Math 40510, Algebraic Geometry

Problem Set 1 Solutions, due February 14, 2018

1. In this problem we explore polynomial rings.

   a) In class we defined the ring $k[x_1, \dots, x_n]$ of polynomials in n variables with coefficients in a field, k. We can similarly define $\mathbb{Z}_6[x_1, \dots, x_n]$ to be the ring of polynomials in n variables with coefficients in $\mathbb{Z}_6$. Prove by example that $\mathbb{Z}_6[x_1, \dots, x_n]$ is not an integral domain.

   Solution: $2x_1$ and $3x_1$ are two non-zero elements of $\mathbb{Z}_6[x_1, \dots, x_n]$ whose product is zero.

   b) Now let k be a field. Prove that if $f, g \in k[x_1, \dots, x_n]$ then $\deg(fg) = \deg(f) + \deg(g)$.

   Solution: Both f and g are linear combinations of monomials in $x_1, \dots, x_n$. Say $\deg(f) = d$ and $\deg(g) = e$.

   Decompose f and g into sums of polynomials

   $$f = f_0 + f_1 + \cdots + f_{d-1} + f_d \quad \text{and} \quad g = g_0 + g_1 + \cdots + g_{e-1} + g_e,$$

   where each f_i and each g_i collects the monomials of degree i together with their coefficients. For example, if $n = 3$ and $f = 5 + 3x - 2z + 4x^2 - 6xy + 7yz - 3x^3 + 2x^2z$ then $d = 3$ and

   $$\begin{aligned} f_0 &= 5 \\ f_1 &= 3x - 2z \\ f_2 &= 4x^2 - 6xy + 7yz \\ f_3 &= -3x^3 + 2x^2z. \end{aligned}$$

   It's clear that $\deg(fg) \leq d + e$ since the highest degree of a monomial that could appear in fg is $d + e$. **We want to show** that $f_d \cdot g_e$ is not zero, so we have $\deg(fg) = d + e$.

   The thing we have to show is that terms don't all cancel out. For example, if $f_d = x^2 + y^2$ and $g_e = x^2 - y^2$ then $f_d \cdot g_e = x^4 + x^2y^2 - x^2y^2 - y^4 = x^4 - y^4$; some terms cancel out, but not all. So let's rewrite what we have to show:

   Want to show if f and g are polynomials such that f is a linear combination of monomials all of degree d (i.e. f is *homogeneous* of degree d) and g is a linear combination of monomials all of degree e (i.e. g is *homogeneous* of degree e) then $fg \neq 0$.

   We'll proceed by induction on n. For $n = 1$, say

   $$f = a_0 + a_1x + \cdots + a_dx^d \quad \text{and} \quad g = b_0 + b_1x + \cdots + b_ex^e$$

   where $a_d, b_e \in k$ and $a_d \neq 0$ and $b_e \neq 0$. Then $fg = a_0b_0 + \cdots + (a_{d-1}b_e + a_db_{e-1})x^{d+e-1} + a_db_ex^{d+e}$. We don't know about other terms, but we do know that $a_db_e \neq 0$ since $a_d \neq 0$ and $b_e \neq 0$. Thus $\deg(fg) = d + e$ and in particular $fg \neq 0$.

   Now the inductive step. Assume that the statement is true for $n - 1$ variables. As we did in class, write f and g as polynomials in x_n with coefficients in $k[x_1, \dots, x_{n-1}]$:

$$\begin{aligned} f &= a_0(x_1, \dots, x_{n-1}) + a_1(x_1, \dots, x_{n-1})x_n + a_2(x_1, \dots, x_{n-1})x_n^2 + \cdots + a_p(x_1, \dots, x_{n-1})x_n^p \\ g &= b_0(x_1, \dots, x_{n-1}) + b_1(x_1, \dots, x_{n-1})x_n + b_2(x_1, \dots, x_{n-1})x_n^2 + \cdots + b_q(x_1, \dots, x_{n-1})x_n^q. \end{aligned}$$

Now we have to be a bit careful, because even though $\deg(f) = d$ and $\deg(g) = e$, it's not necessarily true that $d = p$ or $e = q$. For example, we might have $n = 3$ and

$$\begin{aligned} f &= x^2y^2z^2 + y^4z^2 + x^4yz \\ &= (x^4y)z + (x^2y^2 + y^4)z^2 \\ \\ g &= x^3y^2z^2 + xy^4z^2 + y^4z^3 \\ &= (x^3y^2 + xy^4)z^2 + (y^4)z^3. \end{aligned}$$

Then $d = 6$ and $e = 7$ but $p = 2$ and $e = 3$.

