

THE GEOMETRY OF HILBERT FUNCTIONS

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1. INTRODUCTION

The title of this paper, “The geometry of Hilbert functions,” might better be suited for a multi-volume treatise than for a single short article. Indeed, a large part of the beauty of, and interest in, Hilbert functions derives from their ubiquity in all of commutative algebra and algebraic geometry, and the unexpected information that they can give, very much of it expressible in a geometric way. Most of this paper is devoted to describing just one small facet of this theory, which connects results of Davis (e.g. [Davis]) in the 1980’s, of Bigatti, Geramita and myself (cf. [Bigatti-Geramita-Migliore]) in the 1990’s, and of Ahn and myself (cf. [Ahn-Migliore]) very recently. On the other hand, we have an alphabet soup of topics that play a role here: UPP, WLP, SLP, ACM at the very least. It is interesting to see the ways in which these properties interact, and we also try to illustrate some aspects of this.

There are almost as many different notations for Hilbert functions as there are papers on the subject. We will use the following notation. If I is a homogeneous ideal in a polynomial ring R , we write

$$h_{R/I}(t) := \dim(R/I)_t.$$

If I is a saturated ideal defining a subscheme V of \mathbb{P}^n then we also write this function as $h_V(t)$ or $h_{R/I_V}(t)$.

So where is the geometry? Of course $\dim R_t = \binom{t+n}{n}$, so the information provided by the Hilbert function is equivalent to giving the dimension of the degree t component of I . This dimension is one more than the dimension of the linear system of hypersurfaces of degree t defined by I_t (since this latter dimension is projective). What is the base locus of this linear system? Of course V is contained in this base locus, but it may contain more. The results in this paper (e.g. Theorem 5.3, Theorem 5.4, Theorem 6.2 and Theorem 6.5) can be viewed as describing the dimension, irreducibility and reducedness of this base locus, based on information about the Hilbert function, and other basic properties, of V . We will see that under some situations, just knowing the dimension of this linear system in two consecutive degrees can force

the base locus to contain a hypersurface, or anything smaller. (We will concentrate on the curve case.)

An important starting point for us (and indeed for almost any discussion of Hilbert functions of standard graded algebras) is Macaulay's theorem bounding the growth of the Hilbert function. Once we have this, we need Gotzmann's results about what happens when Macaulay's bound is achieved. These are both discussed in Section 2, as are several other results related to these.

In Section 3 we recall the notions of the Uniform Position Property (UPP) and the Weak Lefschetz Property (WLP) and some of their connections. Subsequent sections, especially Section 6, continue the discussion of UPP. WLP, while often less visible, lurks in the background of many of the results and computations of this paper, and in fact is an important object of study. We include a short discussion of the behavior of WLP in families of points in Section 3, including a new example (Example 3.4) showing how, for fixed Hilbert function, WLP can hold in one component of the postulation Hilbert scheme and not hold in another. See also Theorem 5.6.

The focus in this article is the situation where the first difference of the Hilbert function of a set of points, Z , in projective space \mathbb{P}^n attains the same value in two consecutive degrees: $\Delta h_Z(d) = \Delta h_Z(d+1) = s$. Depending on the relation between d , s and certain invariants of Z , we will get geometric consequences for the base locus. In Section 4 we describe these relations, setting the stage for the main results.

These main results are given in Sections 5 and 6. Here we see that under certain assumptions on d , the condition $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ guarantees that the base locus of the linear system $|I_d|$ is a curve of degree s . This comes from work in [Davis], [Bigatti-Geramita-Migliore] and [Ahn-Migliore]. Other results follow as well. What is surprising here is that the central condition of [Ahn-Migliore], namely that $d > r_2(R/I_Z)$ (see Section 2 for the definition), is much weaker than the central assumption of the comparable results in [Bigatti-Geramita-Migliore], namely $d \geq s$, but the results are very similar. Section 5 focuses on the general results, while Section 6 turns to the question of what can be said about this base locus when the points have UPP.

There are some differences in the results of [Bigatti-Geramita-Migliore] and [Ahn-Migliore] as a result of the differences in these assumptions. Section 7 studies these, and gives examples to show that they are not accidental omissions. Some very surprising behavior is exhibited here.

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2. MAXIMAL GROWTH OF THE HILBERT FUNCTION

We first collect the notation that we will use throughout this paper. Let k be a field of characteristic zero and let $R = k[x_1, \dots, x_n]$.

Definition 2.1. Let $Z \subset \mathbb{P}^{n-1}$ be any closed subscheme with defining (saturated) ideal $I = I_Z$.

(a) The *Hilbert function of Z* is the function

$$h_Z(t) = \dim(R/I_Z)_t$$

We also may write $h_{R/I}(t)$ for this function. If A is Artinian then we write

$$h_A(t) = \dim A_t$$

for its Hilbert function.

(b) We say that Z is *arithmetically Cohen-Macaulay (ACM)* if the coordinate ring R/I_Z is a Cohen-Macaulay ring. Note that if Z is a zero-dimensional scheme then it is automatically ACM. \square

If F is a homogeneous polynomial, by abuse of notation we will also denote by F the hypersurface of \mathbb{P}^{n-1} defined by F .

Definition 2.2. For a homogeneous ideal I we define

$$\alpha = \min\{t \mid I_t \neq 0\},$$

i.e. α is the *initial degree* of I . \square

If $A = R/I$ is a standard graded k -algebra, then there is a famous bound, due to Macaulay (cf. [Macaulay]), that describes the maximum possible growth of the Hilbert function of A from any degree to the next. To give this bound, we need a little preparation.

Definition 2.3. The *i*-binomial expansion of the integer c ($i, c > 0$) is the unique expression

$$c = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j},$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. \square

Note that the assertion that this representation is unique is something that has to be checked!

Definition 2.4. If $c \in \mathbb{Z}$ ($c > 0$) has *i*-binomial expansion as in Definition 2.3, then we set

$$c^{\langle i \rangle} = \binom{m_i + 1}{i + 1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j + 1}.$$

Note that this defines a collection of functions $\langle i \rangle : \mathbb{Z} \rightarrow \mathbb{Z}$. \square

For example, the 5-binomial expansion of 76 is

$$76 = \binom{8}{5} + \binom{6}{4} + \binom{4}{3} + \binom{2}{2},$$

so

$$76^{\langle 5 \rangle} = \binom{9}{6} + \binom{7}{5} + \binom{5}{4} + \binom{3}{3} = 111.$$

Definition 2.5. A sequence of non-negative integers $\{c_i : i \geq 0\}$ is called an *O-sequence* if

$$c_0 = 1 \quad \text{and} \quad c_{i+1} \leq c_i^{\langle i \rangle},$$

for all i . An O-sequence is said to have *maximal growth from degree i to degree $i + 1$* if $c_{i+1} = c_i^{\langle i \rangle}$. \square

The importance of the binomial expansions described above becomes apparent from the following beautiful theorem of Macaulay:

Theorem 2.6 ([Macaulay]). *The following are equivalent:*

- (i) $\{c_i : i \geq 0\}$ is an O-sequence;
- (ii) $\{c_i : i \geq 0\}$ is the Hilbert function of a standard graded k -algebra.

In other words, a sequence of non-negative integers is the Hilbert function of a standard graded k -algebra if and only if the growth from any degree to the next is bounded as above. Remember that the Hilbert

function is eventually equal to the Hilbert polynomial, but until that point, anything is allowed as long as we maintain $c_{i+1} \leq c_i^{(i)}$.

One can ask when such a sequence is the Hilbert function of a *reduced* k -algebra. This was answered by Geramita, Maroscia and Roberts (cf. [Geramita-Maroscia-Roberts]), by introducing the *first difference* of the Hilbert function.

Definition 2.7. Given a sequence of non-negative integers $\underline{c} = \{c_i : i \geq 0\}$, the *first difference* of this sequence is the sequence $\Delta \underline{c} := \{b_i\}$ defined by $b_i = c_i - c_{i-1}$ for all i . (We make the convention that $c_{-1} = 0$, so $b_0 = c_0 = 1$.) We say that \underline{c} is a *differentiable O-sequence* if $\Delta \underline{c}$ is again an O-sequence. By taking successive first differences, we inductively define the k -th difference $\Delta^k \underline{c}$. \square

Remark 2.8. An important fact to remember is that if Z is a zero-dimensional scheme with Artinian reduction A then

$$h_A(t) = \Delta h_Z(t) \text{ for all } t.$$

It follows that $(\Delta h_Z(t) : t \geq 0)$ is a finite sequence of positive integers, called the *h -vector* of Z . Similarly, if V is ACM of dimension d with Artinian reduction A then $h_A(t) = \Delta^{d+1} h_V(t)$ for all t . \square

The following is the theorem of Geramita, Maroscia and Roberts mentioned above. It guarantees the existence of a reduced subscheme of \mathbb{P}^{n-1} with given Hilbert function (remember that $R = k[x_1, \dots, x_n]$), under a simple hypothesis on the Hilbert function.

