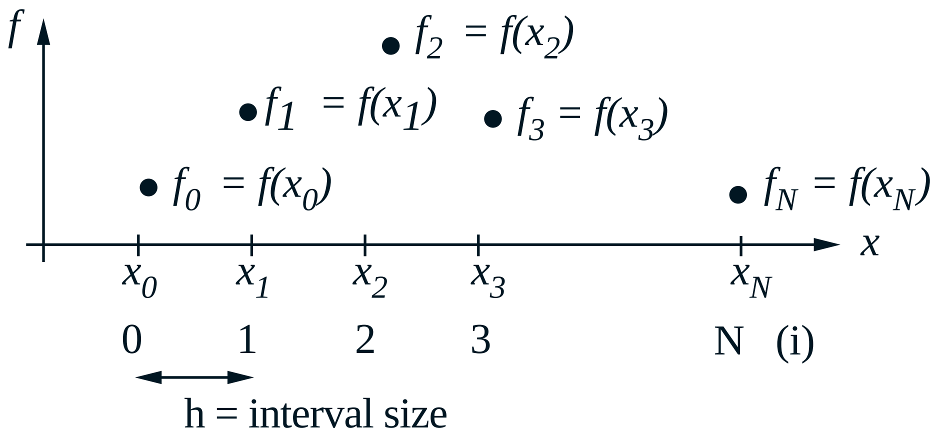
LECTURE 7

NEWTON FORWARD INTERPOLATION ON EQUISPACED POINTS

- Lagrange Interpolation has a number of disadvantages
  - The amount of computation required is large
  - Interpolation for additional values of x requires the same amount of effort as the first value (i.e. no part of the previous calculation can be used)
  - When the number of interpolation points are changed (increased/decreased), the results of the previous computations can not be used
  - Error estimation is difficult (at least may not be convenient)
- Use Newton Interpolation which is based on developing difference tables for a given set of data points
- The N^{th} degree interpolating polynomial obtained by fitting $N + 1$ data points will be identical to that obtained using Lagrange formulae!
  - Newton interpolation is simply ***another*** technique for obtaining the same interpolating polynomial as was obtained using the Lagrange formulae

Forward Difference Tables

- We assume equi-spaced points (not necessary)



- Forward differences are now defined as follows:

$$\Delta^0 f_i \equiv f_i$$ (Zero^th order forward difference)

$$\Delta f_i \equiv f_{i+1} - f_i$$ (First order forward difference)

$$\Delta^2 f_i \equiv \Delta f_{i+1} - \Delta f_i$$ (Second order forward difference)

$$\Delta^2 f_i = (f_{i+2} - f_{i+1}) - (f_{i+1} - f_i)$$

$$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i$$

$$\Delta^3 f_i \equiv \Delta^2 f_{i+1} - \Delta^2 f_i$$ (Third order forward difference)

$$\Delta^3 f_i = (f_{i+3} - 2f_{i+2} + f_{i+1}) - (f_{i+2} - 2f_{i+1} + f_i)$$

$$\Delta^3 f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$$

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$ (k^{th} order forward difference)

- Typically we set up a difference table

i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	f_o	$\Delta f_o = f_1 - f_o$	$\Delta^2 f_o = \Delta f_1 - \Delta f_o$	$\Delta^3 f_o = \Delta^2 f_1 - \Delta^2 f_o$	$\Delta^4 f_o = \Delta^3 f_1 - \Delta^3 f_o$
1	f_1	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$	
2	f_2	$\Delta f_2 = f_3 - f_2$	$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$		
3	f_3	$\Delta f_3 = f_4 - f_3$			
4	f_4				

- Note that to compute higher order differences in the tables, we take forward differences of previous order differences instead of using expanded formulae.