Assume first that $a_p(x_1, \ldots, x_{n-1})$ and $b_q(x_1, \ldots, x_{n-1})$ are both non-zero, and by induction we know $a_p(x_1, \ldots, x_{n-1}) \cdot b_q(x_1, \ldots, x_{n-1}) \neq 0$. Then just as in the case $n = 1$, the coefficient of x_n^{d+e} is not zero, so $fg \neq 0$. Thus $\deg(fg) = d + e$. If no term of f has a positive power of x_n, we simply have $f = a_0(x_1, \ldots, x_{n-1})$, and similarly for g, and the proof still works.

c) Prove that there are $\binom{d+2}{2}$ monomials of degree d in the variables x, y, z. [Your proof should be from scratch, not by using a special case of some formula you find somewhere.]

Solution: Later in the semester we'll give a more general version of this fact, but for now we'll give a more limited proof.

- First, count the monomials that involve only x and y but no power of z:
$$x^d, x^{d-1}y, x^{d-2}y^2, \ldots, xy^{d-1}, y^d.$$
There are $d + 1$ of them.
- Now count the monomials that have z^1:
$$x^{d-1}z, x^{d-2}yz, x^{d-3}y^2z, \ldots, xy^{d-2}z, y^{d-1}z.$$
There are d of those.
- Continue in this way, increasing the power of z. At each step there are one fewer monomials, until we get to
$$xz^{d-1}, yz^{d-1}$$
(of which there are two) and
$$z^d$$
(of which there is one).

So in all we have $(d + 1) + d + (d - 1) + \cdots + 2 + 1$, which is equal to $\binom{d+2}{2}$.

2. In this problem we look at varieties in $\mathbb{R}^n$. (Part c) is only for $\mathbb{R}^2$.)

a) Prove that a single point in $\mathbb{R}^n$ is an affine variety.

Solution: If $P = (a_1, a_2, \ldots, a_n)$ then
$$P = \mathbb{V}(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n).$$

b) Prove that the union of any finite number of points in $\mathbb{R}^n$ is an affine variety. [Hint: Use Lemma 2 of §2 of the book, and extend it to a finite union of varieties using induction.]

Solution: Let $V = \{P_1, P_2, \ldots, P_m\}$. By part a), each P_i is, by itself, an affine variety. This begins the induction. Now assume that the statement is true for $m - 1$ points, i.e. any subset of all but one point of V. So for example, let
$$X = \{P_1, \ldots, P_{m-1}\}$$

and note that $V = X \cup P_m$. By induction, X is an affine variety. By part a), P_m is an affine variety. So by Lemma 2, $V = X \cup P_m$ is also an affine variety.

c) In the next problem you'll show that a certain infinite union of points is not an affine variety. On the other hand, give an example of an infinite set of points in $\mathbb{R}^2$ whose union *is* an affine variety. Justify your answer.

Solution: Let $V = \mathbb{V}(x) \subset \mathbb{R}^2$, i.e. V is the y-axis. V is an affine variety, and it contains infinitely many points.

3. Let
$$X = \{(m, m^3 + 1) \in \mathbb{R}^2 \mid m \in \mathbb{Z}\}.$$
In this problem you'll show that X is *not* an affine variety.

a) Consider the following statement:

$$\begin{array}{l} \textit{If } f(x,y) \textit{ is a polynomial that vanishes at each point of } X \\ \textit{then } f \textit{ vanishes on the whole curve } x^3 - y + 1 = 0. \end{array} \qquad (*)$$

Explain why proving $(*)$ will guarantee that X is not an affine variety.

Solution: Let C be the curve $\mathbb{V}(x^3 - y + 1) \subset \mathbb{R}^2$. Notice that C contains points that are not on X, for example the point $(\pi, \pi^3 + 1)$. Suppose it were true that X were an affine variety, so $X = \mathbb{V}(f_1, \dots, f_s)$ for some polynomials $f_1, \dots, f_s \in \mathbb{R}[x, y]$. That means that

the common vanishing locus of $f_1, \dots, f_s$ *is* precisely X. $\qquad (**)$

If every polynomial f that vanishes at all points of X also vanishes on all of C, then this is true of $f_1, \dots, f_s$, so $(**)$ can't be true – the common vanishing locus contains a lot of other points, such as $(\pi, \pi^3 + 1)$. So this contradiction shows that X is not an affine variety.

b) Prove $(*)$.

