Membership tests for images of algebraic sets by linear projections

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Abstract

Given a witness set for an irreducible variety V and a linear map π, we describe membership tests for both the constructible algebraic set $\pi(V)$ and the algebraic set $\overline{\pi(V)}$. We also provide applications and examples of these new tests including computing the codimension one components of $\overline{\pi(V)} \setminus \pi(V)$. Additionally, we also describe computing the geometric genus of a curve section of an irreducible component of the solution set of a polynomial system and a test for deciding whether a plane quartic curve is a Lüroth quartic.
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Introduction

Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$, an ℓ-dimensional irreducible component $V \subset f^{-1}(0)$, and a linear map $\pi : \mathbb{C}^N \to \mathbb{C}^K$, a "witness set" for $\overline{\pi(V)}$ was constructed in [7] from a witness set for V, hereafter called a *pseudo-witness set* for $\overline{\pi(V)}$. This approach reduces computations on $\overline{\pi(V)}$ to computations on V without using elimination theory to construct a polynomial system g such that $\overline{\pi(V)}$ is an irreducible component of $g^{-1}(0)$.

The main results of this article, presented in §2, are algorithms for performing a numerical membership test for both $\pi(V)$ and $\overline{\pi(V)}$.

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Chevalley's Theorem [4] states that the image of a constructible set, e.g., $\pi(V)$, is a constructible set[1]. Effective symbolic methods for performing computations with constructible sets are discussed in [5, 19].

In §3, we use these membership tests to compute a numerical decomposition of the irreducible components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$ of codimension one in $\overline{\pi(V)}$ and use this to develop an approach for computing the geometric genus of a generic curve section of $\overline{\pi(V)}$.

The necessary background material is presented in §1 which also codifies the properties of our substitute for witness sets into the notion of a pseudo-witness set.

In §4, we present examples using our new membership tests.

1 Background material

We collect some background material in this section. Throughout, we assume $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and define $\mathcal{V}(f)$ to be the set of points in $\mathbb{C}^N$ which f maps to 0. The algebraic set $\mathcal{V}(f)$ is reduced and, in particular, all of the irreducible components of $\mathcal{V}(f)$ have multiplicity one. We let $f^{-1}(0)$ denote $\mathcal{V}(f)$ with its underlying scheme structure, which includes the multiplicity information of the components of $\mathcal{V}(f)$ with regard to f.

1.1 Witness sets

Suppose that $V \subset f^{-1}(0)$ is an ℓ-dimensional irreducible algebraic set of degree d. A *witness set* for V is the triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L}$ consists of ℓ general linear polynomials on $\mathbb{C}^N$ and $W = V \cap \mathcal{V}(\mathcal{L})$. The *witness point set* W consists of d points. A finite set $\mathcal{W}$ with $W \subset \mathcal{W} \subset V$ is called a *witness point superset* for V.

The *multiplicity* of V with respect to f is the multiplicity of any $w \in W$ as a root of $\begin{bmatrix} f \\ \mathcal{L} \end{bmatrix}$. The component V is said to be *generically reduced* with respect to f if the multiplicity of V with respect to f is 1. Otherwise, V is said to be *generically nonreduced*, which we consider in the following section. See [23, Chap. 13] for more details regarding witness sets.

1.2 Deflation

If V is generically nonreduced with respect to f, then the deflation approach of [10] produces a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^m$, with $m \geq n$, such that $F^{-1}(0)$ has an irreducible and generically reduced component $\widehat{V}$ which, as a set, is equal to V. By renaming as necessary, we will assume *without loss of generality* that V is generically reduced with respect to f.

[1] A *constructible subset* of an algebraic set X is any set in the Boolean algebra of subsets of X obtained by starting with algebraic subsets of X and closing up under the operations of finite unions and complementation.

It should be noted that more traditional versions of deflation (see also [8, 11, 12] and [23, §13.3.2, §15.2.2]) change the dimension of the ambient space and may replace V with an algebraic set V' that maps generically one-to-one onto a dense subset of V.

1.3 Randomization

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $1 \leq k \leq n$. For $A \in \mathbb{C}^{k\times(n-k)}$, let

$$\mathcal{R}(f;k) = [I_k \; A] \cdot f$$

where I_k is the $k \times k$ identity matrix. It is a consequence of Bertini's theorem, e.g., [22] or [23, §13.5], that any irreducible codimension k component of $\mathcal{V}(f)$ is an irreducible component of $\mathcal{V}(\mathcal{R}(f;k))$ for a nonempty Zariski open (and hence dense) set of matrices $A \in \mathbb{C}^{k\times(n-k)}$. Thus, we will assume *without loss of generality* that $f : \mathbb{C}^N \to \mathbb{C}^k$ is a polynomial system where $V \subset f^{-1}(0)$ is a codimension k irreducible component.

1.4 Pseudo-witness sets

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $\{f, \mathcal{L}, W\}$ be a witness set for an irreducible and generically reduced component $V \subset f^{-1}(0)$ of dimension ℓ. Suppose that $\pi : \mathbb{C}^N \to \mathbb{C}^K$ is a linear map and $B \in \mathbb{C}^{K\times N}$ such that $\pi(x) = Bx$.

Even though the set $\pi(V)$ might not be an algebraic set, it is very close to an algebraic set. More specifically, $\pi(V)$ is a *constructible algebraic set* which means that it is a member of the Boolean algebra of sets constructed from algebraic sets by the operations of finite unions, finite intersections, and complementation. A typical example is the projection onto (x, y) of $\mathcal{V}(x - yz)$: the image is $\left(\mathbb{C}^2 \setminus \mathcal{V}(y)\right) \cup \{(0, 0)\}$.

The closure of a constructible algebraic set C in the complex topology $\overline{C}$ is the same as the closure of C in the Zariski topology. The same statement holds for the interior C° of C with $\overline{C^\circ} = \overline{C}$. In particular, since the dimensions of $\overline{C}$ and C° are equal, the dimension of C is well-defined. Finally, if $\overline{C}$ is pure k-dimensional, then $\overline{C} \cap L = C^\circ \cap L$ for a general affine linear space L of codimension k. Additional details for constructible algebraic sets is provided in [23, Appendix A].

