $I: \sqrt{J}=I: J$.

## Solution:

Since $J \subseteq \sqrt{J}$, the previous problem gives for free that $I: \sqrt{J} \subseteq I: J$. So we have to prove that $I: J \subseteq I: \sqrt{J}$.

Let $f \in I: J$, so $f g \in I$ for all $g \in J$. We want to show that $f \in I: \sqrt{J}$. Let $h \in \sqrt{J}$. We want to show that $f h \in I$.

Since $h \in \sqrt{J}, h^{r} \in J$ for some $r \geq 1$. Then $f h^{r} \in I$ since $f \in I: J$. This implies $(f h)^{r}=$ $f^{r} h^{r} \in I$. But $I$ is radical, so $f h \in I$ as desired.

