LECTURE 21

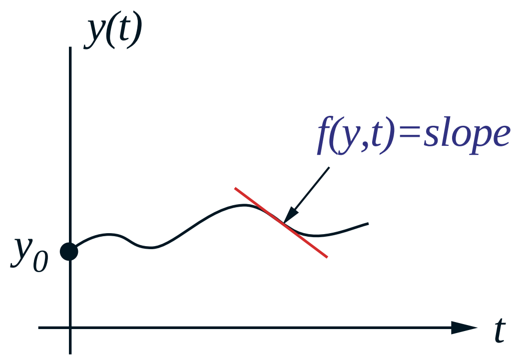
SOLUTIONS TO O.D.E.'S Continued

• Solve

$$\frac{dy}{dt} = f(y, t)$$

$$y(t_o) = y_o \qquad \text{Initial condition}$$

• Note that the solution $y(t)$ is a function of t only while the slope $f(y, t)$ is a function of both y and t.



2nd Order Runge-Kutta Methods

- Use two terms for Φ

$$y_{j+1} = y_j + \Delta t \Phi \qquad (1)$$

$$\Phi = a_1 g_1 + a_2 g_2 \qquad (2)$$

$$\Phi = a_1 f(t_j, y_j) + a_2 f(t_j + p_1 \Delta t, y_j + p_2 \Delta t f(t_j, y_j)) \qquad (3)$$

- Expand the 2nd term in Equation (3) as a Taylor series about (t_j, y_j)

$$f(t_j + p_1 \Delta t, y_j + p_2 \Delta t f(t_j, y_j)) = f(t_j, y_j) + \Delta T \frac{\partial f}{\partial t}\bigg|_{(t_j, y_i)} + \Delta y \frac{\partial f}{\partial y}\bigg|_{(t_j, y_j)}$$

$$+ \frac{1}{2!}\left[(\Delta T)^2 \frac{\partial^2 f}{\partial t^2} + 2(\Delta T)(\Delta y)\frac{\partial^2 f}{\partial y \partial t} + (\Delta y)^2 \frac{\partial^2 f}{\partial y^2}\right]_{(t_j, y_j)} + \dots$$

where $\Delta T \equiv p_1 \Delta t$ and $\Delta y \equiv p_2 \Delta t f(t_j, y_j)$

- Hence the approximation to Φ is expressed as:

$$\Phi = a_1 f_j + a_2 \left\{ f_j + p_1 \Delta t \left.\frac{\partial f}{\partial t}\right|_j + p_2 \Delta t \; f_j \left.\frac{\partial f}{\partial y}\right|_j + O(\Delta t)^2 \right\}$$

  - We neglect terms of 2nd order and higher.

- Hence:

$$\Phi = (a_1 + a_2) f_j + a_2 \Delta t \left\{ p_1 \left.\frac{\partial f}{\partial t}\right|_j + p_2 \; f_j \left.\frac{\partial f}{\partial y}\right|_j \right\} + O(\Delta t)^2$$

- However we recall that:

$$y_{j+1} = y_j + \Delta t \Phi$$

- Substituting in for Φ we obtain an approximating expression for y_{j+1}

$$y_{j+1} = y_j + \Delta t (a_1 + a_2) f_j + a_2 (\Delta t)^2 \left\{ p_1 \left.\frac{\partial f}{\partial t}\right|_j + p_2 \; f_j \left.\frac{\partial f}{\partial y}\right|_j \right\} + O(\Delta t)^3 \qquad (4)$$

- Use Taylor Series to find exact representation for $y_{j+1} = y(t_{j+1})$ by expanding about t_j

$$y_{j+1} = y_j + \Delta t \left.\frac{dy}{dt}\right|_j + \frac{(\Delta t)^2}{2} \left.\frac{d^2y}{dt^2}\right|_j + O(\Delta t)^3 \qquad (5)$$

- However by definition

$$\frac{dy}{dt} = f(t, y) \quad \Rightarrow$$

$$\frac{d^2y}{dt^2} = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) \quad \Rightarrow$$

$$\frac{d^2y}{dt^2} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f$$

- Substituting for $\left.\frac{dy}{dt}\right|_j$ and $\left.\frac{d^2y}{dt^2}\right|_j$ into the Taylor series expanded form of y_{j+1}, Equation (5), we obtain an exact expression for y_{j+1} (to $O(\Delta t)^3$)

$$y_{j+1} = y_j + \Delta t\ f_j + \frac{(\Delta t)^2}{2}\left(\left.\frac{\partial f}{\partial t}\right|_j + \left.\frac{\partial f}{\partial y}\right|_j f_j\right) + O(\Delta t)^3 \qquad (6)$$