- The order of the differences that can be computed depends on how many total data points, $x_o, \ldots, x_N$, are available

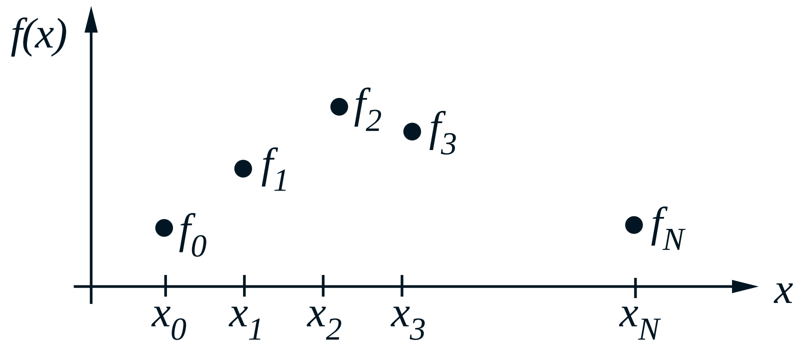
- ***$N + 1$ data points can develop up to N^{th} order forward differences***

Example 1

- Develop a forward difference table for the data given

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$	$\Delta^5 f_i$
0	2	-7	4	5	5	3	1
1	4	-3	9	10	8	4	
2	6	6	19	18	12		
3	8	25	37	30			
4	10	62	67				
5	12	129					

Deriving Newton Forward Interpolation on Equi-spaced Points



- Summary of Steps
    - Step 1: Develop a general Taylor series expansion for $f(x)$ about x_o.
    - Step 2: Express the various order forward differences at x_o in terms of $f(x)$ and its derivatives evaluated at x_o. This will allow us to express the actual derivatives evaluated at x_o in terms of forward differences.
    - Step 3: Using the general Taylor series expansion developed in Step 1, sequentially substitute in for the derivatives evaluated at x_o in terms of forward differences (i.e. substitute in the expressions developed in Step 2).

Step 1

- The Taylor series expansion for $f(x)$ about x_o is:

$$f(x) = f(x_o) + (x - x_o)\frac{df}{dx}\bigg|_{x = x_o} + \frac{1}{2!}(x - x_o)^2\frac{d^2 f}{dx^2}\bigg|_{x = x_o} + \frac{1}{3!}(x - x_o)^3\frac{d^3 f}{dx^3}\bigg|_{x = x_o} + O(x - x_o)^4$$

$$\Rightarrow$$

$$f(x) = f_o + (x - x_o) f_o^{(1)} + \frac{1}{2!}(x - x_o)^2 f_o^{(2)} + \frac{1}{3!}(x - x_o)^3 f_o^{(3)} + O(x - x_o)^4$$

Step 2a

- Express first order forward difference in terms of $f_o,\ f_o^{(1)}, \ldots$

$$\Delta f_o \equiv f_1 - f_o$$

- However since $f_1 = f(x_1)$, we can use the Taylor series given in *Step 1* to express f_1 in terms of f_o and its derivatives:

$$f_1 = f_o + (x_1 - x_o) f_o^{(1)} + \frac{1}{2!}(x_1 - x_o)^2 f_o^{(2)} + \frac{1}{3!}(x_1 - x_o)^3 f_o^{(3)} + O(x_1 - x_o)^4$$

- We note that the spacing between data points is $h \equiv x_1 - x_o$:

$$f_1 = f_o + h f_o^{(1)} + \frac{1}{2!} h^2 f_o^{(2)} + \frac{1}{3!} h^3 f_o^{(3)} + O(h)^4$$

- Now, substitute in for f_1 into the definition of the first order forward differences

$$\Delta f_o = f_o + h f_o^{(1)} + \frac{1}{2!} h^2 f_o^{(2)} + \frac{1}{3!} h^3 f_o^{(3)} + O(h)^4 - f_o$$

$$\Rightarrow$$

$$f_o^{(1)} = \frac{\Delta f_o}{h} - \frac{1}{2!} h f_o^{(2)} - \frac{1}{3!} h^2 f_o^{(3)} - O(h)^3$$

- Note that the first order forward difference divided by h is in fact an approximation to the first derivative to $O(h)$. However, we will use all the terms given in this sequence.

Step 2b

- Express second order forward difference in terms of f_o, $f_o^{(1)}$, …

$$\Delta^2 f_o \equiv f_2 - 2f_1 + f_o$$

- We note that $f_1 = f(x_1)$ was developed in T.S. form in *Step 2a*.