Let $\ell' = \dim \overline{\pi(V)}$. For $i = 1, \ldots, \ell'$, let $b_i \in \mathbb{C}^N$ be general elements in the row span of B and, for $i = \ell' + 1, \ldots, \ell$, let $b_i \in \mathbb{C}^N$ be general elements in $\mathbb{C}^N$. We call the quadruple $\{f, \pi, \mathcal{L}', W'\}$ [7], where

$$\mathcal{L}'(x) = \begin{bmatrix} b_1 \cdot x - 1 \\ \vdots \\ b_\ell \cdot x - 1 \end{bmatrix} \quad \text{and} \quad W' = V \cap \mathcal{V}(\mathcal{L}'),$$

a *pseudo-witness set* for $\pi(V)$ with $\deg \overline{\pi(V)} = |\pi(W)|$.

A pseudo-witness set may be efficiently used to fulfill the same tasks for which a witness set for $\overline{\pi(V)}$ would be used if we had a set of polynomials on $\mathbb{C}^K$ whose solution set contained $\overline{\pi(V)}$ as an irreducible component. One example is using pseudo-witness sets in place of witness sets to work with the numerical irreducible decomposition [20] of the closure of the image of an algebraic map, e.g., [1, §2.1.3].

1.5 Moving linear spaces

The membership tests developed in this article are based on moving linear spaces. Let $\{f, \mathcal{L}, W\}$ be a witness set for an irreducible and generically reduced $V \subset f^{-1}(0)$ of dimension ℓ where $f : \mathbb{C}^N \to \mathbb{C}^{N-\ell}$ and $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^\ell$. Given a system of linear polynomials $\widehat{\mathcal{L}} : \mathbb{C}^N \to \mathbb{C}^\ell$ with $\dim \mathcal{V}(\widehat{\mathcal{L}}) = N - \ell$, we want to compute the set of points $\widehat{W} := V \cap \mathcal{V}(\widehat{\mathcal{L}}) \subset V$ by deforming $\mathcal{L}$ to $\widehat{\mathcal{L}}$ using the "square" homotopy $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ defined by

$$H(x,t) = \begin{bmatrix} f(x) \\ (1-t)\widehat{\mathcal{L}}(x) + t\mathcal{L}(x) \end{bmatrix}. \tag{1}$$

Starting at $t = 1$ with the points in W, continuation allows one to track the path defined by $H(x,t) \equiv 0$ as t goes from 1 to 0. Additional details are provided in [23].

Of the $|W|$ paths tracked using the homotopy H, some of them may diverge as t approaches 0. The set $\widehat{W}$ is the set of endpoints of the paths that converge to a point in $\mathbb{C}^N$ as t approaches 0.

One application of moving linear spaces is the homotopy membership test, first described in [21], which replaced the more expensive interpolation test of [20]. Given a point $y \in \mathbb{C}^N$, let $\widehat{\mathcal{L}} : \mathbb{C}^N \to \mathbb{C}^\ell$ be a system of general linear polynomials such that $y \in \mathcal{V}(\widehat{\mathcal{L}})$. If $\widehat{W}$ is the set of finite endpoints of the homotopy H defined in (1) starting at each point in W, then $y \in V$ if and only if $y \in \widehat{W}$.

1.6 Geometric genus of a curve

In [3], a numerical algorithm is given for computing the geometric genus of an irreducible one-dimensional component $R \subset \mathbb{C}^K$ of the solution set of a polynomial system. The geometric genus of R is the topological genus of the unique smooth compactification of the desingularization of R. Since the desingularizations of a curve and a generically one-to-one image of a curve are isomorphic, deflation of a component will not change its geometric genus. Therefore the component R may be assumed to have multiplicity one. The algorithm of [3], which is based on the Hurwitz theorem, starts with the restriction $p : R \to \mathbb{C}$ of a linear projection $A : \mathbb{C}^K \to \mathbb{C}$.

In that article p is assumed proper, but this is *easily modified* as will be shown below. We also show how the algorithm may be applied to an irreducible curve $R \subset \mathbb{C}^K$ arising as the closure of a constructible set $R' \subset \mathbb{C}^K$.

Let us explain the algorithm of [3].

We may regard A as the product projection $\mathbb{C}^{K-1} \times \mathbb{C} \to \mathbb{C}$. We let $\overline{A}$ be the product projection of $\mathbb{P}^{K-1} \times \mathbb{P}^1 \to \mathbb{P}^1$. Taking the closure $\overline{R}$ of R in $\mathbb{P}^{K-1} \times \mathbb{P}^1$, we have the proper map $\overline{p} := \overline{A}_{\overline{R}}$. Let $s : \widehat{R} \to \overline{R}$ denote the desingularization of $\overline{R}$ and $\widehat{p} : \widehat{R} \to \mathbb{P}^1$

the map $s \circ \overline{p}$. Then, Hurwitz theorem tells us that

$$g = -2\deg(\widehat{p}) + \rho$$

where

1. g is the genus of $\widehat{R}$, which we want to compute;

2. $\deg(\widehat{p})$ is the degree of $\widehat{p}$, which equals the degree of p; and

3. ρ is the ramification of $\widehat{p}$.

From the above we see that we need to compute ρ. Let $\mathcal{R}$ denote the images under $\widehat{p}$ of the branch points of $\widehat{p}$. For any $y \in \mathcal{R}$, let Δ_y denote a contractible set with a continuous and piecewise differentiable boundary, e.g., a disk in a Euclidean patch $\mathbb{C} \subset \mathbb{P}^1$ containing y, such that no points of $\mathcal{R}$ other than y are in Δ_y. Fix a point x_y on the boundary of Δ_y. We have a monodromy transformation $T_y : \widehat{p}^{-1}(x_y) \to \widehat{p}^{-1}(x_y)$ obtained by continuation around the boundary of Δ_y of the paths starting at points of $p^{-1}(x_y)$. The ramification ρ is a sum of contributions ρ_y for the points $y \in \mathcal{R}$.