- Recall that the Runge-Kutta approximation for y_{j+1} was:

$$y_{j+1} = y_j + \Delta t(a_1 + a_2) f_j + (\Delta t)^2\left\{a_2\ p_1\left.\frac{\partial f}{\partial t}\right|_j + a_2\ p_2\ f_j\left.\frac{\partial f}{\partial y}\right|_j\right\} + O(\Delta t)^3 \qquad (4')$$

- Select constants between Equations $(4')$ and (6) such that both are the same

$$\begin{cases} a_1 + a_2 = 1 \\ a_2 p_1 = \frac{1}{2} \\ a_2 p_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} a_1 = 1 - a_2 \\ p_1 = p_2 = \frac{1}{2a_2} \end{cases}$$

- 4 unknowns and 3 equations $\Rightarrow$ Solve in terms of an arbitrary constant a

$$\begin{cases} a_1 = 1 - a \\ a_2 = a \\ p_1 = p_2 = \dfrac{1}{2a} \end{cases}$$

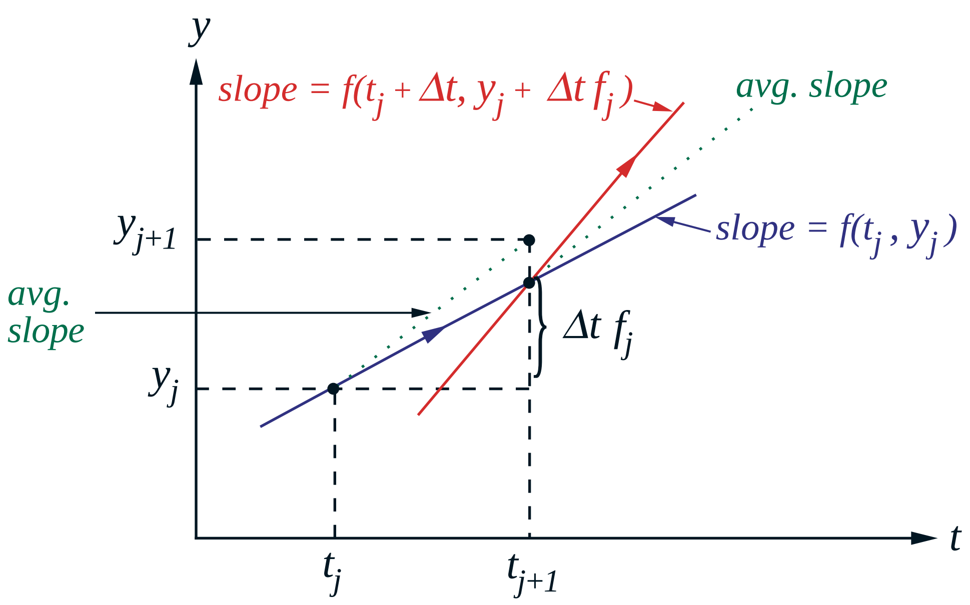
- The local or per time step error is $O(\Delta t)^3$ $\Rightarrow$ These methods will be $O(\Delta t)^2$ accurate

<u>Improved Euler Method (Modified Euler-Cauchy)</u>

- Let $a = \frac{1}{2}$ $\Rightarrow$ $\begin{cases} a_1 = a_2 = \dfrac{1}{2} \\ p_1 = p_2 = 1 \end{cases}$

- Substituting values into the standard form of the Runge-Kutta algorithm, Equations (1) and (3):

$$y_{j+1} = y_j + \frac{1}{2}\Delta t\ f_j + \frac{1}{2}\Delta t\ f(t_j + \Delta t, y_j + \Delta t\ f_j)$$



- Procedure
  - Estimate y_{j+1} by using first order Euler
  - Evaluate the slope at t_{j+1} and the 1st order estimate of $y_{j+1} \rightarrow f(t_j + \Delta t, y_j + \Delta t\ f_j)$
  - Find the point y_{j+1} by taking the average of slope at (t_j, y_j) and at (t_{j+1}, 1st order approximation to y_{j+1})

Modified Euler Method

- Let $a = 1 \Rightarrow a_1 = 0 \quad a_2 = 1 \quad p_1 = p_2 = \frac{1}{2}$

$$\boldsymbol{y_{j+1} = y_j + \Delta t \ f\left(t_j + \frac{\Delta t}{2}, y_j + \frac{\Delta t}{2} \ f_j\right)}$$