Theorem 2.9 ([Geramita-Maroscia-Roberts]). *Let $\underline{c} = \{c_i\}$ be a sequence of non-negative integers, with $c_1 = n$. Then \underline{c} is the Hilbert function of a standard graded k -algebra R/I , with I radical, if and only if \underline{c} is a differentiable O-sequence.*

Note that if I is any saturated ideal (reduced or otherwise), and if L is a general linear form, then we have $[I : L] \cong I(-1)$ as graded modules. It follows that L induces an injection $\times L : ((R/I)(-1))_t \rightarrow (R/I)_t$, i.e. R/I has depth at least 1. Hence the first difference of the Hilbert function of R/I is the Hilbert function of $R/(I, L)$, again a standard graded k -algebra, so the Hilbert function of R/I is a differentiable O-sequence. This shows that Theorem 2.9 would also be true if “radical” were replaced by “saturated” (since if a radical ideal can be constructed for a given differentiable O-sequence, this ideal is also saturated). The real heart of Theorem 2.9 is that any such sequence can be achieved by a radical ideal.

Example 2.10. We should stress that Theorem 2.9 guarantees the existence of a reduced (even non-degenerate) subscheme of \mathbb{P}^n , but it does not guarantee that what we get will be equidimensional. It also does not say anything about higher differences. (Note that even if I is saturated, this is not necessarily true of (I, L) .) We give two examples.

1. Let $I = (x_1, x_2)$ in $k[x_0, x_1, x_2, x_3]$, so I defines a line, λ , in \mathbb{P}^3 . Let F be a homogeneous polynomial of degree 3, non-singular along λ . Let $J = (I^2, F)$. Then J is the saturated ideal of a non-reduced subscheme of degree 2 and genus -2 (cf. [MIgliore2]). Its Hilbert function is the sequence $1, 4, 7, 9, 11, \dots$ (with Hilbert polynomial $2t + 3$). The smallest genus for a reduced equidimensional subscheme of degree 2 is -1 (two skew lines), so this Hilbert function does not exist among reduced, equidimensional curves in \mathbb{P}^3 . However, adding points reduces the genus while not affecting the degree, and in fact this Hilbert function occurs for the union of two skew lines and one point, which is indeed reduced.
2. Consider the sequence $\underline{c} = 1, 4, 10, 17, 26, 35, \dots$ (with Hilbert polynomial $9t - 11 + 1$, so it corresponds to a curve of degree 9 and arithmetic genus 11). Its first difference is $1, 3, 6, 7, 9, 9, \dots$, which is again an O-sequence, so \underline{c} is a differentiable O-sequence. However, the second difference is $1, 2, 3, 1, 2$ which is not an O-sequence. Theorem 2.9 guarantees the existence of a reduced curve with Hilbert function \underline{c} , and indeed it can be achieved by the union $C_1 \cup C_2 \cup P_1 \cup P_2$, where C_1 is a plane curve of degree 3, C_2 is a plane curve of degree 6 not in the plane of C_1 , C_1 and C_2 meet in 3 points, P_1 is a generally chosen point in \mathbb{P}^3 , and P_2 is a generally chosen point in the plane of C_1 .

One sees that finding the reduced subscheme of Theorem 2.9 can be a tricky matter! \square

If $\underline{h} = \{c_i : i \geq 0\}$ is an O-sequence, Theorem 2.6 guarantees that \underline{h} is the Hilbert function of a standard graded k -algebra, say R/I . The following striking result of Gotzmann says what happens if we have maximal growth (see Definition 2.5), i.e. if the equality $c_{d+1} = c_d^{\langle d \rangle}$ holds. Let \bar{I} be the ideal generated by the components of I of degree $\leq d$, so that $\bar{I}_i = I_i$ for all $i \leq d$. We also sometimes write $\bar{I} = \langle I_{\leq d} \rangle$.

Theorem 2.11 ([Gotzmann], [Green1]). *Let $\underline{h} = \{c_i : i \geq 0\}$ be the Hilbert function of R/\bar{I} , and suppose that $c_d = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \dots + \binom{m_j}{j}$ is the d -binomial expansion of c_d . Assume that $c_{d+1} = c_d^{\langle d \rangle}$. Then for*

any $l \geq 1$ we have

$$c_{d+l} = \binom{m_d + l}{d + l} + \binom{m_{d-1} + l}{d - 1 + l} + \cdots + \binom{m_j + l}{j + l}.$$

In particular, the Hilbert polynomial $p_{R/\bar{I}}(x)$ agrees with the Hilbert function $h_{R/\bar{I}}(x)$ in all degrees $\geq d$. It can be written as

$$p_{R/\bar{I}}(x) = \binom{x + m_d - d}{m_d - d} + \binom{x + m_{d-1} - d}{m_{d-1} - (d - 1)} + \cdots + \binom{x + m_j - d}{m_j - j},$$

where $m_d - d \geq m_{d-1} - (d - 1) \leq \cdots \leq m_j - j$.

Remark 2.12. Theorem 2.11 is also known as the *Gotzmann Persistence Theorem*. Since $p_{R/\bar{I}}(x)$ has degree $m_d - d$, this means that the Krull dimension $\dim R/\bar{I} = m_d - d + 1$, and \bar{I} defines a subscheme of dimension $m_d - d$. Its degree can also easily be computed by checking the leading coefficient. Gotzmann also shows that \bar{I} has regularity $\leq d$ and agrees with its saturation in degrees $\geq d$. \square

Remark 2.13. An excellent reference for the Gotzmann Persistence Theorem and related results is section 4.3 of [Bruns-Herzog] (note that this is the revised edition; this material does not appear in the original book). A *much* more detailed and comprehensive expository treatment than that given here can be found in [Iarrobino-Kanev], Appendix C (written by A. Iarrobino and S. Kleiman). This includes a detailed description of the associated Hilbert scheme. \square

Much of the work described in this paper revolves around the following idea. If I is a saturated homogeneous ideal and L is a general linear form, then reducing modulo L gives a new ideal, J , in $S = R/(L)$. Suppose that S/J has maximal growth from degree d to degree $d+1$. This does not necessarily imply that R/I has maximal growth at that spot (see also Remark 4.5). Then what information *can* we get about I if we know that R/J has maximal growth? As a first step we have the following result from [Bigatti-Geramita-Migliore]. We have noted in Remark 2.12 that maximal growth of J implies something about the saturation and regularity of J , thanks to Gotzmann, but we now show that it also says something about the saturation and regularity of $\langle I_{\leq d} \rangle$.

Proposition 2.14. ([Bigatti-Geramita-Migliore] Lemma 1.4, Proposition 1.6) *Let $I \subset R$ be a saturated ideal. Let L be a general linear form, and let $J = \frac{(I, L)}{(L)} \subset S = R/(L)$. Suppose that S/J has maximal growth*

from degree d to degree $d + 1$. Let $\bar{I} = \langle I_{\leq d} \rangle$ as above. Let \bar{I}^{sat} be the saturation of \bar{I} . Then $\bar{I} = \bar{I}^{sat}$, i.e. \bar{I} is a saturated ideal. Furthermore, \bar{I} is d -regular.

Remark 2.15. Again, Proposition 2.14 differs from the Gotzmann results (cf. Theorem 2.11, Remark 2.12) in that S/J is assumed to have maximal growth, but we conclude something about I . But even beyond this, there is another difference between Proposition 2.14 and the part of Gotzmann's work that we have presented (referring to the sources mentioned in Remark 2.13 for a more complete exposition); indeed, in Theorem 2.11 and Remark 2.12 we only conclude that the ideal in question agrees with its saturation in degree d and beyond, while here we make a conclusion about the whole ideal. (The price is that we have to assume that I itself is saturated to begin with, although this does not necessarily hold for J .)

Also, the maximal growth assumption for S/J is necessary. For instance, let I be the saturated ideal of a set, Z , of sixteen general points in \mathbb{P}^3 . Then I has four generators in degree 3 and three generators in degree 4. The four generators in degree 3 do define Z scheme-theoretically, but this is not enough. One can check (e.g. using `macaulay` [Bayer-Stillman]) that if $\bar{I} = \langle I_{\leq 3} \rangle$, then $\bar{I}^{sat} = I \neq \langle I_{\leq 3} \rangle = \bar{I}$. Furthermore, \bar{I} is 5-regular and I itself is 4-regular, and neither is 3-regular. And indeed, the Hilbert function of S/J is 1, 3, 6, 6, 0, so we do not have maximal growth of S/J from degree 3 to degree 4.