Solution: Again by contradiction. Suppose $f \in \mathbb{R}[x, y]$ vanishes at every point of X (i.e. $X \subset \mathbb{V}(f)$).

Consider the intersection of $\mathbb{V}(f)$ and $\mathbb{V}(x^3 - y + 1)$. By Lemma 2, this intersection is an affine variety:
$$\mathbb{V}(f) \cap \mathbb{V}(x^3 - y + 1) = \mathbb{V}(f, x^3 - y + 1).$$
Notice that $X \subset V(f) \cap \mathbb{V}(x^3 - y + 1)$. This intersection is the set of points $(a, b) \in \mathbb{R}^2$ such that
$$f(a, b) = 0 \quad \text{and} \quad a^3 - b + 1 = 0.$$
The second of these equations says that for a point in this intersection, $b = a^3 + 1$. The first of the equations then says that any of these intersection points satisfies
$$f(a, a^3 + 1) = 0.$$
The fact that $X \subset V(f) \cap \mathbb{V}(x^3 - y + 1)$ means that the above equation is satisfied whenever $a \in \mathbb{Z}$.

But $f(t, t^3 + 1)$ is a polynomial in one variable, t. The fact that it vanishes whenever t is an integer says that it has infinitely many roots or else is the zero polynomial. But a non-zero polynomial in one variable has finitely many roots. Thus $f(t, t^3 + 1)$ is the zero polynomial. This means that f vanishes at any point (x, y) such that $y = x^3 + 1$, i.e. it vanishes on the whole curve $\mathbb{V}(x^3 - y + 1)$.

4. In class we showed how to obtain the parametrization for the circle $x^2+y^2=1$. Use the exact same idea (but slightly different algebra) to obtain the parametrization

$$x = \frac{(t-1)^2}{1+t^2}$$

$$y = \frac{2t^2}{1+t^2}.$$

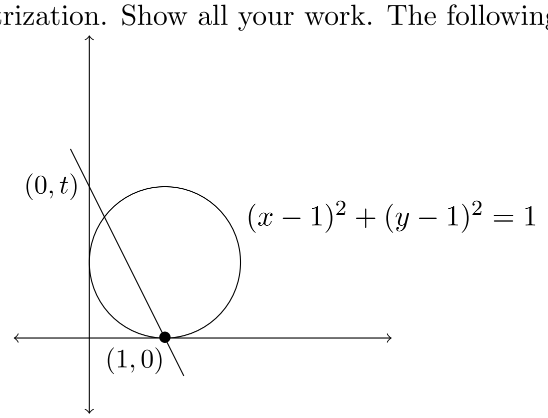
for the circle $(x-1)^2+(y-1)^2=1$. Specifically:

a) Verify that for any value of t in this parametrization, we have $(x-1)^2+(y-1)^2=1$.

Solution:

$$\begin{aligned}
(x-1)^2+(y-1)^2 &= \left(\frac{(t-1)^2}{1+t^2}-1\right)^2+\left(\frac{2t^2}{1+t^2}-1\right)^2 \\
&= \left(\frac{(t^2-2t+1)-(1+t^2)}{1+t^2}\right)^2+\left(\frac{2t^2-(1+t^2)}{1+t^2}\right)^2 \\
&= \left(\frac{-2t}{1+t^2}\right)^2+\left(\frac{t^2-1}{1+t^2}\right)^2 \\
&= \frac{4t^2+t^4-2t^2+1}{(1+t^2)^2} \\
&= \frac{t^4+2t^2+1}{(1+t^2)^2} \\
&= 1.
\end{aligned}$$

b) Derive the above parametrization. Show all your work. The following picture should help.



