LECTURE 5

INTRODUCTION TO INTERPOLATION

- ***Interpolation function: a function that passes exactly through a set of data points.***
- Interpolating functions to interpolate values in tables

x	$\sin(x)$
0.0	0.000000
0.5	0.479426
1.0	0.841471
1.5	0.997495
2.0	0.909297
2.5	0.598472

- In tables, the function is only specified at a limited number or discrete set of independent variable values (as opposed to a continuum function).
- We can use interpolation to find functional values at other values of the independent variable, e.g. $\sin(0.63253)$

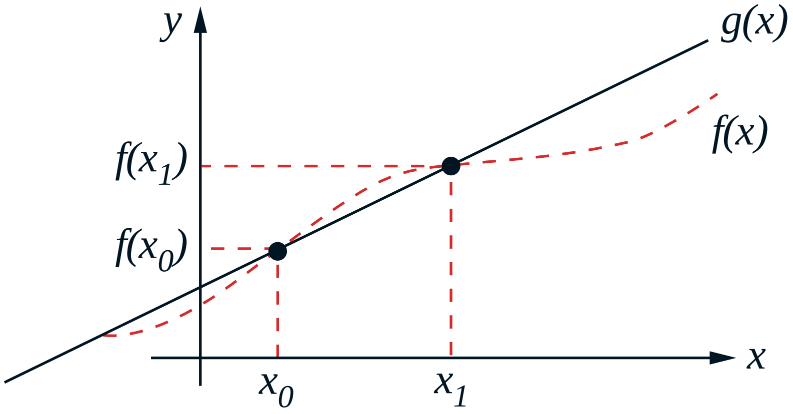
- In numerical methods, like tables, the values of the function are only specified at a discrete number of points! Using interpolation, we can describe or at least approximate the function at every point in space.

- For numerical methods, we use interpolation to
  - Interpolate values from computations
  - Develop numerical integration schemes
  - Develop numerical differentiation schemes
  - Develop finite element methods

- Interpolation is typically not used to obtain a functional description of measured data since errors in the data may lead to a poor representation.
  - Curve fitting to data is handled with a separate set of techniques

Linear Interpolation

- ***Linear interpolation is obtained by passing a straight line between 2 data points***



$f(x)$ = the exact function for which values are known only at a discrete set of data points

$g(x)$ = the interpolated approximation to $f(x)$

x_0 , x_1 = the data points (also referred to as interpolation points or nodes)

- In tabular form:

x_o	$f(x_o)$
$\mathbf{x}$	$\mathbf{g(x)}$
x_1	$f(x_1)$

- If $g(x)$ is a linear function then

$$g(x) = Ax + B \tag{1}$$

where A and B are unknown coefficients

- To pass through points $(x_o, f(x_o))$ and $(x_1, f(x_1))$ we must have:

$$g(x_o) = f(x_o) \quad \Rightarrow \quad Ax_o + B = f(x_o) \tag{2}$$

$$g(x_1) = f(x_1) \quad \Rightarrow \quad Ax_1 + B = f(x_1) \tag{3}$$

  - 2 unknowns and 2 equations $\Rightarrow$ solve for A, B

- Using (2)

$$B = f(x_o) - Ax_o$$

Substituting into (3)

$$Ax_1 + f(x_o) - Ax_o = f(x_1)$$

$$A = \frac{f(x_1) - f(x_o)}{x_1 - x_o}$$

$$B = \frac{f(x_o)x_1 - f(x_1)x_o}{x_1 - x_o}$$

- Substituting for A and B into equation (1)

$$\boldsymbol{g(x) = f(x_o)\frac{(x_1 - x)}{(x_1 - x_o)} + f(x_1)\frac{(x - x_o)}{(x_1 - x_o)}}$$

This is the formula for <u>linear</u> interpolation

Example 1

- Use values at x_o and x_1 to get an interpolated value at $x = 0.632$ using ***linear*** interpolation

Table 1:

x	$f(x) = \sin x$
$x_o = 0.5$	$f(x_o) = 0.47942554$
0.632	$\mathbf{g(0.632) = ?}$
$x_1 = 1.0$	$f(x_1) = 0.84147099$

$$g(0.632) = 0.479425\frac{(1.0 - 0.632)}{(1.0 - 0.5)} + 0.84147099\frac{(0.632 - 0.5)}{(1.0 - 0.5)}$$

$$\mathbf{g(0.632) = 0.57500}$$

Error for Linear Interpolating Functions

- Error is defined as:

$$\boldsymbol{e(x) \equiv f(x) - g(x)}$$

- $e(x)$ represents the difference between the exact function $f(x)$ and the interpolating or approximating function $g(x)$.