- For $f_2 = f(x_2)$ we use the T.S. given in *Step 1* to express f_2 in terms of f_o and derivatives of f evaluated at x_o

$$f_2 = f_o + (x_2 - x_o) f_o^{(1)} + \frac{1}{2!}(x_2 - x_o)^2 f_o^{(2)} + \frac{1}{3!}(x_2 - x_o)^3 f_o^{(3)} + O(x_2 - x_o)^4$$

- We note that $x_2 - x_o = 2h$

$$f_2 = f_o + 2h f_o^{(1)} + \frac{4}{2!}h^2 f_o^{(2)} + \frac{8}{3!}h^3 f_o^{(3)} + O(h)^4$$

- Now substitute in for f_2 and f_1 into the definition of the second order forward difference operator

$$\Delta^2 f_o = f_o + 2hf_o^{(1)} + 2h^2 f_o^{(2)} + \frac{4}{3}h^3 f_o^{(3)} + O(h)^4 - 2f_o - 2hf_o^{(1)} - h^2 f_o^{(2)} - \frac{1}{3}h^3 f_o^{(3)} + O(h)^4 + f_o$$

$$\Rightarrow$$

$$f_o^{(2)} = \frac{\Delta^2 f_o}{h^2} - hf_o^{(3)} + O(h)^2$$

- Note that the second order forward difference divided by h^2 is in fact an approximation to $f_o^{(2)}$ to $O(h)$. However, we will use all terms in the expression.

Step 2c

- Express the third order forward difference in terms of f_o, $f_o^{(1)}\ldots$

$$\Delta^3 f_o \equiv f_3 - 3f_2 + 3f_1 - f_o$$

- We already developed expressions for f_2 and f_1.

- Develop an expression for $f_3 = f(x_3)$ using the T.S. in *Step 1*

$$f_3 = f_o + (x_3 - x_o)f_o^{(1)} + \frac{1}{2!}(x_3 - x_o)^2 f_o^{(2)} + \frac{1}{3!}(x_3 - x_o)^3 f_o^{(3)} + O(x_3 - x_o)^4$$

- Noting that $x_3 - x_o = 3h$

$$f_3 = f_o + 3hf_o^{(1)} + \frac{9}{2}h^2 f_o^{(2)} + \frac{9}{2}h^3 f_o^{(3)} + O(h)^4$$

- Substituting in for f_3, f_2 and f_1 into the definition of the third order forward difference formula.

$$\Delta^3 f_o = f_o + 3hf_o^{(1)} + \frac{9}{2}h^2 f_o^{(2)} + \frac{9}{2}h^3 f_o^{(3)} + O(h)^4 - 3f_o - 6hf_o^{(1)} - \frac{12}{2}h^2 f_o^{(2)} - \frac{24}{3!}h^3 f_o^{(3)} + O(h)^4$$

$$+ 3f_o + 3hf_o^{(1)} + \frac{3}{2}h^2 f_o^{(2)} + \frac{3}{3!}h^3 f_o^{(3)} + O(h)^4 - f_o$$

$$\Rightarrow$$

$$\Delta^3 f_o = h^3 f_o^{(3)} + O(h)^4$$

$$\Rightarrow$$

$$f_o^{(3)} = \frac{\Delta^3 f_o}{h^3} + O(h)$$

- The third order forward difference divided by h^3 is an *O(h)* approximation to $f_o^{(3)}$

Step 3a

- Consider the general T.S. expansion presented in *Step 1* to define $f(x)$ and substitute in for $f_o^{(1)}$ using the result in *Step 2a.*