The number ρ_y equals $\deg(\widehat{p})$ minus the number of orbits of the permutation group on $\widehat{p}^{-1}(x_y)$ generated by T_y. There are two main observations of [3].

The first is that

- ρ_y may be computed using the monodromy transformation $T_y : \overline{p}^{-1}(x_y) \to \overline{p}^{-1}(x_y)$. (We use the same symbol T_y because $p^{-1}(x_y)$ is naturally identified with $\widehat{p}^{-1}(x_y)$ and under this identification, the monodromy transformations are the same.)

Though computing $\mathcal{R}$ is involved, it is straightforward (see [3]) to compute a finite set on $\overline{R}$ that maps to a finite set of $\mathbb{P}^1$ containing $\mathcal{R}$. It suffices to work with this larger set instead of $\mathcal{R}$ is a consequence of the second main observation:

- for any point y not in $\mathcal{R}$, the local monodromy contribution of ρ_y is zero.

Note also that we can work with $p : R \to \mathbb{C}$ as long as we also do a calculation of ρ_∞ by going around a large enough circle on $\mathbb{C}$, so that any point of $\mathcal{R}$ (except possibly for ∞) is contained within the circle.

If p is not proper, we simply need to add to $\mathcal{R}$ the points over which p is not proper, i.e., we add to $\mathcal{R}$ the image under $\overline{A}$ of the set $\overline{R} \cap \left(\mathbb{P}^{K-1} \setminus \mathbb{C}^{K-1}\right) \times \mathbb{C}$. As above, this may add a finite number of extra points without any harm to the final result.

Extension to constructible sets Finally, assume $R' \subset \mathbb{C}^K$ is a constructible set whose closure is an irreducible curve $R \subset \mathbb{C}^K$. We fix a linear projection $A : \mathbb{C}^K \to \mathbb{C}$ and set p equal to A restricted to R. Possibly making a linear change of coordinates, we regard A as the product projection $\mathbb{C}^{K-1} \times \mathbb{C} \to \mathbb{C}$. We let $\overline{A}$ denote the product projection $\mathbb{P}^{K-1} \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\overline{R}$ denote the closure of R in $\mathbb{P}^{K-1} \times \mathbb{P}^1$, we have the proper map $\overline{p} := \overline{A}_{\overline{R}}$. We let $\widehat{p} : \widehat{R} \to \mathbb{P}^1$ denote the composition of the desingularization map $s : \widehat{R} \to \overline{R}$ and $\overline{p}$.

Looking over the argument sketched above for the algorithm to compute g in the case when $R' = R$, we see that the algorithm for a constructible set R' to work, we need to compute

1. the degree of p; and
2. a finite subset of $\mathbb{P}^1$ containing $\widehat{p}(\mathcal{R})$, where $\mathcal{R}$ is the set of branch points of the $\widehat{p}$.

The set $\mathcal{R}$ is contained in the union of $\infty \in \mathbb{P}^1$ and the images under p of

1. the singular points of R;
2. the points $R \setminus R'$;
3. $A\left(\left(\mathbb{P}^{K-1} \setminus \mathbb{C}^K\right) \cap \overline{R}\right)$; and
4. all of the branch points of the algebraic map $p : R \to \mathbb{C}$.

2 Membership tests for projections

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $V \subset f^{-1}(0)$ be an irreducible algebraic set of dimension ℓ. As developed in §1, we may assume without loss of generality that V is generically reduced and $n = N - \ell$.

Let $\pi : \mathbb{C}^N \to \mathbb{C}^K$ be a linear map and $y \in \overline{\pi(\mathbb{C}^N)} \subset \mathbb{C}^K$. We will first use a pseudo-witness set $\{f, \pi, \mathcal{L}, \mathcal{W}\}$ for $\overline{\pi(V)}$ to determine if $y \in \overline{\pi(V)}$ and provide sufficient conditions for deciding if $y \in \pi(V)$. We will then use a witness set $\{f, L, W\}$ for V to determine if $y \in \pi(V)$.

2.1 Basic membership test

Let $\ell' = \dim \overline{\pi(V)}$ and $\mathcal{L} = [\mathcal{L}_1 \ \cdots \ \mathcal{L}_\ell]^T$ such that $\mathcal{L}_1, \ldots, \mathcal{L}_{\ell'}$ are general linear polynomials on $\pi(\mathbb{C}^N)$ and $\mathcal{L}_{\ell'+1}, \ldots, \mathcal{L}_\ell$ are general linear polynomials on $\mathbb{C}^N$. For $i = 1, \ldots, \ell'$, let $\widehat{\mathcal{L}}_i : \mathbb{C}^K \to \mathbb{C}$ be a general linear polynomial such that $y \in \mathcal{V}(\widehat{\mathcal{L}}_i)$ and define

$$\widehat{\mathcal{L}}(x) = \begin{bmatrix} \widehat{\mathcal{L}}_1(\pi(x)) \\ \vdots \\ \widehat{\mathcal{L}}_{\ell'}(\pi(x)) \\ \mathcal{L}_{\ell'+1}(x) \\ \vdots \\ \mathcal{L}_\ell(x) \end{bmatrix}.$$

Consider the homotopy H defined by (1) which deforms $\mathcal{L}$ to $\widehat{\mathcal{L}}$. For each $w \in \mathcal{W}$, let $x_w(t)$ be the path defined by $x_w(1) = w$ and $H(x_w(t), t) \equiv 0$ for $t \in (0, 1]$. There are three possibilities for each path $x_w(t)$ as t approaches 0, namely

1. $x_w(t)$ converges to a point in $\mathbb{C}^N$ yielding that $\pi(x_w(t))$ converges to a point in $\mathbb{C}^K$;
2. $x_w(t)$ diverges but $\pi(x_w(t))$ converges to a point in $\mathbb{C}^K$; or
3. $x_w(t)$ and $\pi(x_w(t))$ both diverge.