- We note that at the interpolating points x_o and x_1

$$e(x_o) = 0$$

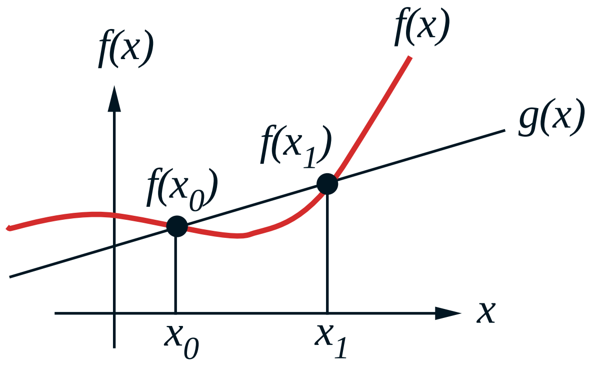
$$e(x_1) = 0$$

- This is because at the interpolating point we have by definition

$$g(x_o) = f(x_o)$$

$$g(x_1) = f(x_1)$$

Derivation of *e(x)*



$$e(x) \equiv f(x) - g(x)$$

Step 1

- Expand $f(x)$ in Taylor Series (T.S.) about x_o

$$f(x) \;=\; f(x_o) + (x - x_o)\left.\frac{df}{dx}\right|_{x\,=\,x_o} + \frac{(x-x_o)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x\,=\,\xi} \qquad \text{where} \quad x_o \le \xi \le x \qquad (4)$$

- The third term is the actual remainder term and represents all other terms in the series since ***it is evaluated at*** $x = \xi$***!***

Step 2

- Express $\left.\frac{df}{dx}\right|_{x = x_o}$ in terms of $f(x_o)$ and $f(x_1)$
- We can accomplish this by simply evaluating the T.S. in (4) at $x = x_1$.

$$f(x_1) = f(x_o) + (x_1 - x_o)\left.\frac{df}{dx}\right|_{x = x_o} + \frac{(x_1 - x_o)^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x = \xi} \tag{5}$$

$$\Rightarrow$$

$$\left.\frac{df}{dx}\right|_{x = x_o} = \frac{f(x_1)}{(x_1 - x_o)} - \frac{f(x_o)}{(x_1 - x_o)} - \frac{(x_1 - x_o)^2}{2!} \cdot \frac{1}{(x_1 - x_o)}\left.\frac{d^2f}{dx^2}\right|_{x = \xi} \tag{6}$$

$$\Rightarrow$$

$$\left.\frac{df}{dx}\right|_{x = x_o} = \frac{f(x_1)}{(x_1 - x_o)} - \frac{f(x_o)}{(x_1 - x_o)} - \frac{(x_1 - x_o)}{2}\left.\frac{d^2f}{dx^2}\right|_{x = \xi} \tag{7}$$

- We note that this is a discrete approximation to the first derivative (a F.D. Formula)

Step 3

- Substitute Equation 7 into T.S. form of $f(x)$, Equation (4).
- This gives us an expression for $f(x)$ in terms of the discrete values $f(x_o)$ and $f(x_1)$.

$$f(x) = f(x_o) + (x - x_o)\left[\frac{f(x_1)}{(x_1 - x_o)} - \frac{f(x_o)}{(x_1 - x_o)} - \frac{(x_1 - x_o)}{2}\frac{d^2 f}{dx^2}\bigg|_{x = \xi}\right] + \frac{(x - x_o)^2}{2}\frac{d^2 f}{dx^2}\bigg|_{x = \xi} \qquad (8)$$

$$\Rightarrow$$

$$f(x) = f(x_o) + \frac{(x - x_o)}{(x_1 - x_o)}f(x_1) - \frac{(x - x_o)}{(x_1 - x_o)}f(x_o) + \left[\frac{(x - x_o)(-x_1 + x_o)}{2} + \frac{(x - x_o)^2}{2}\right]\frac{d^2 f}{dx^2}\bigg|_{x = \xi} \qquad (9)$$

$$\Rightarrow$$

$$f(x) = (x_1 - x_o - x + x_o)\frac{f(x_o)}{(x_1 - x_o)} + (x - x_o)\frac{f(x_1)}{(x_1 - x_o)} + (-x_1 + x_o + x - x_o)\frac{(x - x_o)}{2}\frac{d^2 f}{dx^2}\bigg|_{x = \xi} \qquad (10)$$

$$\Rightarrow$$

$$f(x) = f(x_o)\left[\frac{x_1 - x}{x_1 - x_o}\right] + f(x_1)\left[\frac{x - x_o}{x_1 - x_o}\right] + \frac{(x - x_o)(x - x_1)}{2}\frac{d^2 f}{dx^2}\bigg|_{x = \xi} \qquad (11)$$