- Note that now we are ***not*** evaluating the T.S. at a data point but at any x

$$f(x) = f_o + (x - x_o)\left[\frac{\Delta f_o}{h} - \frac{1}{2!} h f_o^{(2)} - \frac{1}{3!} h^2 f_o^{(3)} - O(h)^3\right]$$

$$+ \frac{1}{2!}(x - x_o)^2 f_o^{(2)} + \frac{1}{3!}(x - x_o)^3 f_o^{(3)} + O(x - x_o)^4$$

$$\Rightarrow$$

$$f(x) = f_o + \frac{x - x_o}{h}\Delta f_o + \frac{1}{2!}[-(x - x_o)h + (x - x_o)^2] f_o^{(2)}$$

$$+ \frac{1}{3!}[-(x - x_o)h^2 + (x - x_o)^3] f_o^{(3)} + O(h)^4$$

Step 3b

- Substitute in for $f_o^{(2)}$ using the expression developed in *Step 2b.*

$$f(x) = f_o + \frac{x-x_o}{h}\Delta f_o + \frac{1}{2!}[-(x-x_o)h + (x-x_o)^2]\left[\frac{\Delta^2 f_o}{h^2} - h f_o^{(3)} + O(h)^2\right]$$

$$+ \frac{1}{3!}[-(x-x_o)h^2 + (x-x_o)^3] f_o^{(3)} + O(h)^4$$

$$\Rightarrow$$

$$f(x) = f_o + \frac{x-x_o}{h}\Delta f_o + \frac{1}{2!}\left[-\frac{(x-x_o)}{h} + \frac{(x-x_o)^2}{h^2}\right]\Delta^2 f_o$$

$$+ \frac{1}{3!}[2(x-x_o)h^2 + (x-x_o)^3 - 3(x-x_o)^2 h] f_o^{(3)} + O(h)^4$$

Step 3c

- Substitute in for $f_o^{(3)}$ from *Step 2c*

$$f(x) = f_o + \frac{(x-x_o)}{h}\Delta f_o + \frac{1}{2!}\left[-\frac{(x-x_o)}{h} + \frac{(x-x_o)^2}{h^2}\right]\Delta^2 f_o$$

$$+ \frac{1}{3!}[2(x-x_o)h^2 + (x-x_o)^3 - 3(x-x_o)^2 h]\frac{\Delta^3 f_o}{h^3} + O(h)^4$$

- Re-arranging the terms in brackets:

$$f(x) = f_o + (x-x_o)\frac{\Delta f_o}{h} + \frac{1}{2!}(x-x_o)[-h + (x-x_o)]\frac{\Delta^2 f_o}{h^2}$$

$$+ \frac{1}{3!}(x-x_o)[2h^2 + (x-x_o)^2 - 3(x-x_o)h]\frac{\Delta^3 f_o}{h^3} + O(h)^4 + HOT$$

$$\Rightarrow$$

$$f(x) = f_o + (x - x_o)\frac{\Delta f_o}{h} + \frac{1}{2!}(x - x_o)[x - (x_o + h)]\frac{\Delta^2 f_o}{h^2}$$

$$+ \frac{1}{3!}(x - x_o)[(x - (x_o + h))(x - (x_o + 2h))]\frac{\Delta^3 f_o}{h^3} + O(h)^4 + HOT$$

- Also considering higher order terms ***and*** noting that $x_o + h = x_1$, $x_o + 2h = x_2$

  and $f(x) = g(x) + e(x)$

$$g(x) = f_o + (x - x_o)\frac{\Delta f_o}{h} + \frac{1}{2!}(x - x_o)(x - x_1)\frac{\Delta^2 f_o}{h^2} + \frac{1}{3!}(x - x_o)(x - x_1)(x - x_2)\frac{\Delta^3 f_o}{h^3}$$

$$+ \dots + \frac{1}{N!}(x - x_o)(x - x_1)(x - x_2)\dots(x - x_{N-1})\frac{\Delta^N f_o}{h^N}$$

- **This is the N^{th} degree polynomial approximation to $N + 1$ data points and is identical to that derived for Lagrange interpolation or Power series (only the form in which it is presented is different).**