Solution: For any given t, the equation of the line through $(0,t)$ and $(1,0)$ is $y-t=-t(x-0)$, i.e.

$$t(x-1)+y=0, \quad \text{or} \quad y=-t(x-1).$$

This line meets the circle in two points, one of which is always $(1,0)$. We have to find the other point. So we have to solve the system of equations

$$\begin{aligned}(x-1)^2+(y-1)^2&=1\\ y&=-t(x-1).\end{aligned}$$

So we substitute this latter value of y into the first equation. We get

$$(x-1)^2 + (-t(x-1)-1)^2 = 1$$

$$(x-1)^2 + t^2(x-1)^2 + 2t(x-1) + 1 = 1$$

$$(x-1)[x-1+t^2(x-1)+2t] = 0$$

If $x - 1 = 0$ we already know about this intersection point. So it's the other factor that we're interested in:

$$x - 1 + t^2(x-1) + 2t = 0$$

$$(x-1)(1+t^2) + 2t = 0$$

$$x - 1 = \frac{-2t}{1+t^2}$$

$$x = \frac{-2t}{1+t^2} + 1$$

$$x = \frac{-2t+1+t^2}{1+t^2}$$

$$x = \frac{(t-1)^2}{1+t^2}.$$

This gives x. For y we have

$$\begin{aligned} y &= -t(x-1) \\ &= -t\left(\frac{(t-1)^2}{1+t^2} - 1\right) \\ &= -t\left(\frac{t^2-2t+1-(1+t^2)}{1+t^2}\right) \\ &= \frac{2t^2}{1+t^2} \end{aligned}$$

as desired.

c) In particular, which point of the circle is missed by this parametrization?

Solution: It's missing the point $(1, 2)$. In part b) you can see that this point corresponds to a vertical line, i.e. to $t = \infty$, which has no slope.

5. Let V be the parabola in $\mathbb{R}^2$ given by the equation $y = x^2$. Let $P = (a, a^2)$ be a point of V. (I don't mean that you should choose a specific value of a.)

a) Find a polynomial f so that $V = \mathbb{V}(f)$. [Hint: this is as easy as it looks. Don't look for anything tricky here.]

Solution: $f = y - x^2$.

b) Find a polynomial ℓ so that $\mathbb{V}(\ell)$ is the tangent line to V at P.

Solution: We use methods from calculus. Since $\frac{d}{dx}x^2 = 2x$, the slope of the tangent line at P is $2a$. So the tangent line is

$$y - a^2 = 2a(x-a), \qquad \text{i.e.} \qquad y = 2ax - a^2.$$

So $\ell = 2ax - y - a^2$.

c) Prove directly that $\langle \ell, f\rangle$ is not a *radical* ideal. That is, find a polynomial g such that some power of g is in $\langle \ell, f\rangle$ but g itself is not. Be sure to show all your work: prove that some power of g is in this ideal (what power?), and prove that g itself is not in the ideal. [Hint: look at vertical lines for one possible answer.]

Solution:

$$\begin{aligned}
\langle 2ax - y - a^2, y - x^2\rangle &= \langle y - x^2, (2ax - y - a^2) + (y - x^2)\rangle \\
&= \langle y - x^2, 2ax - x^2 - a^2\rangle \\
&= \langle y - x^2, x^2 - 2ax + a^2\rangle \\
&= \langle y - x^2, (x-a)^2\rangle
\end{aligned}$$

Take $g = x - a$. Then we have just shown that $g^2 \in \langle \ell, f\rangle$. We have to show that g itself is not in $\langle \ell, f\rangle$. But $\langle \ell, f\rangle = \langle y - x^2, (x-a)^2\rangle$, and the equation

$$h_1(y - x^2) + h_2(x-a)^2 = x - a$$

can be rewritten as

(1) $$(h_2 - h_1)x^2 - 2ah_2x + h_1y + h_2a^2 = x - a,$$

Looking at the constant term we get $h_2a^2 = -a$. (No matter what h_1 and h_2 are, there can't be any other constant terms in this equation.) This gives either $a = 0$ or $h_2 = -\frac{1}{a}$. Take the first case, $a = 0$. Then

$$(h_2 - h_1)x^2 + h_1y = x,$$

No matter what h_1 and h_2 are, there is no term on the left that has only x in it (i.e. neither has x^2 nor y). So this is impossible. So we can assume $a \neq 0$ and $h_2 = -\frac{1}{a}$. Substituting for h_2 in (1) gives, after a little computation,

$$\left(-\frac{1}{a} - h_1\right)x^2 + 2x + h_1y = x.$$

No matter what h_1 is, the only term on the left that has x and nothing else is $2x$, which is not equal to x. So this is impossible too.

d) If $I = \langle \ell, f\rangle$, find $\mathbb{V}(I)$ and find $\mathbb{I}(\mathbb{V}(I))$. [Note that you can do this part even if you did not get part c). However, I would like you to justify your answer. No full credit if you find the right ideal but don't give a proof.]