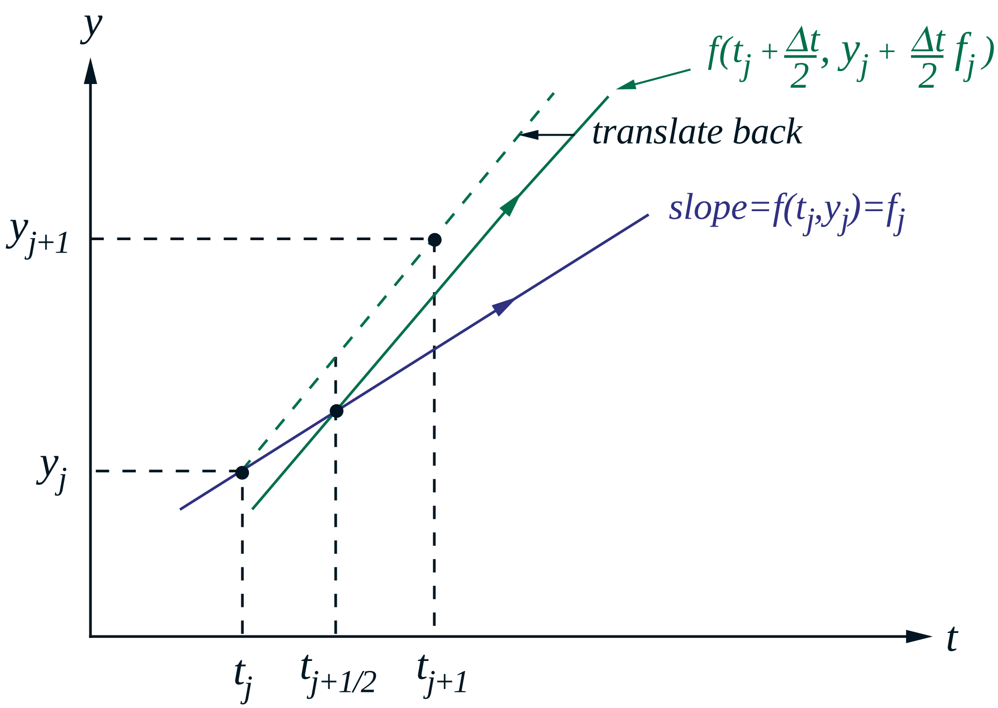
- Implementation of this formula may be viewed as follows:

$$t_{j+\frac{1}{2}} = t_j + \frac{\Delta t}{2} \quad \leftarrow \text{evaluate first}$$

$$y^*_{j+\frac{1}{2}} = y_j + \frac{\Delta t}{2} \ f(t_j, y_j) \quad \leftarrow \text{evaluate second}$$

$$y_{j+1} = y_j + \Delta t \ f\left(t_{j+\frac{1}{2}}, y^*_{j+\frac{1}{2}}\right) \quad \leftarrow \text{final step}$$

- Note that the 2nd order formulae require 2 evaluations of the function $f(t, y)$



- Procedure
  - Using the slope at point (t_j, y_j), estimate the value of y at point $t_{j+\frac{1}{2}}$ using simple first order Euler
  - Evaluate the slope at the halfway point ($t_{j+\frac{1}{2}}$, lst order approximation for $y_{j+\frac{1}{2}}$)
  - Use this slope at the halfway point to predict y_{j+1}

4th Order Runge-Kutta (often referred to as the "Runge-Kutta formula")

- Sum of coefficients equals 1.0 since we're taking an average slope.

$$y_{j+1} = y_j + \Delta t\left[\frac{g_1}{6} + \frac{g_2}{3} + \frac{g_3}{3} + \frac{g_4}{6}\right]$$