- Note that the $N+1$ data point are ***exactly*** fit by $g(x)$

$$g(x_o) = f_o$$

$$g(x_1) = f_o + (x_1 - x_o)\frac{f_1 - f_o}{h} = f_o + h\left(\frac{f_1 - f_o}{h}\right) = f_1$$

$$g(x_2) = f_o + (x_2 - x_o)\left(\frac{f_1 - f_o}{h}\right) + \frac{1}{2}(x_2 - x_o)(x_2 - x_1)\frac{1}{h^2}(f_2 - 2f_1 + f_o)$$

$$\Rightarrow$$

$$g(x_2) = f_o + \frac{2h}{h}(f_1 - f_o) + \frac{(2h)h}{2h^2}(f_2 - 2f_1 + f_o)$$

$$\Rightarrow$$

$$g(x_2) = f_o + 2f_1 - 2f_o + f_2 - 2f_1 + f_o = f_2$$

- ***In general***

$$\boldsymbol{g(x_i) = f_i \qquad i = 0, N}$$

- It can be readily shown that the error at any x is: (by carrying through error terms in the T.S.)

$$e(x) = f(x) - g(x) = \frac{(x-x_o)(x-x_1)\dots(x-x_N)}{(N+1)!} f^{(N+1)}(\xi) \qquad x_o < \xi < x_N$$

- This error function is identical to that for Lagrange Interpolation (since the polynomial approximation is the same).

- However we note that $f^{N+1}(x)$ can be approximated as (can be shown by T.S.)

$$f^{(N+1)}(x_o) \cong \frac{\Delta^{N+1} f_o}{h^{N+1}}$$

- In fact if $f^{(N+1)}(x)$ does not vary dramatically over the interval

$$f^{(N+1)}(\xi) \cong \frac{\Delta^{N+1} f_o}{h^{N+1}}$$

- Thus the error can be estimated as

$$e(x) \cong \frac{(x-x_o)(x-x_1)\dots(x-x_N)}{(N+1)!} \frac{\Delta^{N+1} f_o}{h^{N+1}}$$

- Notes
  - Approximation for $e(x)$ is equal to the term that would follow the last term in the N^{th} degree polynomial series for $g(x)$
  - If we have $N+2$ data points available and develop an N^{th} degree polynomial approximation with $N+1$ data points, we can then easily estimate $e(x)$. This was not as simple for Lagrange polynomials since you then needed to compute the finite difference approximation to the derivative in the error function.
  - If the exact function $f(x)$ is a polynomial of degree $M \leq N$, then $g(x)$ will be an (almost) exact representation of $f(x)$ (with small roundoff errors).
  - Newton Interpolation is much more efficient to implement than Lagrange Interpolation. If you develop a difference table ***once***, you can
    - Develop various order interpolation functions very quickly (since each higher order term only involves one more product)
    - Obtain error estimates very quickly

Example 2

- For the data and forward difference table presented in *Example 1*.
  - *(a)* Develop $g(x)$ using 3 points ($x_o = 2, x_1 = 4$ and $x_2 = 6$) and estimate $e(x)$
  - *(b)* Develop $g(x)$ using 4 points ($x_o = 2, x_1 = 4, x_2 = 6, x_3 = 8$) and estimate $e(x)$
  - *(c)* Develop $g(x)$ using 3 different points ($x_o = 6, x_1 = 8, x_2 = 10$)

(Part a)

- 3 data points $\Rightarrow N = 2$

$$g_3(x) = f_o + (x - x_o)\frac{\Delta f_o}{h} + \frac{1}{2!}(x - x_o)(x - x_1)\frac{\Delta^2 f_o}{h^2}$$

with

$$x_o = 2 \quad x_1 = 4 \quad x_2 = 6 \quad \text{and} \quad h = 2$$

- Note that the “3” designation in $g_3(x)$ indicates *N+1=3* data points

- f_o, Δf_o and $\Delta^2 f_o$ are obtained by simply picking values off of the difference table (across the row $i = 0$)

$$f_o = -7 \quad \Delta f_o = 4 \quad \Delta^2 f_o = 5$$

$$\mathbf{g_3(x) = -7 + (x-2)\frac{4}{2} + \frac{1}{2!}(x-2)(x-4)\frac{5}{4}}$$

- The error can be estimated as:

$$e_3(x) = \frac{(x-x_o)(x-x_1)(x-x_2)}{3!}\frac{1}{h^3}\Delta^3 f_o$$

- Simply substitute in for x_o, x_1, x_2, h and pick off $\Delta^3 f_o = 5$ from the table in *Example 1*

$$\mathbf{e_3(x) = \frac{(x-2)(x-4)(x-6)}{6} \cdot \frac{1}{2^3} \cdot 5 = (x-2)(x-4)(x-6)\frac{5}{48}}$$

