

## LECTURE 14

### PRACTICAL ISSUES IN APPLYING FINITE DIFFERENCE APPROXIMATIONS

#### Derivatives of Variable Coefficients

- Consider the term

$$\frac{d}{dx} \left( g(x) \frac{df(x)}{dx} \right)$$

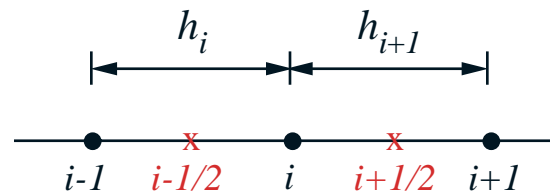
where  $g$  is a coefficient which is a function of  $x$

- Proceed in steps. First let:

$$u \equiv g(x) \frac{df}{dx}$$

- Now evaluate:

$$\frac{du}{dx} = u_i^{(1)}$$



$$u_i^{(1)} = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\frac{1}{2}(h_i + h_{i+1})}$$

- Now evaluate  $u_{i+\frac{1}{2}}$  and  $u_{i-\frac{1}{2}}$

$$u_{i+\frac{1}{2}} = g_{i+\frac{1}{2}} f_{i+\frac{1}{2}}^{(1)} = g_{i+\frac{1}{2}} \left\{ \frac{f_{i+1} - f_i}{h_{i+1}} \right\}$$

$$u_{i-\frac{1}{2}} = g_{i-\frac{1}{2}} f_{i-\frac{1}{2}}^{(1)} = g_{i-\frac{1}{2}} \left\{ \frac{f_i - f_{i-1}}{h_i} \right\}$$

- Hence

$$u_i^{(1)} = f_{i+1} \left[ \frac{g_{i+1/2}}{h_{avg} h_{i+1}} \right] - f_i \left[ \frac{g_{i+1/2}}{h_{avg} h_{i+1}} + \frac{g_{i-1/2}}{h_{avg} h_i} \right] + f_{i-1} \left[ \frac{g_{i-1/2}}{h_{avg} h_i} \right]$$

where  $h_{avg} \equiv \frac{h_i + h_{i+1}}{2}$

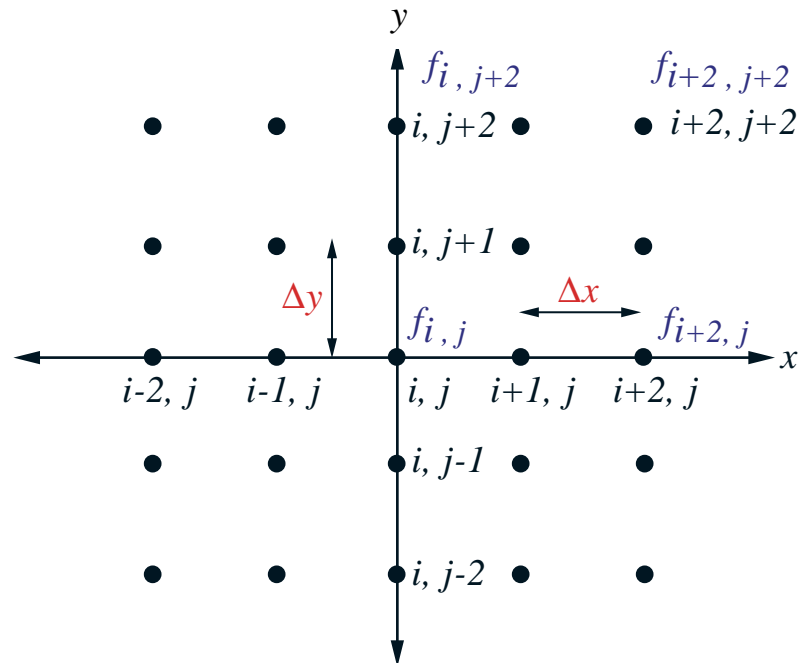
- When  $h = h_{i+1} = h_i$ , then the approximation reduces to:

$$u_i^{(1)} = \frac{1}{h^2} \left[ f_{i+1} g_{i+\frac{1}{2}} - f_i \left( g_{i+\frac{1}{2}} + g_{i-\frac{1}{2}} \right) + f_{i-1} g_{i-\frac{1}{2}} \right]$$

- Example Applications with variable coefficients:
  - Fluids: spatially varying viscosities/diffusivities
  - Solids: spatially varying elasticities