- The first part of Equation (11) is simply the linear interpolation formula. The second part is in fact the error. Thus:

$$e(x) \equiv f(x) - g(x)$$

$$\Rightarrow$$

$$e(x) \equiv f(x_o)\left[\frac{x_1 - x}{x_1 - x_o}\right] + f(x_1)\left[\frac{x - x_o}{x_1 - x_o}\right] + \frac{(x - x_o)(x - x_1)}{2}\left.\frac{d^2 f}{dx^2}\right|_{x = \xi}$$

$$-f(x_o)\left[\frac{x_1 - x}{x_1 - x_o}\right] - f(x_1)\left[\frac{x - x_o}{x_1 - x_o}\right]$$

$$\Rightarrow$$

$$e(x) = \frac{(x - x_o)(x - x_1)}{2}\left.\frac{d^2 f}{dx^2}\right|_{x = \xi} \qquad x_o \le \xi \le x_1$$

- If we assume that the interval $[x_o, x_1]$ is small, then the second derivative won't change dramatically in the interval!

$$\left.\frac{d^2 f}{dx^2}\right|_{x = \xi} \cong \left.\frac{d^2 f}{dx^2}\right|_{x = x_o} \cong \left.\frac{d^2 f}{dx^2}\right|_{x = x_1} \cong \left.\frac{d^2 f}{dx^2}\right|_{x = x_m} \qquad \text{where} \qquad x_m \equiv \frac{x_o + x_1}{2}$$

- Thus we typically evaluate the derivative term in the error expression using the midpoint in the interval

$$\boldsymbol{e(x) \cong \frac{1}{2}(x - x_o)(x - x_1)\left.\frac{d^2 f}{dx^2}\right|_{x = x_m}}$$

- Another problem is that we typically don't know the second derivative at the midpoint of the interval, x_m
- However using finite differencing formulae we can approximate this derivative knowing the functional values at the interpolating points
- Maximum error occurs at the midpoint for linear interpolation (where $(x - x_o)\ (x - x_1)$ is the largest)

$$\max|e(x)|_{x_0 < x < x_1} \cong \frac{1}{2}(x_m - x_o)(x_m - x_1)\left.\frac{d^2 f}{dx^2}\right|_{x = x_m}$$

- However

$$h \equiv x_1 - x_o$$

and

$$\frac{h}{2} = x_m - x_o \quad \text{and} \quad \frac{h}{2} = x_1 - x_m$$

- Thus

$$\max|e(x)|_{x_0 < x < x_1} = \frac{h^2}{8}\frac{d^2 f}{dx^2}\bigg|_{x_m}$$

- Notes on Error for linear interpolation
  - The error expression has a polynomial and a derivative portion.
  - Maximum error occurs approximately at the midpoint between x_o and x_1
  - Error increases as the interval h increases
  - Error increases as $f^{(2)}(x)$ increases. Again note that $f^{(2)}(x)$ can be approximated with finite difference (F.D.) formulae if at least 3 surrounding functional values are available. (We will discuss F.D. formulae later.)

Example 2

- Compute an error estimate for the problem in Example 1.
- Recall we found that

$$g(0.632) = 0.57500$$

- Error is estimated as:

$$e(x) \cong \frac{1}{2}(x - x_o)(x - x_1)\left.\frac{d^2 f}{dx^2}\right|_{x = x_m}$$

- Since $x = 0.750$ is the midpoint at the interval $[0.5, 1.0]$, we have

$$e(0.632) \cong \frac{1}{2}(0.632 - 0.5)(0.632 - 1.0)\left.\frac{d^2 f}{dx^2}\right|_{x = 0.750}$$

$$\Rightarrow$$

$$e(0.632) \cong -0.024288\left.\frac{d^2 f}{dx^2}\right|_{x = 0.75}$$

- Since we have not yet extensively discussed approximating derivatives using discrete values, we will compute $\left.\frac{d^2 f}{dx^2}\right|_{x\,=\,0.75}$ using analytical methods:

$$\left.\frac{d^2 f}{dx^2}\right|_{x\,=\,0.75} = -\sin(0.750) = -0.68164$$

- Substituting in the value for $\left.\frac{d^2 f}{dx^2}\right|_{x\,=\,0.75}$, we obtain an estimate for the error:

$$\boldsymbol{e(0.632) \cong (-0.024288)(-0.68164) = 0.016555}$$

- Computing the actual error (the actual solution - the estimated error):

$$E(x) = \sin(x) - g(x)$$

$$\boldsymbol{E(0.632) = \sin(0.632) - 0.57500 = 0.01576}$$

- **The estimated error, $e(x)$ is a good approximation of the actual error $E(x)$!**