Consider the related sets:

1. $C_y = \{\lim_{t\to 0} \pi(x_w(t)) \mid w \in \mathcal{W} \text{ and } \lim_{t\to 0} x_w(t) \text{ converges}\}$; and
2. $P_y = \{\lim_{t\to 0} \pi(x_w(t)) \mid w \in \mathcal{W} \text{ and } \lim_{t\to 0} \pi(x_w(t)) \text{ converges}\}$.

Clearly, $C_y \subset P_y \subset \overline{\pi(V)}$. The following lemma yields a membership test for $\overline{\pi(V)}$ using P_y and sufficient conditions for deciding if $y \in \pi(V)$ using C_y.

Lemma 1. *With the setup described above, we have the following tests:*

1. *$y \in \overline{\pi(V)}$ if and only if $y \in P_y$;*
2. *if $y \in C_y$, then $y \in \pi(V)$; and*
3. *if $C_y = P_y$ or $\dim \overline{\pi(V)} = 1$, then $y \in C_y$ if and only if $y \in \pi(V)$.*

Proof. Define $\mathcal{L}_y(z) = \begin{bmatrix} \widehat{\mathcal{L}}_1(z) \\ \vdots \\ \widehat{\mathcal{L}}_{\ell'}(z) \end{bmatrix}$. By genericity, $\overline{\pi(V)} \cap \mathcal{V}(\mathcal{L}_y)$ consists of finitely many points. It follows from [14] that $P_y = \overline{\pi(V)} \cap \mathcal{V}(\mathcal{L}_y)$. Since $y \in \mathcal{V}(\mathcal{L}_y)$, we know that $y \in P_y$ if and only if $y \in \overline{\pi(V)}$.

If $y \in C_y$, then there exists $w \in \mathcal{W}$ and $\alpha \in V \subset \mathbb{C}^N$ such that $\alpha = \lim_{t\to 0} x_w(t)$ and $y = \pi(\alpha) = \lim_{t\to 0} \pi(x_w(t))$. Since $\alpha \in V$, this implies $y = \pi(\alpha) \in \pi(V)$.

The only part remaining for the final statement is showing that $y \in \pi(V)$ implies $y \in C_y$. If $C_y = P_y$, this follows from the first statement. If $\dim \overline{\pi(V)} = 1$, we know that $y \in \pi(V)$ implies that $V \cap \pi^{-1}(y)$ is pure-dimensional of dimension $\dim V - 1$ since V is irreducible. Therefore, this case also follows from [14]. $\square$

Remark 2. *If $w_1, w_2 \in \mathcal{W}$ such that $\pi(w_1) = \pi(w_2)$, then $\pi(x_{w_1}(t)) = \pi(x_{w_2}(t))$ for all $t \in (0, 1]$. In particular, we only need to track at most $\deg \overline{\pi(V)} = |\pi(W)|$ paths in order to determine if $y \in \overline{\pi(V)}$.*

Remark 3. *We note that the membership test of this section immediately applies to a wide class of projections of quasialgebraic sets[2]. For example, consider the product*

[2] A quasialgebraic set is a set of the form $A \setminus B$, where A and B are algebraic subsets of $\mathbb{P}^N$.

projection $\pi_{\mathbb{P}} : \mathbb{P}^{N-k} \times \mathbb{P}^K \to \mathbb{P}^K$*. Let* $X \subset \mathbb{P}^{N-k} \times \mathbb{P}^K$ *be a quasialgebraic set. Let* $y \in \mathbb{P}^K$ *be a point that we wish to check is in* $\overline{\pi_{\mathbb{P}}(X)}$*. Choose a generic Euclidean patch* $U \subset \mathbb{P}^{N-k} \times \mathbb{P}^K$*, i.e., choose generic hyperplanes* $H_v \subset \mathbb{P}^{N-K}$ *and* $H_h \subset \mathbb{P}^K$ *and let*

$$U = \left(\mathbb{P}^{N-K} \setminus H_v\right) \times \left(\mathbb{P}^K \setminus H_h\right).$$

Then with probability one, $y \in \left(\mathbb{P}^K \setminus H_h\right)$ *and if* $y \in \overline{\pi_{\mathbb{P}}(X)} = \pi_{\mathbb{P}}(\overline{X})$*, i.e., if there is an* $x \in \overline{X}$ *going to* y*, then* $x \in \left(\mathbb{P}^{N-K} \setminus H_h\right)$*.*

2.2 Advanced membership test

We see from Lemma 1 that the one remaining case is deciding if $y \in \pi(V)$ given that $y \in P_y \subset \overline{\pi(V)}$, $y \notin C_y$, and $\dim \overline{\pi(V)} > 1$. The advanced membership test is based on the fact that $y \in \pi(V)$ if and only if $V \cap \pi^{-1}(y)$ is nonempty. That is, one simply computes the intersection of V with the linear space $\pi^{-1}(y)$. This can be accomplished starting with a witness set for V together with slice moving which we perform following a regenerative cascade approach [9]. Since we only need to decide if $V \cap \pi^{-1}(y)$ is empty, the test simply cascades down through the dimensions under consideration and terminates when either a point in $V \cap \pi^{-1}(y)$ is found or all of the possible dimensions are empty. As above, let $\ell = \dim V$ and $\ell' = \dim \overline{\pi(V)}$. Then, since $\ell - \ell'$ is the general fiber dimension, the possible fiber dimensions under consideration are $\ell - 1, \ell - 2, \ldots, \ell - \ell'$. Thus, this test tracks at most $\ell' \cdot \deg V$ paths. If we have already verified that $y \notin C_y$ from §2.1, then we do not need to consider the general fiber dimension. In this case, this test tracks at most $(\ell' - 1) \cdot \deg V$ additional paths.