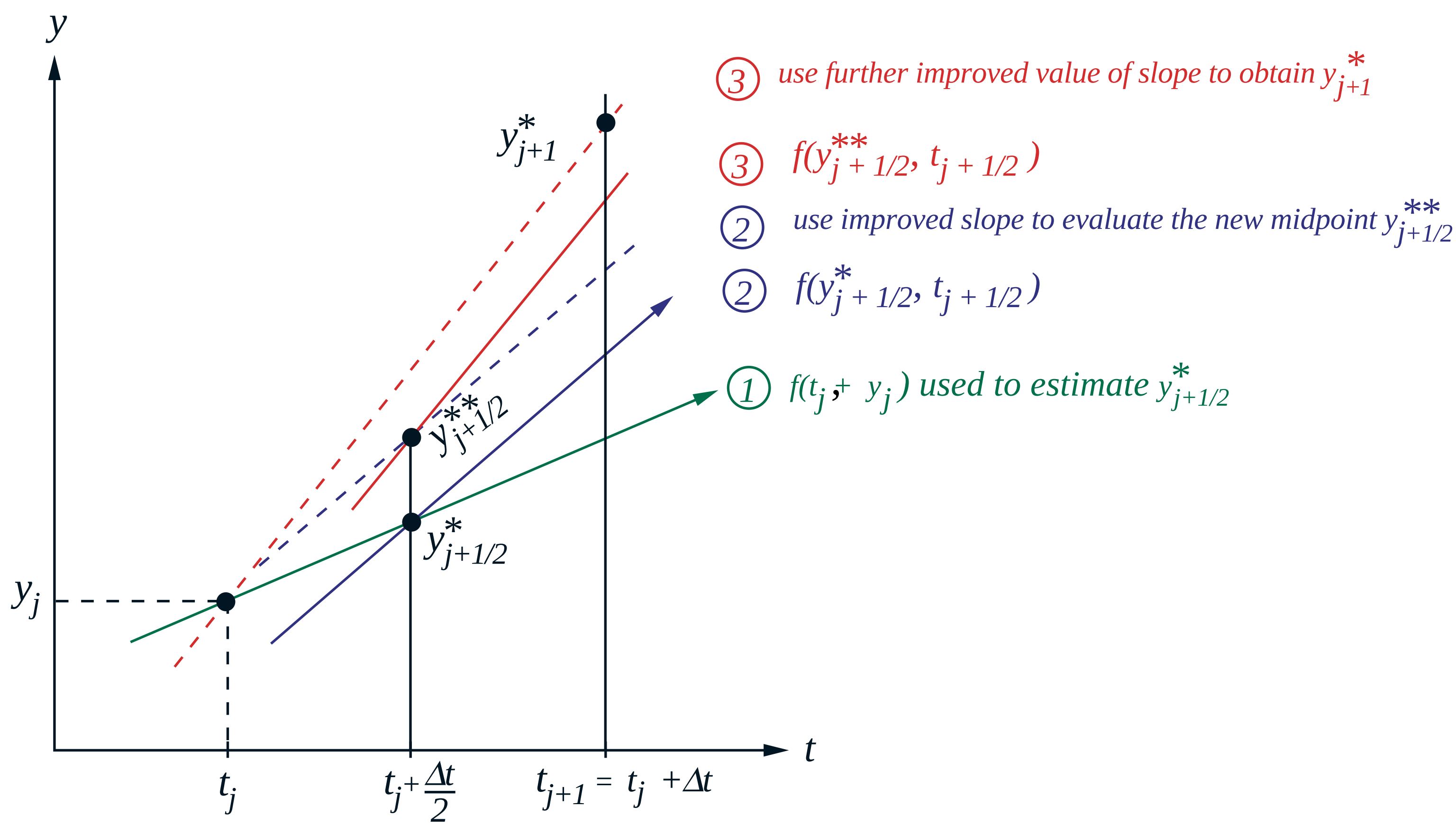
- Procedure as follows:

$$y^*_{j+\frac{1}{2}} = y_j + \frac{\Delta t}{2}\, f(y_j, t_j)$$

$$y^{**}_{j+\frac{1}{2}} = y_j + \frac{\Delta t}{2}\, f\left(y^*_{j+\frac{1}{2}}, t_{j+\frac{1}{2}}\right)$$

$$y^*_{j+1} = y_j + \Delta t\, f\left(y^{**}_{j+\frac{1}{2}}, t_{j+\frac{1}{2}}\right)$$

$$\boldsymbol{y_{j+1} = y_j + \Delta t\left[\frac{1}{6}\, f(y_j, t_j) + \frac{1}{3}\, f\left(y^*_{j+\frac{1}{2}}, t_{j+\frac{1}{2}}\right) + \frac{1}{3}\, f\left(y^{**}_{j+\frac{1}{2}}, t_{j+\frac{1}{2}}\right) + \frac{1}{6}\, f(y^*_{j+1}, t_{j+1})\right]}$$



- Notes
  - Find the point y_{j+1} by using the weighted average of the 4 slopes
  - Note that there are other coefficients possible for 4th order Runge-Kutta
  - We require 4 evaluations of the slope for this 4th order method → 4 times the work of the 1st order Runge-Kutta Method

Summary of Runge-Kutta Methods

- Self starting and interval Δt can be changed at any time without complications
- Very easy to program
- Comparable if not better accuracy than other methods
- Require much more computer time than other methods of comparable accuracy since
  - Number of functional evaluations is proportional to the accuracy
  - Functional evaluations can not be reused
- Local truncation errors are difficult and expensive to obtain (easier for other methods)

- Qualitative basis for verifying accuracy of solutions → use 2 different time steps (similar to Romberg integration)
  - Can estimate truncation error as:

$$E_{j+1} \cong \frac{\hat{y}_{j+1} - y_{j+1}}{2^{-k} - 1}$$

where

$\hat{y}_{j+1} \Rightarrow$ solution found using $\Delta t/2$ (therefore 2 steps)

$y_{j+1} \Rightarrow$ solution found using Δt (1 step)

k = order of the method

  - Need to run the solution using two different time steps!