Despite this, the results in [Ahn-Migliore] show that statements along the lines of Proposition 2.14 can be obtained even weakening the maximal growth assumption. \square

3. UPP AND WLP

A very important property of (many) reduced sets of points in projective space is the following:

Definition 3.1. A reduced set of points $Z \subset \mathbb{P}^{n-1}$ has the *Uniform Position Property (UPP)* if, for any $t \leq |Z|$, all subsets of t points have the same Hilbert function, which necessarily is the truncated Hilbert function. \square

The last comment follows from the fact that it was also shown in [Geramita-Maroscia-Roberts] that given any reduced set Z of, say, d points with known Hilbert function, and truncating this function at any value $k < d$, there is a subset X of Z consisting of k points whose

Hilbert function is this truncated function. Hence if Z has UPP, then *all* subsets have this truncated Hilbert function.

While it is known that the general hyperplane section of an irreducible curve has UPP (cf. [Harris-Eisenbud]), at least in characteristic zero, much remains open. An important open question is to determine all possible Hilbert functions of sets of points in projective space with UPP. It is known in \mathbb{P}^2 but it is open even in \mathbb{P}^3 . See [Geramita-Migliore] for a discussion of several of the many papers that have contributed to this question for \mathbb{P}^2 .

For any given Hilbert function, there may or may not be a set of points with UPP having that Hilbert function. One of the contributions of the papers discussed here is toward showing some conditions on the Hilbert function that prohibit the existence of points with UPP having that function. If there *is* a set of points with UPP having a given Hilbert function, somehow “most” sets of points with that Hilbert function have UPP: consider the postulation Hilbert scheme parameterizing sets of points with that Hilbert function; then in the component containing the given set of points, there is an open subset corresponding to points with UPP. (We make this more precise in the discussion about WLP below.)

Another important property is the following.

Definition 3.2. An Artinian algebra A has the *Weak Lefschetz Property (WLP)* if, for a general linear form L , the map $\times L : A_t \rightarrow A_{t+1}$ has maximal rank, for all t . We say that A has the *Strong Lefschetz Property (SLP)* if for every d , and for a general form F of degree d , the map $\times F : A_t \rightarrow A_{t+d}$ has maximal rank, for all t . If Z is a set of points, we sometimes say that Z has WLP or SLP if its general Artinian reduction does \square

Remark 3.3. We have the following comments about WLP and SLP:

1. The statement that SLP holds for an ideal of any number of general forms is equivalent to the well-known Fröberg conjecture. See [Anick], [Migliore-Miró-Roig1], [Migliore-Miró-Roig2], [Migliore-Miró-Roig-Nagel].
2. *Every* Artinian complete intersection in $k[x_1, x_2, x_3]$ has WLP (cf. [Harima-Migliore-Nagel-Watanabe]). It is also known that SLP (and hence WLP) holds for *every* ideal in $k[x_1, x_2]$ (cf. [Iarrobino1], [Harima-Migliore-Nagel-Watanabe]), and it (and hence Fröberg’s conjecture) holds for an ideal of general forms in $k[x_1, x_2, x_3]$ (cf. [Anick]). \square

We now begin a short digression about the behavior of WLP in families of reduced zero-dimensional schemes. Fix a Hilbert function, H , that corresponds to a zero-dimensional scheme, and consider the h -vector, $\underline{h} = (a_0, a_1, a_2, \dots, a_s)$, associated to that Hilbert function (i.e. its first difference ΔH – cf. Remark 2.8). Let $d = \sum_i a_i$ be the degree of the zero-dimensional scheme. Consider the postulation Hilbert scheme, $\mathcal{H}_{\underline{h}} := \text{Hilb}^H(\mathbb{P}^n)$, parameterizing zero-dimensional schemes in \mathbb{P}^n with Hilbert function H (inside the punctual Hilbert scheme, $\text{Hilb}^d(\mathbb{P}^n)$, of all zero-dimensional schemes in \mathbb{P}^n with the given degree). It is known that the closure, $\overline{\mathcal{H}}_{\underline{h}}$, of $\mathcal{H}_{\underline{h}}$ may have several irreducible components – see for instance Richert (cf. [Richert], and use Hartshorne’s lifting procedure, cf. [Hartshorne], [Migliore-Nagel], on the Artinian monomial ideals that Richert gives), Ragusa-Zappalà (cf. [Ragusa-Zappalà]) or Kleppe (cf. [Kleppe], Remark 27). We will consider only those components of $\overline{\mathcal{H}}_{\underline{h}}$ for which the general element is reduced.

One can show that, like UPP, WLP is an open condition in the sense that in any component of $\overline{\mathcal{H}}_{\underline{h}}$, an open subset (possibly empty) corresponds to zero-dimensional schemes with WLP. We will now give an example that answers (positively) the following question. Namely, does there exist a Hilbert function with h -vector \underline{h} , and components \mathcal{H}_1 and \mathcal{H}_2 of the corresponding $\overline{\mathcal{H}}_{\underline{h}}$, for which

- $\underline{h} = (a_0, a_1, a_2, \dots, a_s)$ is unimodal, and in fact satisfies
- $$(1) \quad a_0 < a_1 < a_2 \cdots < a_t \geq a_{t+1} \geq \cdots \geq a_s.$$

for some t (this is a technical necessity for WLP– cf. [Harima-Migliore-Nagel-Watanabe], Remark 3.3);

- the general element of \mathcal{H}_1 corresponds to a reduced, zero-dimensional scheme with WLP;
- *no* element of \mathcal{H}_2 corresponds to a zero-dimensional scheme with WLP (i.e. the open subset referred to above is empty)?

The following example answers this question.

Example 3.4. We will give an example of ideals I_1 and I_2 of reduced zero-dimensional schemes in \mathbb{P}^3 , both with h -vector

$$\underline{h} = (1, 3, 6, 9, 11, 11, 11),$$

such that the Artinian reduction of R/I_1 has WLP and the Artinian reduction of R/I_2 does not. Furthermore, the Betti diagrams for R/I_1 and R/I_2 are (respectively)

total:	1	14	24	11		total:	1	14	25	12
0:	1	-	-	-		0:	1	-	-	-
1:	-	-	-	-		1:	-	-	-	-
2:	-	1	-	-		2:	-	1	-	-
3:	-	1	-	-		3:	-	1	1	-
4:	-	1	2	-		4:	-	2	2	1
5:	-	-	-	-		5:	-	-	-	-
6:	-	11	22	11		6:	-	10	22	11

Clearly these Betti diagrams allow no minimal element in the sense of Richert (cf. [Richert]) or Ragusa-Zappalà (cf. [Ragusa-Zappalà]). Furthermore, neither can be a specialization of the other, so they correspond to different components of $\overline{\mathcal{H}}_h$.

For I_1 we begin with a line in \mathbb{P}^3 and link, using a complete intersection of type $(3, 4)$, to a smooth curve, C , of degree 11. Since a line is ACM, the same is true of C by liaison. We let Z_1 be a set of 52 general points on C , and let I_1 be the homogeneous ideal of Z_1 . It is easy to check that R/I_1 has the desired h -vector, using liaison computations (see for instance [Migliore1]) to compute the Hilbert function of C and then the fact that Z_1 is chosen generically on C so that the Hilbert function is the truncation. The rows 0 to 5 in the Betti diagram come only from C , and then the last row is forced from the Hilbert function. Because the Hilbert function of Z_1 agrees with that of C up to and including degree 6, the WLP for Z_1 follows from the Cohen-Macaulayness of C . Hence the general element of the corresponding component has WLP.

For I_2 we start with the ring $R = k[x, y, z]$ (dropping subscripts on the variables for convenience). Consider the monomial ideal J consisting of $(x^3, x^2y^2, x^2yz^2, z^5)$ together with all the monomials of degree 7. One can check on `macaulay` (cf. [Bayer-Stillman]) that R/J is Artinian with Hilbert function \underline{h} and with the Betti diagram above to the right. Using the lifting procedure for monomial ideals (cf. [Hartshorne], [Migliore-Nagel]), we lift J to the ideal I_2 of a reduced zero-dimensional scheme, Z_2 . Now, it is clear from the Betti diagram that R/J has a socle element in degree 4. This means that the map from $(R/J)_4$ to $(R/J)_5$ (both of which are 11-dimensional) induced by a general linear form necessarily has a kernel, and so it is neither injective nor surjective. Hence Z_2 does not have WLP. But from the Betti diagram, it is clear from semicontinuity that the general element (hence every element) of the component corresponding to Z_2 similarly fails to have WLP.