(Part b)

- 4 data points $x_o = 2,\ x_1 = 4,\ x_2 = 6,\ x_3 = 8 \ \Rightarrow\ N = 3$

- Simply add the next term to the series for $g_3(x)$ in *Part a*:

$$g_4(x) = g_3(x) + \frac{1}{3!}(x-x_o)(x-x_1)(x-x_2)\frac{\Delta^3 f_o}{h^3}$$

  - We note that the term we are adding to $g_3(x)$ is actually $e_3(x)$

- Pick off $\Delta^3 f_o = 5$ from the table in *Example 1* and substitute in

$$\mathbf{g_4(x) = g_3(x) + \frac{1}{3!}(x-2)(x-4)(x-6) \cdot \frac{5}{8}}$$

- The error is estimated as

$$e_4(x) = \frac{(x-x_o)(x-x_1)(x-x_2)(x-x_3)}{4!}\frac{1}{h^4}\Delta^4 f_o \qquad \Rightarrow$$

$$\mathbf{e_4(x) = (x-2)(x-4)(x-6)(x-8)\frac{3}{384}}$$

(Part c)

- 3 data points $x_o = 6$, $x_1 = 8$ and $x_3 = 10$ $\Rightarrow$ *N=2*

- We must shift i in the table such that $x_o = 6$ etc.

$$g_{3/s}(x) = f_o + (x - x_o)\frac{\Delta f_o}{h} + \frac{1}{2!}(x - x_o)(x - x_1)\frac{\Delta^2 f_o}{h^2}$$

- Pick off f_o, Δf_o and $\Delta^2 f_o$ from the same difference table with a shifted index

$$f_o = 6, \quad \Delta f_o = 19, \quad \Delta^2 f_o = 18$$

- Substituting

$$g_{3/s}(x) = 6 + (x - 6)\frac{19}{2} + \frac{1}{2!}(x - 6)(x - 8)\frac{18}{2^2}$$

Newton Backward Interpolation

- Newton backward interpolation is essentially the same as Newton forward interpolation except that backward differences are used

- Backward differences are defined as:

$$\nabla^o f_i \equiv f_i$$ Zeroth order backward difference

$$\nabla f_i = f_i - f_{i-1}$$ First order backward difference

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$$ Second order backward difference

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1}$$ k^{th} order backward difference

- For $N + 1$ data point which are fitted with an N^{th} degree polynomial

$$g(x) = f_N + (x - x_N)\frac{\nabla f_N}{h} + \frac{1}{2!}(x - x_N)(x - x_{N-1})\frac{\nabla^2 f_N}{h^2}$$

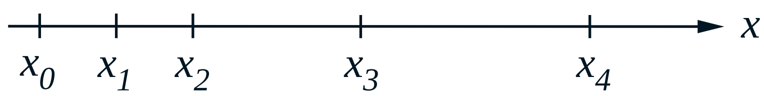
$$+ \frac{1}{3!}(x - x_N)(x - x_{N-1})(x - x_{N-2})\frac{\nabla^3 f_N}{h^3}$$

$$+ \frac{1}{N!}(x - x_N)(x - x_{N-1})\ldots(x - x_1)\frac{\nabla^N f_N}{h^N}$$

- Note that we are really expanding about the right most point to the left. Therefore we must develop f_N, ∇f_N etc. in the difference table

Newton Interpolation on Non-uniformly Spaced Data Points

- Newton interpolation can be readily extended to deal with non-uniformly spaced data points



- The difference table for non-uniformly spaced nodes is developed and an appropriate interpolation formula is developed and used

SUMMARY OF LECTURE 7

- Newton formulae can be obtained by manipulating Taylor series
- Newton interpolating function is related to easily computed forward/backward differences
- Error is readily established and estimated from the difference table (as long as you have one more data point than used in interpolation)
- Newton interpolation through $N + 1$ data points gives the same N^{th} degree polynomial as Lagrange interpolation
- Newton interpolation is more efficient than Lagrange interpolation and is easily implemented