Let $\{f, L, W\}$ be a witness set for V where $L = [L_1, \ldots, L_\ell]^T$ and $\widehat{\mathcal{L}}_1, \ldots, \widehat{\mathcal{L}}_{\ell'}$ be general linear polynomials on $\mathbb{C}^K$ such that $y \in \mathcal{V}(\widehat{\mathcal{L}}_i)$. For $i = 0, \ldots, \ell'$, define

$$\mathcal{M}_i(x) = \begin{bmatrix} \widehat{\mathcal{L}}_1(\pi(x)) \\ \vdots \\ \widehat{\mathcal{L}}_i(\pi(x)) \\ L_{i+1}(x) \\ \vdots \\ L_\ell(x) \end{bmatrix}.$$

We have $\mathcal{M}_0 = L$ and define $S_0 = W$. If $0 \le i < \ell'$ such that S_i is known, we compute S_{i+1} as follows. Let W_{i+1} be the finite endpoints of the modified homotopy H defined by (1) which deforms $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$ with start points S_i. Let G_{i+1} be the subset of points of W_{i+1} which π maps to y and $S_{i+1} = W_{i+1} \setminus G_{i+1}$. In particular, it follows from [9, §2] that each point in S_{i+1} is a nonsingular root of $\begin{bmatrix} f \\ \mathcal{M}_{i+1} \end{bmatrix}$ and G_{i+1} is a witness point superset for the pure $(i+1)$-codimensional component of $V \cap \pi^{-1}(y)$.

Lemma 4. *With the setup described above, $y \in \pi(V)$ if and only if $G_i \neq \emptyset$ for some $i \in \{1, \ldots, \ell'\}$. Moreover, if $y \notin C_y$, where C_y is defined as in §2.1, then $y \in \pi(V)$ if and only if $G_i \neq \emptyset$ for some $i \in \{1, \ldots, \ell' - 1\}$.*

Proof. This follows from the above discussion together with [9, Lemma 2.2 & Theorem 2.3] applied to this context. □

Remark 5. *By working with generic Euclidean patches as in Remark 3, the membership test of this section extends to a wide class of projections of quasiprojective sets. We will use the version for the restriction of the product projection $\pi_{\mathbb{P}} : \mathbb{P}^{N-K} \times \mathbb{P}^K \to \mathbb{P}^K$ in the next section.*

3 Codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$

In order to compute more detailed invariants of $\pi(V)$, it may be necessary to have a numerical irreducible decomposition of $\overline{\overline{\pi(V)} \setminus \pi(V)}$, i.e., a numerical irreducible decomposition of $\overline{\pi(V)} \setminus \pi(V)^\circ$, where the set $\mathcal{C}^\circ$ is the largest Zariski open set contained in the constructible algebraic set $\mathcal{C}$. The results of this article allow us to compute the decomposition of the codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$. As an illustration, we describe how to use this partial decomposition to compute a basic invariant of $\overline{\pi(V)}$.

3.1 Decomposition of codimension one components

Assume we have the standard setup, i.e., $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and $V \subset f^{-1}(0)$ is an irreducible ℓ-dimensional component. By §1.2 and §1.3, we may assume without loss of generality that V is generically reduced and $n = N - \ell$, respectively. For simplicity, assume that $\pi : \mathbb{C}^N \to \mathbb{C}^K$ is a linear projection onto the last K coordinates. Note that the projection π extends to the product projection $\pi_{\mathbb{P}} : \mathbb{P}^{N-K} \times \mathbb{P}^K \to \mathbb{P}^K$. Let $V_{\mathbb{P}}$ denote the closure of V in $\mathbb{P}^{N-K} \times \mathbb{P}^K$.

Define $\mathcal{P} = \mathbb{P}^{N-K} \times \mathbb{C}^K$ and $\mathcal{E} = \left(\mathbb{P}^{N-K} \setminus \mathbb{C}^{N-K}\right) \times \mathbb{C}^K$. Let $V_{\mathcal{P}}$ denote the closure of V in $\mathcal{P}$ and $\pi_{\mathcal{P}}$ denote the restriction of $\pi_{\mathbb{P}}$ to $\mathcal{P}$. We have the following:

1. $\pi_{\mathbb{P}}(V_{\mathbb{P}})$ is the closure of $\pi(V)$ in $\mathbb{P}^K$; and
2. $\overline{\pi(V)}$, the closure of $\pi(V)$ in $\mathbb{C}^K$, equals $\pi_{\mathcal{P}}(V_{\mathcal{P}})$.

We first consider the case when $\dim \overline{\pi(V)} = 1$. In this case, all fibers of $\pi_{\mathcal{P}}$ restricted to $V_{\mathcal{P}}$, i.e., $\pi_{\mathcal{P}|V_{\mathcal{P}}}$, are of pure dimension $\dim \overline{V} - 1$. Since $\dim(V_{\mathcal{P}} \cap \mathcal{E}) = \dim \overline{V} - 1$, we conclude that the irreducible components of the fibers of $\pi_{\mathcal{P}|V_{\mathcal{P}}}$ over the finite set $\overline{\pi(V)} \setminus \pi(V)$ are irreducible components of $V_{\mathcal{P}} \cap \mathcal{E}$. That is, we can first compute the numerical irreducible decomposition of $V_{\mathcal{P}} \cap \mathcal{E}$ and then use the membership test of this article to determine the points of $\overline{\pi(V)} \setminus \pi(V)$.

Now assume that $\dim \overline{\pi(V)} \geq 2$. It is a consequence of a vanishing theorem of Picard-Kodaira type [18, Theorem 3.42] that, if $\dim \overline{\pi(V)} \geq 2$, then for a general hyperplane

$\mathcal{H}$ of $\mathbb{C}^K$, $H = \pi_{\mathcal{P}}^{-1}(\mathcal{H}) \cap V_{\mathcal{P}}$ is irreducible. Since a general linear space of codimension $\dim \overline{\pi(V)} + 1$ meets $\overline{\pi(V)}$ in an irreducible curve and meets each codimension one component A of $\overline{\overline{\pi(V)} \setminus \pi(V)}$ in $\deg A$ points, we have constructed a pseudo-witness point set for the union of codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$. This shows in particular that for each codimension one irreducible component A of $\overline{\overline{\pi(V)} \setminus \pi(V)}$, there is a $\dim V - 1$ component of $V_{\mathcal{P}} \cap \mathcal{E}$ surjecting onto A.