## Two Dimensional Finite Difference Approximations

- Define nodes in a two-dimensional plane with
  - $i$  = spatial index in the  $x$ -direction
  - $j$  = spatial index in the  $y$ -direction



- When taking partial derivatives w.r.t.  $x$ , we hold  $y$  constant

$$\left. \frac{\partial f}{\partial x} \right|_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}$$

and

$$\left. \frac{\partial^2 f}{\partial x^2} \right|_{i,j} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta x^2}$$

- Similarly

$$\left. \frac{\partial^2 f}{\partial y^2} \right|_{i,j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta y^2}$$

**Example 1**

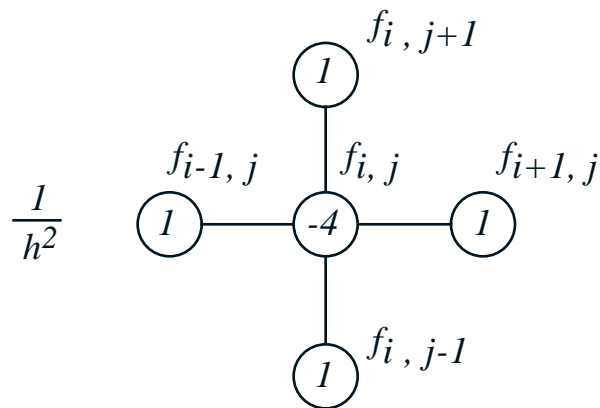
- Consider Laplace's equation and  $\Delta x = \Delta y = h$ :

$$\nabla^2 f|_{i,j} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$\Rightarrow$

$$\nabla^2 f|_{i,j} = \frac{1}{h^2}(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j})$$

- This may be viewed as follows in terms of a molecule:



**Example 2**

- Let's consider:

$$\nabla^4 f = \nabla^2(\nabla^2 f) = \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}$$

- Draw a molecule diagram for the 2nd order central difference formulation.

- Obtain  $\frac{\partial^4 f}{\partial x^4}$  by finding  $\frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 f}{\partial x^2} \right)$

$$\frac{\partial^4 f}{\partial x^4} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 f}{\partial x^2} \right)$$

$\Rightarrow$

$$\frac{\partial^4 f}{\partial x^4} = \frac{\partial^2}{\partial x^2} \left( \frac{1}{\Delta x^2} (f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) \right)$$

$\Rightarrow$

$$\frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^2} \left( \frac{\partial^2}{\partial x^2}(f_{i+1,j}) - 2 \frac{\partial^2}{\partial x^2}(f_{i,j}) + \frac{\partial^2}{\partial x^2}(f_{i-1,j}) \right)$$

$$\Rightarrow$$

$$\begin{aligned} \frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^2} & \left( \frac{1}{\Delta x^2}(f_{i+2,j} - 2f_{i+1,j} + f_{i,j}) \right. \\ & - \frac{2}{\Delta x^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) \\ & \left. + \frac{1}{\Delta x^2}(f_{i,j} - 2f_{i-1,j} + f_{i-2,j}) \right) \end{aligned}$$

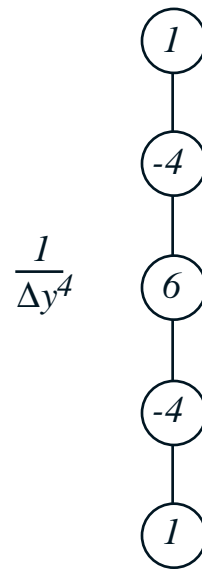
$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^4} = \frac{1}{\Delta x^4} (f_{i-2,j} - 4f_{i-1,j} + 6f_{i,j} - 4f_{i+1,j} + f_{i+2,j})$$

$$\frac{1}{\Delta x^4} \begin{array}{c} \textcircled{1} \text{---} \textcircled{-4} \text{---} \textcircled{6} \text{---} \textcircled{-4} \text{---} \textcircled{1} \end{array}$$

- Similarly:

$$\frac{\partial^4 f}{dy^4} = \frac{1}{\Delta y^4} (f_{i,j-2} - 4f_{i,j-1} + 6f_{i,j} - 4f_{i,j+1} + f_{i,j+2})$$



- Evaluation of the mixed derivative:

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 f}{\partial x^2} \right)$$

$$\Rightarrow$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial y^2} \left( \frac{1}{\Delta x^2} (f_{i-1,j} - 2f_{i,j} + f_{i+1,j}) \right)$$