Incidentally, the lex-segment ideal corresponding to this Hilbert function (and hence having maximal Betti numbers) has Betti diagram

total:	1	18	31	14
0:	1	-	-	-
1:	-	-	-	-
2:	-	1	-	-
3:	-	1	1	-
4:	-	2	4	2
5:	-	2	3	1
6:	-	12	23	11

One can check that this is a specialization of both of the Betti diagrams above.

It would be nice to find an example where \underline{h} is actually the h -vector of a complete intersection. We remark that it is possible to have a reduced zero-dimensional scheme with the Hilbert function of a complete intersection, that does not have WLP. For example, for the h -vector $(1, 3, 4, 4, 3, 1)$, which is that of a complete intersection of type $(2, 2, 4)$, one can take the union, Z , of a general set of points in the plane with h -vector $(1, 2, 3, 4, 3, 1)$ (easily produced with liaison) and two general points in \mathbb{P}^3 , to produce a set of points with the desired h -vector. One can check that the Artinian reduction of R/I_Z has a socle element in degree 2, so the multiplication from the component in degree 2 to the component in degree 3 (both of which are 4-dimensional) induced by a general linear form has no chance to be injective or surjective, i.e. Z does not have WLP. We know that a complete intersection in \mathbb{P}^3 has Artinian reduction with WLP (cf. [Harima-Migliore-Nagel-Watanabe]), but we do not know if this example is contained in the same component as the complete intersection or not. \square

Remark 3.5. In the preceding example, note that Z_1 has UPP, while Z_2 is very far from having UPP, being the lifting of a monomial ideal. This motivates the following natural questions:

1. Does a set of points with UPP automatically have WLP?
2. Does the general hypersurface section of a smooth curve necessarily have WLP (we know that it does have UPP, at least in characteristic zero)?
3. Does the general hyperplane section of a smooth curve necessarily have WLP? Of course if the curve is in \mathbb{P}^3 then the hyperplane section is in \mathbb{P}^2 , so the Artinian reduction is an ideal in $k[x_1, x_2]$, and WLP and SLP are automatic (as we mentioned above). So

the first place that this question is interesting is for a smooth curve in \mathbb{P}^4 .

4. Does the Artinian reduction of every reduced, arithmetically Gorenstein set of points have WLP?
 - It is not true that every Artinian Gorenstein ideal in codimension ≥ 5 has WLP, because the h -vector can fail to satisfy condition (1) on page 10 (cf. for instance [Bernstein-Iarrobino], [Boij-Laksov]). But the question is open for the Artinian reduction of arithmetically Gorenstein points in any codimension. (Part of what is missing is knowledge of what Artinian algebras lift to reduced sets of points.)
 - It is an open question whether all height 3 Artinian Gorenstein ideals have WLP. It is true for all height 3 complete intersections (cf. [Harima-Migliore-Nagel-Watanabe]).

In [Ahn-Migliore] we were able to answer the first three of the above questions in the negative, essentially with one example (cf. [Ahn-Migliore], Examples 6.9 and 6.10). We omit the details here, but the basic idea is as follows. We consider a smooth arithmetically Buchsbaum curve whose deficiency module

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t))$$

is one-dimensional in degrees 3 and 4. Any linear form induces a homomorphism from $H^1(\mathbb{P}^3, \mathcal{I}_C(3))$ to $H^1(\mathbb{P}^3, \mathcal{I}_C(4))$. The condition “arithmetically Buchsbaum” means that this map is zero for all linear forms L . It is known that such a smooth curve exists (cf. [Bolondi-Migliore]). Now, we put a lot of points on such a curve, placed generically, and call the resulting set of points Z . Note that I_Z agrees with I_C in low degrees. We then play with the coordinate ring of C , the coordinate ring of Z , the cohomology of \mathcal{I}_C and the cohomology of \mathcal{I}_Z , and in the end we show that the Artinian reduction of R/I_Z does not have WLP. Tweaking this example somewhat (looking at the cone over C and taking a general hypersurface and hyperplane section) answers the remaining questions. \square

Some of the results given below, in particular those coming from the paper [Ahn-Migliore], are given in terms of *reduction numbers*, which we now define.

Definition 3.6 (adaptation of Hoa-Trung [Hoa-Trung]). Let $I \subset R$ be a homogeneous ideal. Let $m \geq \dim R/I$. The m -reduction number

of R/I is

$$\begin{aligned} r_m(R/I) &= \min\{k \mid x_{n-m}^{k+1} \in \text{Gin}(I)\} \\ &= \min\{k \mid h_{R/(I+J)} \text{ vanishes in degree } k+1\} \end{aligned}$$

where J is an ideal generated by m general linear forms. We will use the second line as the definition, and we include the first line only for completeness. (For the definition of, and results on, the generic initial ideal, $\text{Gin}(I)$, see for instance [Green2].) \square

Example 3.7. Let Z be a complete intersection of type $(3, 3, 4)$ in \mathbb{P}^3 and let $I = I_Z$. Note $\dim R/I = 1$. Let L_1, L_2, L_3 be general linear forms. It is known that any Artinian reduction of R/I has WLP (cf. [Harima-Migliore-Nagel-Watanabe]).

We have

$$\begin{aligned} h_{R/(I+(L_1))} &: 1 \ 3 \ 6 \ 8 \ 8 \ 6 \ 3 \ 1 &\Rightarrow r_1(R/I) = 7. \\ h_{R/(I+(L_1, L_2))} &: 1 \ 2 \ 3 \ 2 &\Rightarrow r_2(R/I) = 3 \quad (\text{by WLP}) \\ h_{R/(I+(L_1, L_2, L_3))} &: 1 \ 1 \ 1 &\Rightarrow r_3(R/I) = 2 \quad (\text{by WLP}) \end{aligned}$$

The first use of WLP came because we had a complete intersection; the second came because we were in a ring with two variables. So we see that having WLP is extremely helpful in computing the reduction numbers. \square

4. SETTING THE STAGE

The main goal of this paper is to sketch a progression of results on Hilbert functions for sets of points, starting in \mathbb{P}^2 (mostly due to Davis, cf. [Davis]), then moving to some of the generalizations of these results to higher projective space obtained by Bigatti, Geramita and myself (cf. [Bigatti-Geramita-Migliore]), and finally recent extensions of some of these results obtained by Ahn and myself (cf. [Ahn-Migliore]). In Section 5 we will focus on the general case, while in Section 6 we will specialize to the case of UPP. For simplicity of exposition we will usually focus on reduced schemes, but many of these results extend to the non-reduced case.

Let Z be a zero-dimensional scheme. The central assumption in this paper will be the following:

$$(2) \quad \Delta h_Z(d) = \Delta h_Z(d+1) = s, \text{ for some } d, s.$$

In both Section 5 and Section 6, we will be focusing on two situations:

- (a) $d \geq s$ (which was used in [Bigatti-Geramita-Migliore])

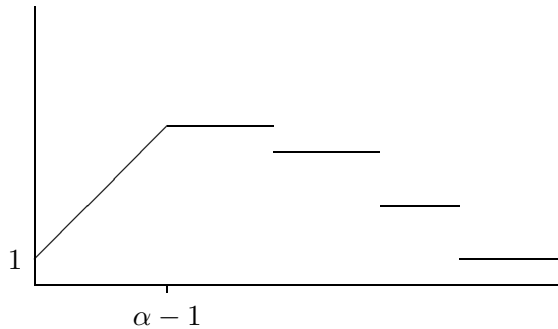
(b) $d > r_2(R/I_Z)$ (which was used in [Ahn-Migliore]).

We will see that (a) corresponds to a certain kind of maximal growth (see Remark 4.4), while (b) in general does not (see Remark 4.2). This is the striking aspect of the results centered around (b) (see especially Theorem 5.4 and Theorem 6.5), that strong results can be obtained even without the power of the Gotzmann machinery behind us. In this section we make a series of remarks in preparation for the discussion in the coming sections.

Remark 4.1. In Example 3.7 we have (2), but neither $d \geq s$ nor $d > r_2(R/I_Z)$ holds! (Neither do the conclusions that we will mention in the coming sections.) \square

Remark 4.2. What is the connection between (a) and (b) above? If $d \geq s$ then it can be shown that $d > r_2(R/I_Z)$ (see [Ahn-Migliore] Remark 4.4). But in general $d > r_2(R/I_Z)$ does *not* imply $d \geq s$, and indeed it can happen that $d > r_2(R/I_Z)$ but d is much smaller than s . In Example 3.7, $s = 8$ and $r_2(R/I_Z) = 3$. \square

Remark 4.3. We now remark that in \mathbb{P}^2 , condition (2) essentially always corresponds to case (a) (there is only one, very special, exception). Assume that $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme. For $d \leq \alpha - 1$ we have $\Delta h_Z(d) = d + 1$ and for $d \geq \alpha$ we have that Δh_Z is non-increasing, so in particular $\Delta h_Z(d) \leq \alpha$.