We may compute a numerical irreducible decomposition of the $\dim \overline{V} - 1$ components $\mathcal{A}$ of $V_{\mathcal{P}} \cap \mathcal{E}$ and then use the membership test for the images of the witness sets of these components to check which components have images in $\overline{\overline{\pi(V)} \setminus \pi(V)}$. For the irreducible components $\mathcal{A}$ with an image A in $\overline{\overline{\pi(V)} \setminus \pi(V)}$, the results of [7] using the map $\pi_{\mathcal{A}} : \mathcal{A} \to A$ yield a pseudo-witness set of A. Finally, we need to compute the dimensions of the fibers over the images of one point from the witness set of $\mathcal{A}$: those A of codimension one are precisely the ones where the fiber dimension is $\dim V - \dim \overline{\pi(V)}$.

3.2 Application to the geometric genus of a curve section

As an application, we describe how we may use the pseudo-witness set for the codimension one boundary components to compute the geometric genus g of a general curve section of $\overline{\pi(V)} \subset \mathbb{C}^K$. The number g is, by definition, equal to the genus of the desingularization $s : \widehat{R} \to R$ of the intersection $R \subset \mathbb{C}^K$ of $\overline{\pi(V)}$ and a general affine linear space L of $\mathbb{C}^K$ of dimension $K + 1 - \dim \overline{\pi(V)}$. Topologically, g is the usual genus of the unique smooth compactification of $\widehat{R}$. Equivalently, it is the genus of the desingularization of the closure in $\mathbb{P}^K$ of the intersection of $\overline{\pi(V)}$ and a general affine linear space of $\mathbb{C}^K$ of dimension $K + 1 - \dim \overline{\pi(V)}$.

Given a general affine linear space L of $\mathbb{C}^K$ of dimension $K + 1 - \dim \overline{\pi(V)}$, we take $R = \overline{\pi(V)} \cap L$ and $R' = \pi(V) \cap L$. Using [18, Theorem 3.42], we can again reduce down to the case that $\dim \overline{\pi(V)} = 1$. We take the map $p : R \to \mathbb{C}$ to be the restriction to R to any linear projection from $\mathbb{C}^K$ to $\mathbb{C}$. By taking the intersection of V with a general affine linear space of codimension $\ell - 1$, we may, by renaming if necessary, assume that V is one-dimensional. Let $q : V \to R$ be the map obtained by composing the restriction of π to V with the map p.

For Q, we take the union of the following sets:

1. the image under q of the branchpoints of q;
2. the image under q of the singular points of V;
3. the image under $p \circ \pi_{\mathcal{P}}$ of the set $V_{\mathcal{P}} \cap \mathcal{E}$ using the notation from §3.1; and
4. the image under p of the points in $\overline{\overline{\pi(V)} \setminus \pi(V)}$.

The first three items require only standard computations. The last item follows from the computation of the finite set $\overline{\overline{\pi(V)} \setminus \pi(V)}$, which was computed in §3.1. Note the third item is a finite set of points containing the points over which q is not proper and therefore also the points over which p is not proper.

To see that this set Q suffices, note that all the singular points of $\overline{\pi(V)}$ and branchpoints of p are either over $\overline{\overline{\pi(V)} \setminus \pi(V)}$ or in the image of the branchpoints of $p \circ \pi_{\mathcal{P}}$ and the singular set of $V_{\mathcal{P}}$.

The last item needed is the ability to track paths. We note that the paths on $\overline{\pi(V)}$ which are contained in $\pi(V)^\circ$ may be tracked using the pseudo-witness set of $\overline{\pi(V)}$.

4 Examples

We conclude by demonstrating the membership tests and codimension one decomposition on illustrative examples and then report on a more advanced example. The linear slice moving computations reported here were performed using Bertini v1.3.1 [2].

4.1 A parameterized circle

Consider the rational parameterization $(x(s), y(s)) = \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}\right)$ of an open dense subset the unit circle. Clearing denominators, this parameterization yields the system

$$f(s, x, y) = \begin{bmatrix} x(1+s^2) - (1-s^2) \\ y(1+s^2) - 2s \end{bmatrix}$$

with the accompanying projection $\pi(s, x, y) = (x, y)$ defined by the matrix $B = [0 \;\; I_2]$ where I_2 is the 2×2 identity matrix. It is easy to verify that $V = f^{-1}(0)$ is a irreducible curve of degree 3 that is generically reduced with respect to f. We will first use a witness set for V to construct a pseudo-witness set for $\overline{\pi(V)}$ and then determine if $z_j \in \pi(V)$ and $z_j \in \overline{\pi(V)}$ where

$$z_1 = (0, 1), \quad z_2 = (-1, 0), \quad z_3 = (\sqrt{2}, i), \quad \text{and} \quad z_4 = (1 + i, 1/3 - i/2)$$

with $i = \sqrt{-1}$. Finally, we will compute $\overline{\overline{\pi(V)} \setminus \pi(V)}$.

Pseudo-witness set construction: Let $\{f, L, W\}$ be a witness set for V where $L : \mathbb{C}^3 \to \mathbb{C}$ is a general linear polynomial and $|W| = 3$. Since, for any $w \in W$,

$$\begin{bmatrix} Jf(w) \\ B \end{bmatrix}$$

is full rank, where $Jf(w)$ is the Jacobian matrix of f evaluated at w, Lemma 3 of [7] yields that $\dim \overline{\pi(V)} = 1$. Let $\mathcal{L}(s, x, y) = \alpha x + \beta y - 1$ where $\alpha, \beta \in \mathbb{C}$ are random, which is a linear polynomial in the image of π. Consider the three paths defined by modifying the homotopy H from (1) to move from L to $\mathcal{L}$ starting at the three points in W. Two paths converge with their endpoints mapping to distinct points under π. This implies that the degree of $\overline{\pi(V)}$ is 2. If $\mathcal{W}$ is the set consisting of these two endpoints, then $\{f, \pi, \mathcal{L}, \mathcal{W}\}$ is a pseudo-witness set for $\overline{\pi(V)}$.

j	$\lvert C_{z_j}\rvert$	$z_j \in C_{z_j}$?	$\lvert P_{z_j}\rvert$	$z_j \in P_{z_j}$?	Result from Lemma 1
1	2	Yes	2	Yes	$z_1 \in \pi(V)$
2	1	No	2	Yes	$z_2 \in \overline{\pi(V)} \setminus \pi(V)$
3	2	Yes	2	Yes	$z_3 \in \pi(V)$
4	2	No	2	No	$z_4 \notin \overline{\pi(V)}$

Table 1: Summary of basic membership test for unit circle

Basic membership test: For each $j = 1, \ldots, 4$, let $z_j = (z_j^x, z_j^y)$ and consider the linear polynomial $\widehat{\mathcal{L}}_j(s, x, y) = \alpha(x - z_j^x) + \beta(y - z_j^y)$. The basic membership test described in §2.1 uses a modification of the homotopy H from (1) to move from $\mathcal{L}$ to $\widehat{\mathcal{L}}_j$ starting with the two points in $\mathcal{W}$. Since $\dim \overline{\pi(V)} = 1$, Lemma 1 provides membership tests for both $\pi(V)$ and $\overline{\pi(V)}$. Table 1 summarizes the results. Here, the sets C_{z_j} and P_{z_j} are the sets as in §2.1 arising from this basic membership test.

Codimension one components: The codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$ correspond to the points in $\pi_{\mathcal{P}}(V_{\mathcal{P}} \cap \mathcal{E})$ (as defined in §3.1). By working on a random patch in $\mathbb{P}^1$, this reduces to tracking paths in $\mathbb{C}^4$. We homogenize f and L with respect to s, namely

$$F(s_0, s_1, x, y) = s_0^2 \cdot f\left(\frac{s_1}{s_0}, x, y\right) \quad \text{and} \quad M(s_0, s_1, x, y) = s_0 \cdot L\left(\frac{s_1}{s_0}, x, y\right),$$

and fix an affine patch in $\mathbb{P}^1$ defined by the equation $P(s_0, s_1, x, y) = p_0 s_0 + p_1 s_1 - 1$ where $p_0, p_1 \in \mathbb{C}$ are random. Let $\widehat{M}$ be a general linear form and $M_0(s_0, s_1, x, y) = s_0$. Starting with the points

$$S = \left\{ \left(\frac{1}{p_1 + p_2 s}, \frac{s}{p_1 + p_2 s}, x, y \right) \;\middle|\; (s, x, y) \in W \right\},$$

we first compute the finite endpoints of the homotopy H from (1) modified to deform from $[M, P]^T$ to $[\widehat{M}, P]^T$ and then use those as start points as we deform from $[\widehat{M}, P]^T$ to $[M_0, P]^T$. The resulting finite endpoints correspond to the points in $V_{\mathcal{P}} \cap \mathcal{E}$. In particular, the first had three finite endpoints while only one of the three paths converged for the second. This endpoint corresponds to the point $(0, 1, -1, 0) \in \mathbb{P}^1 \times \mathbb{C}^2$. Since this point projects to $(-1, 0)$ under $\pi_{\mathcal{P}}$ (as defined in §3.1), we know that $\overline{\overline{\pi(V)} \setminus \pi(V)} = \{(-1, 0)\}$.

4.2 A two-dimensional constructible set

Consider the example from §1.4, namely the image of $V = f^{-1}(0)$ under the projection $\pi(s, x, y) = (x, y)$ where $f(s, x, y) = x - sy$. The projection π is defined by the matrix $B = [0 \;\; I_2]$ where I_2 is the 2×2 identity matrix. Clearly, V is an irreducible surface of degree 2 that is generically reduced with respect to f. After constructing a pseudo-witness set for $\overline{\pi(V)}$, we will use the membership tests to determine if $p_j \in \pi(V)$ where

$$p_1 = (1, 1), \quad p_2 = (0, 0), \quad \text{and} \quad p_3 = (1, 0),$$

j	$\lvert C_{p_j}\rvert$	$p_j \in C_{p_j}$?	$\lvert P_{p_j}\rvert$	$p_j \in P_{p_j}$?	Result from Lemma 1
1	1	Yes	1	Yes	$p_1 \in \pi(V)$
2	1	Yes	1	Yes	$p_2 \in \pi(V)$
3	0	No	1	Yes	$p_3 \in \overline{\pi(V)}$, inconclusive on $\pi(V)$

Table 2: Summary of basic membership test

and then compute a decomposition of the codimension one components of $\overline{\overline{\pi(V)} \setminus \pi(V)}$.

Pseudo-witness set construction: Let $\{f, L, W\}$ be a witness set for V where $L : \mathbb{C}^3 \to \mathbb{C}^2$ is a system of general linear polynomials and $|W| = 2$. Since, for any $w \in W$,

$$\begin{bmatrix} \nabla f(w)^T \\ B \end{bmatrix}$$

is full rank, where $\nabla f(w)$ is the gradient of f evaluated at w, Lemma 3 of [7] yields that $\dim \overline{\pi(V)} = 2$. Therefore, $\overline{\pi(V)} = \mathbb{C}^2$ and $\deg \overline{\pi(V)} = 1$. For random $\alpha, \beta \in \mathbb{C}$, let

$$\mathcal{L}(s, x, y) = \begin{bmatrix} x - \alpha \\ y - \beta \end{bmatrix}.$$

A pseudo-witness set for $\overline{\pi(V)}$ is the quadruple $\{f, \pi, \mathcal{L}, \mathcal{W}\}$ where $\mathcal{W} = \{(\alpha/\beta, \alpha, \beta)\}$.