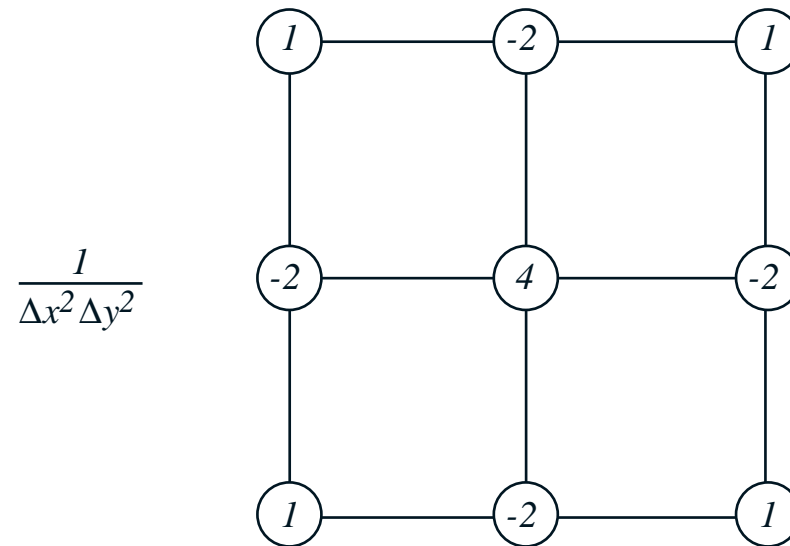
$$\Rightarrow$$

$$\begin{aligned} \frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{1}{\Delta x^2} & \left[ \frac{1}{\Delta y^2} (f_{i-1,j-1} - 2f_{i-1,j} + f_{i-1,j+1}) \right. \\ & - \frac{2}{\Delta y^2} (f_{i,j-1} - 2f_{i,j} + f_{i,j+1}) \\ & \left. + \frac{1}{\Delta y^2} (f_{i+1,j-1} - 2f_{i+1,j} + f_{i+1,j+1}) \right] \end{aligned}$$

$\Rightarrow$ 

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{1}{\Delta x^2 \Delta y^2} [f_{i-1,j-1} - 2f_{i-1,j} + f_{i-1,j+1} \\ - 2f_{i,j-1} + 4f_{i,j} - 2f_{i,j+1} \\ + f_{i+1,j-1} - 2f_{i+1,j} + f_{i+1,j}]$$

- This then produces the following module



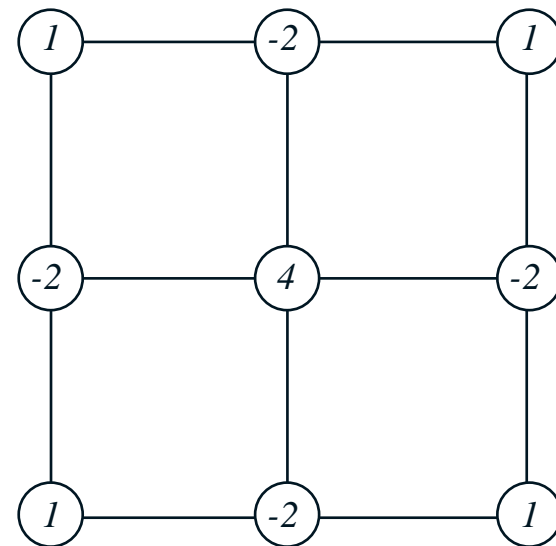
- We can also obtain this module as follows:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\Delta x^2} [ 1 \text{---} -2 \text{---} 1 ]$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{\Delta y^2} \begin{array}{c} 1 \\ | \\ -2 \\ | \\ 1 \end{array}$$

- Therefore

$$\frac{\partial^2}{\partial y^2} \left( \frac{\partial^2 f}{\partial x^2} \right) = \frac{1}{\Delta x^2 \Delta y^2} \begin{array}{c} 1 \\ | \\ -2 \\ | \\ 1 \end{array} \times \begin{array}{c} 1 \text{---} -2 \text{---} 1 \end{array} = \frac{1}{\Delta x^2 \Delta y^2}$$



- To obtain  $\nabla^4 f$ , we must add up modules:

$$\nabla^4 f = \frac{\partial^4 f}{\partial x^4} + 2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}$$

- If  $\Delta x = \Delta y = h$  then:

