RESEARCH STATEMENT

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1. Introduction

I am an applied mathematician who has specialized in numerical algebraic geometry, an area that uses numerical methods for computing and manipulating the solution set of polynomial systems. Since polynomial systems naturally arise in engineering, economics, and science, numerical algebraic geometry and my research are interdisciplinary in nature. In particular, I have articles related to numerical analysis [4, 6], computational algebraic geometry [1, 2, 8], and software development [5], along with articles containing applications in kinematics [4, 6, 9] and partial differential equations [9]. I am also a member of the development team of Bertini [3, 5], a leading edge software package in numerical algebraic geometry.

The remainder of this document provides a general background for numerical algebraic geometry, discusses my contributions, and presents my current and future research in the following three areas: large scale computing methods in numerical algebraic geometry, further development of numerical algebraic geometry, and applications utilizing numerical algebraic geometry. In particular, § 2 provides background information on numerical algebraic geometry and my contributions to this field, § 3 describes my contributions utilizing numerical algebraic geometry in computational algebraic geometry, and § 4 describes my current and future research.

2. Contributions within numerical algebraic geometry

Numerical algebraic geometry combines numerical analysis with algebraic geometry to compute and manipulate algebraic sets. This young field runs parallel to symbolic methods, e.g., Gröbner bases, in that both symbolic and numerical methods could be applied to many similar problems. Numerical methods are better suited for many applications in engineering and science since they can handle limited numerical accuracy and are easily parallelized.

The field of numerical algebraic geometry started by computing the isolated solutions of a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$ using continuation. Homotopy continuation works by continuously deforming a polynomial system with known solutions to a polynomial system with unknown solutions. A properly constructed homotopy implicitly defines continuous solution paths which can be tracked numerically using predictor–corrector methods, such as Euler prediction and Newton correction methods. For many of these paths, standard IEEE double precision arithmetic (16 digits) is sufficient, but, on some paths, the Jacobian can become ill-conditioned. To maintain numerical integrity on these paths, higher precision arithmetic (> 16 digits) is needed. To handle the ill-conditioning, the approach that we took in [4] was to analyze homotopy continuation and apply numerical linear algebra methods to create rules for when the precision needed to be changed along the path. After implementation and testing of this method, we discovered that the step size for the prediction method can be changed along with the precision to create a more robust and efficient tracking method. This new approach is described in [6] and is currently being used in Bertini.

Bertini [3, 5] is a software package written in C that provides an implementation of many algorithms in numerical algebraic geometry. It is a leading edge software package in the field and is under ongoing development. The beta version of Bertini has been publically available for over a year, Bertini 1.0

has been available for over 6 months, and Bertini 1.1 is expected to be released shortly. Bertini has been downloaded and used by researchers from many different fields all over the world.

In the past 12 years, building from the ability to compute isolated solutions, algorithms have been developed to compute and manipulate positive-dimensional algebraic sets using generic linear space sections of complimentary dimension [15]. Two important steps in computing a numerical irreducible decomposition, see [16] for more details, are to compute a witness superset for each dimension and to filter out the "junk points" from this set. In [9], we describe an efficient equation-by-equation method, called regeneration, for computing witness supersets. This article also presents examples that show that regeneration can be used to efficiently compute the isolated nonsingular solutions for systems arising in applications since regeneration can exploit that these systems typically have fewer nonsingular solutions than an *a priori* root count would suggest, e.g., [17].

For many problems, the standard junk point filtering method based on membership testing is a bottleneck when computing the numerical irreducible decomposition. In [1], we exploit the scheme structure at a solution to perform junk point filtering that computational evidence suggests eliminates the bottleneck. By using a numerical-symbolic approach for analyzing the scheme structure, the local dimension test presented in this article computes the maximum dimension of the irreducible components passing through the given solution.

3. Contributions utilizing numerical algebraic geometry techniques

Numerical algebraic geometric techniques can be used to perform computations in algebraic geometry. Aside from computing the local dimension at a solution [1], as mentioned in § 2, two other contributions in this direction are the decomposition of the rank-deficiency set of a matrix of multivariate polynomials and computations related to sheaf cohomology.

For a polynomial system $f(x)$ and a matrix $A(x)$ with polynomial entries, [2] describes an efficient numerical-symbolic method utilizing grassmanians to decompose the algebraic sets $S_k(x) = \{x \mid f(x) = 0, \text{ rank } A(x) \leq k\}$. This method has applications in computing the singular set of a polynomial system, the degeneracy set of an algebraic map, and the support of a module.

Let C be a union of reduced, irreducible pure-dimensional curves, D be a reduced set of points on C, and S be a reduced, connected, locally Cohen-Macaulay pure-dimensional surface. By using a numerical-symbolic approach, [8] presents algorithms which computes the dimension of the first cohomology of any twist of the ideal sheaf of C, computes the dimension of the first and second cohomology of any twist of the ideal sheaf of S, and solves the Reimann-Roch problem of computing the dimension of the space of divisors on C that are linearly equivalent to D.

4. Current and future research

4.1. **Large scale computing methods in numerical algebraic geometry.**

Regeneration [9], parallelization of other basic algorithms in numerical algebraic geometry, and the growth of parallel computing clusters has allowed me to solve problems that were considered impractical just a few years ago. By developing an efficient method for solving large scale polynomial systems, we can create algorithms that replace linear solving methods with polynomial solving methods.

The equation-by-equation approach of regeneration works by systematically building the solution set of the given polynomial system from smaller and easier to solve systems. The drawback of regeneration lies in the complexity of computing isolated singular solutions and generically nonreduced components. Recently, I have reformulated regeneration to address this issue. An article discussing this reformulation, called regenerative cascade, is in preparation and an initial implementation is complete. Computational experiments show that it is many times faster than traditional methods on moderately sized systems.

The regenerative methods along with many algorithms in numerical algebraic geometry are naturally parallelizable. Our research group has a cluster that consists of 240 processing cores that we have been using to develop a parallel implementation, using MPI, of many of the basic algorithms in numerical algebraic geometry. These implementations will be available in the upcoming release of Bertini 1.1. This experience has allowed me to understand the challenges and implementation changes needed for solving large scale problems on increasingly larger parallel clusters.

With parallel implementations of regeneration and regenerative cascade, Bertini will be the leader in large scale computing methods in numerical algebraic geometry. With the goal of staying a leading edge software package in numerical algebraic geometry, I am working with the other developers to formulate a plan for Bertini 2.0. This version, written in C++, will be more user-friendly and scriptable allowing users to write modules utilizing the core of Bertini, which will expand the capabilities of Bertini and its use. With easy access to core functions in Bertini, we will work with algebraic geometers to develop and implement methods that utilize the strengths of both numerical and symbolic approaches.

4.2. **Development in numerical algebraic geometry.** There are many exciting research projects in the field of numerical algebraic geometry. For example, some research projects that I will pursue include developing methods for computing and manipulating real surfaces, understanding where deflation [10, 11, 16] is not one-to-one, and developing a practical method for computing exceptional sets building upon the foundation of [14].

Two more research projects in numerical algebraic geometry are computing the local dimension of a solution with respect to a given algebraic set and numerically computing Hilbert functions for a collection of irreducible components. For a given solution to a polynomial system, the local dimension algorithm presented in [1], as mentioned in § 2, computes the maximum dimension of the irreducible components of the polynomial system passing through the solution by analyzing its scheme structure. A natural extension to the local dimension algorithm that I will investigate is to analyze the scheme structure at a solution to compute the local dimension with respect to an implicitily defined algebraic subset of the solution set of the polynomial system.

Computing the local dimension is an example of how numerical methods are very effective at computing local information. Another example is computing the Hilbert function for a zero-scheme defined by a collection of isolated solutions for a polynomial system. We are preparing an article that presents a numerical algorithm to compute the Hilbert function, regularity, and standard monomials for the zero-scheme defined by that collection of isolated solutions by using a dual basis [12, 7, 1] constructed at each solution. Building from this method, I will investigate numerical methods for computing the Hilbert function, Hilbert polynomial, and regularity for a collection of irreducible components for a polynomial system.

4.3. **Applications using numerical algebraic geometry.** My research in numerical algebraic geometry is motivated by problems arising from real-world applications. Two such applications are partial differential equations modeling biological phenomena and dynamical systems modeling combustion reactions.

Using a fine mesh, discretizing a system of nonlinear (algebraic) partial differential equations (PDEs) produces a large scale polynomial system. Typically, solving these systems is impractical using traditional solving methods, e.g., [9, § 9.4]. These polynomial systems are naturally sparse due to the discretization and, for PDEs arising in mathematical biology, typically only have a few solutions that are biologically meaningful. By using regenerative techniques and exploiting the sparsity, we have conducted experiments to confirm that numerical algebraic geometry can compute time-evolution and steady-state solutions for nonlinear PDEs arising in mathematical biology.

An example is a model for the dorsal-ventral patterning of a zebrafish. For given parameter values, it is known that the model has multiple steady-state solutions, which means that different initial conditions can lead to different dorsal-ventral patterns. We used polynomial systems to compute 7 steady-state solutions and found that 3 of these are stable, which are biologically meaningful. I will build upon these experiments to develop and implement numerical algebraic geometric based solving methods for nonlinear PDEs as well as research applying numerical algebraic geometry to free-boundary problems.

Another application where polynomial systems naturally arise is dynamical systems. Working with engineers in the Department of Aerospace and Mechanical Engineering at the Unversity of Notre Dame, we setup and solved polynomial systems whose solutions yielded the equilibrium point and critical points of a dynamical system modeling a combustion reaction in a closed reactive system. An article in prepartion describes how to use these solution points to construct the one-dimensional slow invariant manifold (SIM) for these systems. The SIM provides engineers information on a system's dynamics as it relaxes toward equilibrium. These polynomial systems, typical of chemical reactions, were poorly scaled. By utilizing SCLGEN [13] and devloping other numerical scaling techniques, we were able to compute the solutions to these polynomial systems accurately and reliably. I will build upon this experience to develop and implement robust methods to solve poorly scaled polynomial systems.

Above all, the training and research that I have in algebraic geometry, parallel and scientific computing, and numerical analysis, as well as interest in many applications, has provided me a foundation to continue performing new and exciting research for many years to come.

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