Of course there may not be any flat part at all, but the point is that the function cannot increase past degree $\alpha - 1$. In any case we have

$$[\Delta h_Z(d) = \Delta h_Z(d + 1) = s] \Rightarrow \begin{cases} d = s - 1, & \text{if } d = \alpha - 1 \text{ and } s = \alpha \\ d \geq s, & \text{if } d \geq \alpha \end{cases}$$

In the first case in the above calculation, we have that $\dim(I_Z)_\alpha = 1$. Notice that in this case I_Z is zero in degree $d = \alpha - 1$, but $(I_Z)_{d+1} = (I_Z)_\alpha$ defines a hypersurface (or equivalently a curve) in \mathbb{P}^2 . \square

Remark 4.4. Assume that $Z \subset \mathbb{P}^n$ is a zero-dimensional scheme. If $d \geq s$ then the condition $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ implies that the growth of the Artinian reduction from degree d to degree $d+1$ is maximal in the sense of Definition 2.5. Indeed, the d -binomial expansion of s is

$$s = \binom{d}{d} + \cdots + \binom{d-s+1}{d-s+1}$$

so

$$s^{(d)} = \binom{d+1}{d+1} + \cdots + \binom{d-s+2}{d-s+2} = s$$

as claimed. \square

Remark 4.5. In practice, it usually is the case that Δh_Z having maximal growth from degree d to degree $d+1$ does *not* imply that h_Z itself has maximal growth from degree d to degree $d+1$. If it did, Gotzmann's results (Theorem 2.11 and Remark 2.12) would immediately apply to h_Z . The interest in the results described below is that similar powerful conclusions come from this maximal growth of the first difference (see Proposition 2.14 and Remark 2.15). And as we will see, even more surprising is that similar results can be deduced at times even when the first difference does not have maximal growth. \square

5. GENERAL RESULTS

A by-now classical result of Davis (cf. [Davis]) is the following. Note that there is no uniformity assumption on Z . In the next section we will discuss the refinements that are possible when we assume UPP.

Theorem 5.1 (Davis). *Let $Z \subset \mathbb{P}^2$ be a zero-dimensional scheme. If $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for $d \geq s$, then $(I_Z)_d$ and $(I_Z)_{d+1}$ both have a GCD, F , of degree s . If Z is reduced then so is F . The polynomial F defines a $\begin{cases} \text{hypersurface} \\ \text{curve} \end{cases}$ in \mathbb{P}^2 . If $Z_1 \subset Z$ is the subscheme of Z lying*

on F (defined by $[I_Z + (F)]^{\text{sat}}$) and Z_2 is the "residual" scheme defined by $[I_Z : F]$, then there are formulas relating the Hilbert functions of I_Z, I_{Z_1} and I_{Z_2} .

Remark 5.2. It is worth mentioning that the Artinian version of Theorem 5.1 was proved earlier, by A. Iarrobino (cf. [Iarrobino2], page 56). \square

The paper [Bigatti-Geramita-Migliore] began with the observation that Davis' result, Theorem 5.1, was really about maximal growth of the function $\Delta h_Z(t)$ from degree d to degree $d + 1$, thanks to Remark 4.3 and Remark 4.4 above. These remarks show that in \mathbb{P}^2 , Δh_Z can have two different kinds of maximal growth: either h_Z takes on the value of the polynomial ring (before the initial degree of the ideal), or else Δh_Z takes the constant value $s \geq d$. This latter is the only interesting case.

Notice that since Z is a zero-dimensional scheme, eventually the Hilbert function is constant, so eventually Δh_Z is zero. So the condition that $\Delta h_Z(d) = \Delta h_Z(d + 1) = s$ directly gives us information not so much about I_Z as about $(I_Z)_{\leq d}$, the ideal generated by the components of degree $\leq d$. We have to deduce properties of I_Z from this, via Proposition 2.14. Gotzmann's results tell us that in a general Artinian reduction of R/I_Z (which has Hilbert function Δh_Z), the degree d component of the ideal $J = \frac{(I_Z, L)}{(L)}$ agrees with the degree d component of the saturated ideal of a subscheme of degree s in the line defined by L (cf. Remark 2.12). Since this corresponds to a hyperplane section by L , that means that the base locus of the linear system $|I_d|$ contains a curve of degree s , which is the GCD in question. The results about the subscheme of Z lying on this GCD and the "residual" subscheme come from some algebraic arguments chasing exact sequences.

As noted in the way we phrased Davis' theorem, this GCD can be viewed as a curve, or as a hypersurface in \mathbb{P}^2 . These correspond, respectively, to the interpretations (that happen to coincide in this case) that the maximal growth for Δh_Z takes the constant value $s \leq d$, and it takes the largest possible growth short of being equal to the Hilbert function of the whole polynomial ring itself.

In [Bigatti-Geramita-Migliore] we extended this to \mathbb{P}^{n-1} , and we called these kinds of maximal growth *growth like a curve*, and *growth like a hypersurface*, respectively. We proved many results for $Z \subset \mathbb{P}^{n-1}$. In particular, we showed that viewing F in Theorem 5.1 as a curve, and viewing it as a hypersurface, both extend in separate directions when we move to higher projective spaces.

In this paper we focus on the former. We now give a summary of the results in the "curve" direction found in [Bigatti-Geramita-Migliore] that do not assume UPP.

Theorem 5.3 ([Bigatti-Geramita-Migliore]). *Let $Z \subset \mathbb{P}^{n-1}$ be a reduced zero-dimensional scheme. Assume that $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for some $d \geq s$. Then*

- (a) $\langle (I_Z)_{\leq d} \rangle$ is the **saturated** ideal of a curve V of degree s . V is not necessarily unmixed, but it is **reduced** and **d -regular**.

Let C be the unmixed one-dimensional part of V . Let Z_1 be the subscheme of Z lying on C (defined by $[I_Z + I_C]^{sat}$) and Z_2 the residual scheme (defined by $[I_Z : I_C]$). Then

- (b) $\langle (I_{Z_1})_{\leq d} \rangle = I_C$ and I_C is d -regular. (Part (a) was for V . This one is not surprising, since V consists of C plus some points.)
(c) There are formulas relating the Hilbert functions.

How might these results be improved? Recall that $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for $d \geq s$ guarantees that Δh_Z has maximal growth from degree d to degree $d+1$.

1. Weaken the condition $\Delta h_Z(d) = \Delta h_Z(d+1) = s$, e.g. to the condition $\Delta h_Z(d) = \Delta h_Z(d+1)+1$ (possibly even maintaining the assumption $d \geq s$). It may be that at least “usually” something similar will hold. But there will be important differences. This is still an open direction, although in her thesis Susan Cooper (cf. [Cooper]), a student of Tony Geramita, is working on questions related to this.
2. Weaken the assumption $d \geq s$. This is the approach we take. We will assume only $d > r_2(R/I_Z)$. This is a beautiful idea, conceived by my co-author, Jeaman Ahn. It is very striking to see how much carries over to this case.

Theorem 5.4 ([Ahn-Migliore]). *Let $Z \subset \mathbb{P}^{n-1}$ be a reduced zero-dimensional scheme. Assume that $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for some $d > r_2(R/I_Z)$. Then*

- (a) $\langle (I_Z)_{\leq d} \rangle$ is the saturated ideal of a curve V of degree s . V is not necessarily unmixed or reduced, but it is d -regular.

Let C be the unmixed one-dimensional part of V . Let Z_1 be the subscheme of Z lying on C (defined by $[I_Z + I_C]^{sat}$) and Z_2 the residual scheme (defined by $[I_Z : I_C]$). Then

- (b) $\langle (I_{Z_1})_{\leq d} \rangle = I_C$ and C is d -regular.
(c) There are formulas relating the Hilbert functions.
(d) **If we also assume that $h^1(\mathcal{I}_{C_{red}}(d-1)) = 0$ then V is reduced and $C = C_{red}$ is d -regular.**

Remark 5.5. We stress that in (a), we do not necessarily obtain that V is reduced. This is the first new twist. If V is not reduced, then the

top dimensional part, C , is not necessarily reduced. This curve C is d -regular. If it is not reduced, however, it is supported on a reduced curve, C_{red} . Surprisingly, C_{red} being d -regular does not follow from C being d -regular, and there are examples where it is not true. But with the extra assumption, (d) delivers this conclusion. Example 7.3 below illustrates what can happen. \square

Theorem 5.4 combines different results of [Ahn-Migliore]. The proofs heavily use generic initial ideals and results of Green, Bayer, Stillman, Galligo, etc.

We end this section with a result that incorporates WLP and also higher differences of the Hilbert function and higher reduction number.

Theorem 5.6 ([Ahn-Migliore] Theorem 6.7). *Let Z be a zero-dimensional subscheme of \mathbb{P}^{n-1} , $n > 3$, with WLP. Suppose that*

$$\Delta^2 h_Z(d) = \Delta^2 h_Z(d+1) = s$$

for $r_2(R/I_Z) > d > r_3(R/I_Z)$. Then $\langle (I_Z)_{\leq d} \rangle$ is a saturated ideal defining a two-dimensional subscheme of degree s in \mathbb{P}^{n-1} , and it is d -regular.

6. RESULTS ON UNIFORM POSITION

We now investigate the effects of assuming that our points have UPP. We begin again in \mathbb{P}^2 .

Theorem 6.1. *Let $Z \subset \mathbb{P}^2$ be a reduced set of points with UPP. Then we have*

- (a) ([Geramita-Maroscia], [Maggioni-Ragusa]) *The component of I_Z of least degree, α , contains an irreducible form (hence the general such form is irreducible). In particular, this holds if there is only one such form, up to scalar multiple. This is in fact true not only in \mathbb{P}^2 but also in \mathbb{P}^{n-1} .*
- (b) ([Harris]) *Δh_Z is of decreasing type, i.e.*

$$\text{if } \Delta h_Z(d) > \Delta h_Z(d+1) \text{ then } \Delta h_Z(t) > \Delta h_Z(t+1)$$

for all $t \geq d$ as long as $\Delta h_Z(t) > 0$.

- (c) *If $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ then $s = \alpha$ (see Remark 4.3), $(I_Z)_t = (F)_t$ for all $t \leq d+1$, and the points of Z all lie on the irreducible curve defined by F .*

As before, this theorem was also extended in [Bigatti-Geramita-Migliore] to \mathbb{P}^{n-1} for the case of “maximal growth like a hypersurface,” but here we focus on extending it for “maximal growth like a curve.”

(See the discussion preceding Theorem 5.3.) In the following we repeat part of Theorem 5.3 and underline the new features.

Theorem 6.2 ([Bigatti-Geramita-Migliore]). *Let $Z \subset \mathbb{P}^{n-1}$ be a reduced zero-dimensional scheme with UPP. Assume that $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for some $d \geq s$. Then*

- (a) $\langle (I_Z)_{\leq d} \rangle$ is the saturated ideal of a curve V of degree s . V is reduced, d -regular, unmixed and irreducible.
- (b) Since V is unmixed, its top dimensional part C is equal to V . Hence $Z \subset C$.
- (c) $\langle (I_Z)_{\leq d} \rangle = I_C$ and I_C is d -regular.

Remark 6.3. The hardest part is irreducibility, and it strongly uses the assumption $d \geq s$. \square

Still under the hypothesis of [Bigatti-Geramita-Migliore] that $d \geq s$, the following extension of “decreasing type” was observed in [Ahn-Migliore]. This completes the picture of extending the \mathbb{P}^2 result to higher projective space under the assumption of “maximal growth like a curve.”

Corollary 6.4 ([Ahn-Migliore]). *If $Z \subset \mathbb{P}^{n-1}$ has UPP and $\Delta h_Z(d) = \Delta h_Z(d+1) > \Delta h_Z(d+2)$ for some $d \geq s$, then $\Delta h_Z(t) > \Delta h_Z(t+1)$ for all $t \geq d+1$ as long as $\Delta h_Z(t) > 0$.*

We now turn to the situation of [Ahn-Migliore], where we only assume that $d > r_2(R/I_Z)$. It is somewhat surprising to see what we retain and what we lose. Compare this with Theorem 5.4 (part of which is repeated here).

Theorem 6.5 ([Ahn-Migliore]). *Let $Z \subset \mathbb{P}^{n-1}$ be a reduced zero-dimensional scheme with UPP. Assume that $\Delta h_Z(d) = \Delta h_Z(d+1) = s$ for some $d > r_2(R/I_Z)$. Then*

- (a) $\langle (I_Z)_{\leq d} \rangle$ is the saturated ideal of an unmixed curve, V , of degree s , and it is d -regular. V is not necessarily reduced or irreducible.
- (b) Since V is unmixed, its top dimensional part C is equal to V . Hence $Z \subset C$.
- (c) $\langle (I_Z)_{\leq d} \rangle = I_C$ and I_C is d -regular.
- (d) **If we also assume that $h^1(\mathcal{I}_{C_{red}}(d-1)) = 0$ then V is reduced and $C = C_{red}$ is d -regular.**

Remark 6.6. Why might we have hoped that “reduced and irreducible” would still hold in part (a) of Theorem 6.5? Recall the following.

- We saw in Theorem 6.1 that if $Z \subset \mathbb{P}^{n-1}$ is a set of points in \mathbb{P}^{n-1} with UPP, it is known that a general element of smallest degree

α is reduced and irreducible. When this element is unique (up to scalar multiple), say F , we have $\langle (I_Z)_{\leq \alpha} \rangle$ is the saturated ideal of a reduced, irreducible hypersurface which is α -regular. This is the base locus in degree α , and just having UPP is enough to guarantee the irreducibility.

- Recall from Theorem 6.2 that if $d \geq s$ then it is true that the base locus, C , is reduced and irreducible.

In the next section we will give examples to show that in fact “reduced and irreducible” does not necessarily hold (as opposed to merely being a gap in the theorem). \square

7. EXAMPLES

These examples will serve to illustrate that some of our results above are apparently close to optimal. We omit some details, and refer the reader to [Ahn-Migliore]. We will see that

1. In contrast to Corollary 6.4, for a set of points in \mathbb{P}^{n-1} with UPP satisfying $\Delta h_Z(d) = \Delta h_Z(d+1) > \Delta h_Z(d+2)$ for $d > r_2(R/I_Z)$ (instead of $d \geq s$), it is not necessarily true that Δh_Z is strictly decreasing beyond this point. This is shown in Example 7.3.
2. In Theorem 6.5, when we assume only $d > r_2(R/I_Z)$, it is in fact true (as claimed) that V is not necessarily reduced or irreducible. This stands in contrast to Theorem 6.2, when we assumed $d \geq s$. This is shown in Example 7.3.
3. Upon realizing that in Theorem 6.5 V is not necessarily reduced and irreducible, one might hope that at least the points of Z are restricted to one component of V that is reduced and irreducible. Examples 7.4 and 7.5 show that even this is not true.
4. Examples 7.6 and 7.7 illustrate some of the difficulties of extending these results past the case $d > r_2(R/I_Z)$, showing that these results are optimal in a sense.

We begin with an example of Chardin and D’Cruz. It settles an old question of independent interest.

Example 7.1 ([Chardin-D’Cruz]). Consider the family of complete intersection ideals

$$I_{m,n} := (x^m t - y^m z, z^{n+2} - x t^{n+1}) \subset k[x, y, z, t].$$

Then for $m, n \geq 1$, $\text{reg}(I_{m,n}) = m + n + 2$ while $\text{reg}(\sqrt{I_{m,n}}) = mn + 2$. Hence the regularity of the radical may be much larger than the regularity of the ideal itself!! In particular, if we take $m = n = 4$

then we obtain the following: $I_{4,4}$ is a complete intersection of type $(5, 6)$. As such, it has degree 30, and has a Hilbert function whose first difference is

$$\begin{array}{c|cccccccccccccc} \text{deg} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \Delta h & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 27 & 29 & 30 & 30 & \dots \end{array}$$

On the other hand, $\sqrt{I_{4,4}}$ can be computed on a computer program, e.g. `macaulay` (cf. [Bayer-Stillman]). It has degree 26 (hence $I_{4,4}$ is not reduced), and its Hilbert function has first difference

$$\begin{array}{c|cccccccccccccccccc} \text{deg} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline \Delta h & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 27 & 29 & 29 & 29 & 29 & 28 & 28 & 28 & 27 \\ \\ \text{deg} & 16 & 17 & 18 & 19 & 20 & 21 \\ \hline \Delta h & 27 & 27 & 26 & 26 & 26 & 26 & \dots \end{array}$$

In [Chardin-D’Cruz], the authors were not so concerned with the geometry of this curve. We will modify this slightly, taking a more geometric approach, in Example 7.3. \square

Remark 7.2. Example 7.1 shows that a curve $C \subset \mathbb{P}^3$ can have the property that $\text{reg}(\sqrt{I_C})$ may be (much) larger than $\text{reg}(I_C)$. However, it seems to still be an open question whether this can happen, for instance, for a smooth curve. Chardin and D’Cruz also study the surface case, where other interesting phenomena occur. For curves, see also Ravi [Ravi]. \square

Example 7.3 ([Ahn-Migliore] Example 5.7). In this example we recall that k has characteristic zero. We will be basing our example on the case $m = n = 4$ of the example of Chardin and D’Cruz. It is obtained by taking a geometric interpretation. Let $R = k[x, y, z, t]$.

Let $I_\lambda = (z, t)$. Let $F \in (I_\lambda)_5$ be a homogeneous polynomial that is smooth along λ . Let $I' = I_\lambda^5 + (F)$. I' is the saturated ideal of a non-ACM curve of degree 5 corresponding to the divisor $D := 5\lambda$ on F . In particular, viewed as a subscheme of \mathbb{P}^3 , D has degree 5.

Now, choose a general element C in the linear system $|6H - D|$ on F . C is smooth and irreducible. Furthermore, C has degree 25 (by liaison) and Hilbert function with first difference

$$\begin{array}{c|cccccccccccccccccc} \text{deg} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \hline \Delta h & 1 & 3 & 6 & 10 & 15 & 20 & 24 & 27 & 29 & 29 & 29 & 29 & 28 & 28 & 28 & 27 \\ \\ \text{deg} & 16 & 17 & 18 & 19 & 20 & 21 \\ \hline \Delta h & 27 & 27 & 26 & 26 & 26 & 25 & \dots \end{array}$$

We now let Z consist of sufficiently many points of C , chosen generally, so that I_Z agrees with I_C up to and including degree 21. Z has UPP (since C is smooth), and one checks that $r_2(R/I_Z) = r_2(R/I_C) = 8$. Taking $d = 9$, we see that Theorem 6.5 applies. We obtain that $\langle (I_Z)_{\leq 9} \rangle$ is the saturated ideal of an unmixed curve V of degree 29 consisting of the union of C and a subcurve of D of degree 4 supported on λ , hence V is neither irreducible nor reduced. Taking $d = 12$, $d = 15$, $d = 18$ slices away at the non-reduced part, and taking $d = 21$ gives just C .

A computation on `macaulay` reveals that in fact the regularity of I_C is 21. On the other hand, the curve V of degree 29 has ideal I_V with regularity 9. Actually, it is worth recording the Betti diagram of I_C and I_V , because they display a surprising pattern: For I_V we have

total:	1	3	2
0:	1	-	-
1:	-	-	-
2:	-	-	-
3:	-	-	-
4:	-	1	-
5:	-	1	-
6:	-	-	-
7:	-	-	-
8:	-	1	2

while for I_C we have

total:	1	7	10	4
0:	1	-	-	-
1:	-	-	-	-
2:	-	-	-	-
3:	-	-	-	-
4:	-	1	-	-
5:	-	1	-	-
6:	-	-	-	-
7:	-	-	-	-
8:	-	1	2	-
9:	-	-	-	-
10:	-	-	-	-
11:	-	1	2	1
12:	-	-	-	-
13:	-	-	-	-
14:	-	1	2	1
15:	-	-	-	-
16:	-	-	-	-
17:	-	1	2	1
18:	-	-	-	-
19:	-	-	-	-

20: - 1 2 1

Notice that the diagram for I_V is a subdiagram of the diagram for I_C , and that there is a striking simplicity to the diagram for I_C . Notice also that V is ACM, while C is not. \square

Note that in the previous example, the points of Z all lie on one reduced, irreducible curve (C). It is simply the case that V contains another component, which happens to also be non-reduced. However, now we give another example to show that it is not necessarily true that all the points lie on one irreducible component of the base locus.

Example 7.4 ([Ahn-Migliore] Example 5.8). In Example 7.3, instead of choosing “sufficiently many” points on C , instead choose Z to consist of 192 general points on C and one general point of λ . The first difference of the Hilbert function of Z is

deg	0	1	2	3	4	5	6	7	8	9	10	11
Δh_Z	1	3	6	10	15	20	24	27	29	29	29	0

Note that this is exactly the same as what we would have if we had taken 193 general points of C . Again, $r_2(R/I_Z) = 8$. The base locus of $(I_C)_9$ and $(I_C)_{10}$ is exactly the non-reduced and reducible curve of degree 29 mentioned in Example 7.3, so the Hilbert function of Z is the truncation of the Hilbert function given above. And by the general choice of the points, this will continue to be true regardless of which subsets we take. Hence Z has UPP and satisfies $\Delta h_Z(d) = \Delta h_Z(d + 1) = s$ for some $d > r_2(R/I_Z)$, but not all of the points of Z lie on one reduced and irreducible component of the curve of degree s obtained by our result (since one point lies on the non-reduced component). \square

Example 7.5. [Ahn-Migliore] Example 5.9] We produced Example 7.4 by taking almost all of the points on C , and just one off of C (in fact, it was on λ). In this example we show a surprising instance where half of the points of V are on one irreducible component and the other half are on another irreducible component, but we still have UPP. We again omit some of the technical details.

Let Q be a smooth quadric surface in \mathbb{P}^3 , and as usual by abuse of notation we use the same letter Q to denote the quadratic form defining this surface. Let C_1 be a *general* curve on Q of type $(1, 15)$, and let C_2 be a *general* curve on Q of type $(15, 1)$. Hence both C_1 and C_2 are smooth rational curves of degree 16, and $C := C_1 \cup C_2$ is the complete intersection of Q and a form of degree 16. Note that C is arithmetically Cohen-Macaulay, but C_1 and C_2 are not.

It is not difficult to compute the Hilbert functions of these curves. We record their first differences (of course there is no difference between behavior of C_1 and behavior of C_2 ; this is important in the argument given in [Ahn-Migliore]):

degree	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Δh_C	1	3	5	7	9	11	13	15	17	19	21	23	25	27
Δh_{C_i}	1	3	5	7	9	11	13	15	17	19	21	23	25	27

degree	14	15	16	17	18
Δh_C	29	31	32	32	32
Δh_{C_i}	29	16	16	16	16

We now observe:

1. These first differences (hence the ideals themselves) agree through degree 14, and in fact the only generator before degree 15 is Q .
2. By adding these values, we see that $h_C(18) = 352$ and $h_C(15) = 241$.
3. Since C and C_i are curves, these values represent the Hilbert functions of $I_C + (L)$ and $I_{C_i} + (L)$ for a general linear form L .
4. $r_2(R/I_C) = 16$ since C is an arithmetically Cohen-Macaulay curve.

Let Z_1 (respectively Z_2) be a general set of 176 points on C_1 (respectively C_2). So $Z := Z_1 \cup Z_2$ is a set of 352 points whose Hilbert function agrees with that of C through degree 18. In particular, we have $\Delta h_Z(17) = \Delta h_Z(18) = 32 = \deg C$. Furthermore, $r_2(R/I_Z) = 16$ by our observation 4. above. Hence Theorem 6.5 applies, and we indeed have that the component of I_Z in degree 17 defines C . However, C is not irreducible. In [Ahn-Migliore] Example 5.9, it is shown that Z has the Uniform Position Property. Thus there is no chance of showing that all the points must lie on a unique irreducible component in Theorem 6.5, under our hypothesis that $d > r_2(R/I_Z)$ (as was done in [Bigatti-Geramita-Migliore] when $d \geq s$).

To show UPP, it is enough to show that the union of any choice of t_1 points of Z_1 (i.e. t_1 general points of C_1 for $t_1 \leq 176$) and t_2 points of Z_2 (i.e. t_2 general points of C_2 for $t_2 \leq 176$) has the truncated Hilbert function. For example, if $t_1 = 150$ and $t_2 = 160$ then we have to show that $\Delta H(R/I_Z)$ has values

$$1 \ 3 \ 5 \ 7 \ 9 \ 11 \ 13 \ 15 \ 17 \ 19 \ 21 \ 23 \ 25 \ 27 \ 29 \ 31 \ 32 \ 22 \ 0.$$

Notice that we know that some subset has this Hilbert function, by [Geramita-Maroscia-Roberts]. We have to show that *all* subsets have

this Hilbert function. See [Ahn-Migliore], Example 5.9, for the details. \square

Example 7.6. We saw that the condition $d \geq s$ of [Bigatti-Geramita-Migliore] was improved in [Ahn-Migliore] to $d > r_2(R/I_Z)$. There was some loss in the strength of the result, but a surprising amount of it did go through. One might wonder if the condition $d > r_2(R/I_Z)$ can be further improved. But in fact, we saw already in Example 3.7 and Remark 4.1 that this is not the case.

Similarly, we have the following examples:

7 general points in \mathbb{P}^3 have h -vector 1 3 3
 16 general points in \mathbb{P}^3 have h -vector 1 3 6 6
 30 general points in \mathbb{P}^3 have h -vector 1 3 6 10 10
 etc.

In each case, $r_2(R/I_Z)$ has the “expected” value, say d . We have $\Delta h_Z(d) = \Delta h_Z(d+1)$ for $d = r_2(R/I_Z)$, but clearly the base locus of $(I_Z)_d$ is not one-dimensional. See Remark 2.15 as well.

Similar examples can easily be found in higher projective spaces. A different sort of example can be found in Example 3.7. \square

Example 7.7. We return to Example 3.4 to see how the theorems mentioned above apply to that example. Recall that we have two ideals, I_1 and I_2 , both with h -vector $(1, 3, 6, 9, 11, 11, 11)$. Taking $s = 11$, it is clear that the results from [Bigatti-Geramita-Migliore] (Theorem 5.3 and Theorem 6.2) do not apply because of the hypothesis $d \geq s$. As for the results of [Ahn-Migliore] (Theorem 5.4 and Theorem 6.5), one can check that $r_2(R/I_1) = 4$ while $r_2(R/I_2) = 5$. Hence both results from [Ahn-Migliore] apply to I_1 , taking $d = 5$, but not to I_2 . And indeed, we have seen that $\langle (I_1)_{\leq 5} \rangle$ is the saturated ideal of the curve C described in that example. However, it can easily be checked that $\langle (I_2)_{\leq 5} \rangle$ is saturated, but it defines a curve of degree 10 rather than 11 (and it is not unmixed, although the unmixed part is ACM; in fact it is a complete intersection of type $(2, 5)$). \square

REFERENCES

- [Ahn-Migliore] J. Ahn and J. Migliore, *Some Geometric Results arising from the Borel Fixed Property*, preprint 2004.
- [Anick] D. Anick, *Thin Algebras of embedding dimension three*, *J. Algebra* **100** (1986), 235–259.
- [Bayer-Stillman] D. Bayer and M. Stillman, *Macaulay: A system for computation in algebraic geometry and commutative algebra*. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from <ftp://math.harvard.edu> via anonymous ftp.

- [Bernstein-Iarrobino] D. Bernstein and A. Iarrobino, *A nonunimodal graded Gorenstein Artin algebra in codimension five*, *Comm. Alg.* **20(8)** (1992), 2323–2336.
- [Bigatti-Geramita-Migliore] A. Bigatti, A.V. Geramita and J. Migliore, *Geometric Consequences of Extremal Behavior in a Theorem of Macaulay*, *Trans. Amer. Math. Soc.* **346** (1994), 203–235.
- [Boij-Laksov] M. Boij and D. Laksov, *Nonunimodality of Graded Gorenstein Artin Algebras*, *Proc. Amer. Math. Soc.* **120** (1994), 1083–1092.
- [Bolondi-Migliore] G. Bolondi and J. Migliore, *Buchsbaum Liaison Classes*, *J. Alg.* **123** (1989), 426–456.
- [Bruns-Herzog] W. Bruns and J. Herzog, “Cohen-Macaulay Rings, revised edition,” *Cambridge studies in advanced mathematics #39*, Cambridge University Press, Cambridge, 1998.
- [Chardin-D’Cruz] M. Chardin and C. D’Cruz, *Castelnuovo-Mumford regularity: examples of curves and surfaces*, *J. Algebra* **270** (2003), no. 1, 347–360.
- [Cooper] S. Cooper, Ph.D. thesis, Queen’s University, 2005.
- [Davis] E.D. Davis, *Complete Intersections of Codimension 2 in \mathbb{P}^r : The Bezout-Jacobi-Segre Theorem Revisited*, *Rend. Sem. Mat. Univers. Politecn. Torino*, **43**, 4 (1985), 333–353.
- [Geramita-Maroscia] A.V. Geramita and P. Maroscia, *The Ideal of Forms Vanishing at a Finite Set of Points in \mathbb{P}^n* , *J. Alg.* **90** (1984), 528–555.
- [Geramita-Maroscia-Roberts] A.V. Geramita, P. Maroscia and L. Roberts, *The Hilbert Function of a Reduced k -Algebra*, *J. London Math. Soc.* **28** (1983), 443–452.
- [Geramita-Migliore] A.V. Geramita and J. Migliore, *Hyperplane Sections of a Smooth Curve in \mathbb{P}^3* , *Comm. Alg.* **17** (1989), 3129–3164.
- [Gotzmann] G. Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes* *Math. Z.* **158** (1978), 61–70.
- [Green1] M. Green, *Restrictions of Linear Series to Hyperplanes, and some results of Macaulay and Gotzmann*, SLN 1389, *Algebraic Curves and Projective Geometry*, Proceedings Trento, 1988, Springer-Verlag.
- [Green2] M. Green, *Generic Initial Ideals*, in *Six lectures on Commutative Algebra*, (Elias J., Giral J.M., Miró-Roig, R.M., Zarzuela S., eds.), *Progress in Mathematics* **166**, Birkhäuser, 1998, 119–186.
- [Harima-Migliore-Nagel-Watanabe] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The Weak and Strong Lefschetz Properties for Artinian k -algebras*, *J. Algebra* **262** (2003), 99–126.
- [Harris] J. Harris, *The Genus of Space Curves*, *Math. Ann.* **249** (1980), 191–204.
- [Harris-Eisenbud] J. Harris, D. Eisenbud, *Curves in Projective Space*, *Sem. de Mathématiques Supérieures– Université de Montréal* (1982).
- [Hartshorne] R. Hartshorne, *Connectedness of the Hilbert scheme*, *Math. Inst. des Hautes Etudes Sci.* **29** (1966), 261–304.
- [Hoa-Trung] L.T. Hoa and N.V. Trung, *Borel-fixed ideals and reduction number*, *J. Algebra* **270** (2003), no. 1, 335–346.
- [Iarrobino1] A. Iarrobino, “Associated graded algebra of a Gorenstein Artin algebra,” *Memoirs Amer. Math. Soc.* **107** (1994).
- [Iarrobino2] A. Iarrobino, “Punctual Hilbert Schemes,” *Memoirs Amer. Math. Soc.* **10**, no. 188 (1977).

- [Iarrobino-Kanev] A. Iarrobino and V. Kanev, “Power Sums, Gorenstein Algebras, and Determinantal Loci,” Springer LNM **1721** (1999).
- [Kleppe] J. Kleppe, *Maximal families of Gorenstein algebras*, to appear in Trans. Amer. Math. Soc.
- [Macaulay] F.S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. Lond. Math. Soc. **26** (1927) 531–555.
- [Maggioni-Ragusa] R. Maggioni and A. Ragusa, *Connections between Hilbert Function and Geometric Properties for a Finite Set of Points in \mathbb{P}^2* , Le Matematiche **34**, Fasc. I-II (1984), 153–170.
- [Migliore1] J. Migliore, “Introduction to Liaison Theory and Deficiency Modules,” Birkhäuser, Progress in Mathematics 165, 1998; 224 pp. Hardcover, ISBN 0-8176-4027-4.
- [Migliore2] J. Migliore, *On Linking Double Lines*, Trans. Amer. Math. Soc. **294** (1986), 177–185.
- [Migliore-Miró-Roig1] J. Migliore and R. Miró-Roig, *On the minimal free resolution of $n + 1$ general forms*, Trans. Amer. Math. Soc. **355** No. 1 (2003), 1–36.
- [Migliore-Miró-Roig2] J. Migliore and R. Miró-Roig, *Ideals of general forms and the ubiquity of the Weak Lefschetz Property*, J. Pure Appl. Algebra **182** (2003), 79–107.
- [Migliore-Miró-Roig-Nagel] J. Migliore, R. Miró-Roig and U. Nagel, *Minimal Resolution of relatively compressed level algebras*, to appear in J. Algebra.
- [Migliore-Nagel] J. Migliore and U. Nagel, *Lifting Monomial Ideals*, Comm. Algebra **28** (2000) (special volume in honor of Robin Hartshorne), 5679–5701.
- [Ragusa-Zappalà] A. Ragusa and G. Zappalà, *On the reducibility of the postulation Hilbert scheme*, Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo LIII (2004), pp. 401–406
- [Ravi] M.S. Ravi, *Regularity of Ideals and their Radicals*, manuscripta math. **68** (1990), 77–87.
- [Richert] B. Richert, *Smallest graded Betti numbers*, J. Algebra **244** (2001), 236–259.

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