Solution: $\mathbb{V}(I)$ is the common vanishing locus of ℓ and f, i.e. the total intersection of the parabola and the tangent line at P. Since (from calculus) we know that the parabola is always concave up, the tangent line meets the parabola *only* at the point P, so $\mathbb{V}(I) = \{P\}$.

So we just have to find $\mathbb{I}(P)$. Remember that $P = (a, a^2)$. We'll show that

$$\mathbb{I}(P) = \langle x - a, y - a^2\rangle.$$

(Remember that a is a constant, so $x - a$ and $y - a^2$ are both linear polynomials.) The inclusion $\supseteq$ is clear, so we just have to show $\subseteq$.

If $a = 0$ we actually showed this in class. If $a \neq 0$ the idea is the same: by writing x as $(x-a)+a$ and y as $(y-a^2)+a^2$, we can convert any polynomial in x and y into a polynomial in $x-a$ and $y-a^2$. Then a polynomial p that vanishes at P has to have zero constant term when written in terms of $x-a$ and $y-a^2$, so it is in $\langle x-a, y-a^2\rangle$.

6. In class we mentioned that if k is a field then $k[x_1,\dots,x_{n-1}][x_n] \cong k[x_1,\dots,x_n]$. Give a proof of this fact. In particular, you should

   a) find a function $\phi : k[x_1,\dots,x_{n-1}][x_n] \to k[x_1,\dots,x_n]$ [Hint: don't try to do anything too fancy. For example, $(3x+y)z+(4xy+5y^3)z^2$ is both an element of $k[x,y][z]$ and of $k[x,y,z]$];

   Solution: Let $f \in k[x_1,\dots,x_{n-1}][x_n]$. So
   $$f = g_0(x_1,\dots,x_{n-1}) + g_1(x_1,\dots,x_{n-1})x_n + \cdots + g_d(x_1,\dots,x_{n-1})x_n^d$$
   for some non-negative integer d. So f can be viewed naturally as an element of $k[x_1,\dots,x_n]$ just by multiplying out all the terms. Define $\phi(f) = f$ in this way.

   b) show that ϕ is a ring homomorphism,

   Solution: $\phi(f+g) = \phi(f)+\phi(g) = f+g$ and $\phi(fg) = \phi(f)\phi(g) = fg$ are both immediate from the definition.

   c) show that ϕ is injective,

   Solution: Again from the definition, $f \in \ker\phi$ if and only if $\phi(f) = 0$ if and only if $f = 0$.

   d) and show that ϕ is surjective.

   Solution: By separating out the x_n's, any polynomial in $k[x_1,\dots,x_n]$ can be expressed as a polynomial in $k[x_1,\dots,x_{n-1}][x_n]$.

   (Your proof of this whole problem should take very few lines. Just convince me that you understand what's going on.)

7. Consider the infinite family of polynomials $f_1, f_2, f_3, \dots$ with
   $$f_i = 3x^i + 5y^{i+7} - (i^2+3)x^{i-2}y \in \mathbb{R}[x,y] \quad \text{(where } i = 1,2,3,\dots).$$
   Prove that there is some integer N so that every f_j with $j > N$ can be written as a linear combination of $f_1, f_2, \dots, f_N$. [Hint: the form of the f_i is a red herring. Also, I do *not* want to know specifically what N is.]

   Solution: Consider the chain of ideals
   $$\langle f_1\rangle \subseteq \langle f_1, f_2\rangle \subseteq \langle f_1, f_2, f_3\rangle \subseteq \cdots$$
   Since $k[x,y]$ is Noetherian, this chain stabilizes. That is, there is some N so that
   $$\langle f_1,\dots,f_N\rangle = \langle f_1,\dots,f_N, f_{N+1},\dots,f_j\rangle$$
   for any $j \geq N+1$. So in particular f_j can be written as a linear combination of $f_1,\dots,f_N$.