Basic membership test: Even though $\overline{\pi(V)} = \mathbb{C}^2$ and hence $p_j \in \overline{\pi(V)}$, we can still use the basic membership test of §2.1 to determine which points to further investigate using the advanced membership test of §2.2. For each $j = 1, 2, 3$, we used the system of linear polynomials $\widehat{\mathcal{L}}_j(s, x, y) = (x, y) - p_j$. Table 2 summarizes the results. Here, the sets C_{p_j} and P_{p_j} are the sets as in §2.1 arising from this basic membership test.

Advanced membership test: Since the basic membership test was inconclusive for deciding if $p_3 = (1, 0) \in \pi(V)$, we now apply the advanced membership test of §2.2. Let $L = [L_1, L_2]^T$ where L is the linear system in the witness set $\{f, L, W\}$ for V. For $i = 1, 2$, let $\widehat{\mathcal{L}}_i(s, x, y) = r_{i1}(x - 1) + r_{i2}y$ for random $r_{ij} \in \mathbb{C}$ and consider

$$\mathcal{M}_0 = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad \mathcal{M}_1 = \begin{bmatrix} \widehat{\mathcal{L}}_1 \\ L_2 \end{bmatrix}, \quad \text{and} \quad \mathcal{M}_2 = \begin{bmatrix} \widehat{\mathcal{L}}_1 \\ \widehat{\mathcal{L}}_2 \end{bmatrix}.$$

Starting with $S_0 = W$, tracking the paths for the modified homotopy H from (1) that deforms from $\mathcal{M}_0$ to $\mathcal{M}_1$ produces two points, neither of which projects to p_3. Since we have already performed the basic membership test and found that $p_3 \notin C_{p_3}$, Lemma 4 provides that $p_3 \notin \pi(V)$. If the basic test was not already performed, one would need to track the two paths arising from moving $\mathcal{M}_1$ to $\mathcal{M}_2$. Since both of these paths diverge, the same conclusion is reached.

Since the endpoint of the path for $p_2 = (0, 0)$ was singular when performing the basic membership test, it is instructive to perform the advanced membership test on this point as well. In this case, we take $\widehat{\mathcal{L}}_i(s, x, y) = r_{i1}x + r_{i2}y$. The deformation from $\mathcal{M}_0$ to

$\mathcal{M}_1$ also produces two points, one of which does project to p_2. Therefore, we know that $V \cap \pi^{-1}(p_2)$ contains a line. Tracking from the other point as $\mathcal{M}_1$ moves to $\mathcal{M}_2$ produces another point on this line. Therefore, $V \cap \pi^{-1}(p_2)$ is a line, namely $\{(s, 0, 0) \mid s \in \mathbb{C}\}$.

Codimension one components: We now turn to computing the curves in $\mathbb{C}^2$ contained in $\overline{\pi(V)} \setminus \pi(V)$ which correspond to the curves in $\pi_{\mathcal{P}}(V_{\mathcal{P}} \cap \mathcal{E})$ (as defined in §3.1). As in §4.1, we perform this computation on a random patch in $\mathbb{P}^1$ which reduces to tracking paths in $\mathbb{C}^4$. We homogenize f and L with respect to s, namely

$$F(s_0, s_1, x, y) = s_0 \cdot f\left(\frac{s_1}{s_0}, x, y,\right) = s_0 x - s_1 y \quad \text{and} \quad M(s_0, s_1, x, y) = s_0 \cdot L\left(\frac{s_1}{s_0}, x, y\right),$$

and fix an affine patch in $\mathbb{P}^1$ defined by the equation $P(s_0, s_1, x, y) = p_0 s_0 + p_1 s_1 - 1$ where $p_0, p_1 \in \mathbb{C}$ are random. Let $\widehat{M} = [\widehat{M}_1, \widehat{M}_2]^T$ be a system of two general linear forms and $M_0(s_0, s_1, x, y) = [s_0, \widehat{M}_2(s_0, s_1, x, y)]^T$. Starting with the points

$$S = \left\{ \left(\frac{1}{p_1 + p_2 s}, \frac{s}{p_1 + p_2 s}, x, y \right) \;\middle|\; (s, x, y) \in W \right\},$$

we first compute the finite endpoints of the homotopy H from (1) modified to deform from $[M, P]^T$ to $[\widehat{M}, P]^T$ and then use those as start points as we deform from $[\widehat{M}, P]^T$ to $[M_0, P]^T$. We can use the resulting finite endpoints to produce a witness set for the curves in $V_{\mathcal{P}} \cap \mathcal{E}$. In particular, the first had two finite endpoints while only one of the two paths converged for the second. This endpoint corresponds to the point $(0, 1, x^*, 0) \in \mathbb{P}^1 \times \mathbb{C}^2$ where $x^* \in \mathbb{C}$. This point projects to $(x^*, 0)$ under $\pi_{\mathcal{P}}$ (as defined in §3.1) which forms a witness point set for the line in $\overline{\pi(V)} \setminus \pi(V)$, namely $\mathcal{V}(y)$.

4.3 Lüroth hypersurface

Classically, a plane quartic is called a Lüroth quartic if it contains the ten vertices of a complete pentalateral [13]. The closure of the set of classical Lüroth quartics is a hypersurface $\mathcal{H}$ in the space of plane quartics, called the Lüroth hypersurface, that was showed by Morley in 1919 to have degree 54 [15]. The degree 54 polynomial equation defining this hypersurface is called the Lüroth invariant and, according to Ottaviani [16], it is still unknown. Without the defining equation, deciding if a given quartic lies on the Lüroth hypersurface requires another approach. The approach in [16] provides a partial test which builds on classical results of White and Miller [24]. We will use a pseudo-witness set for $\mathcal{H}$ and the membership test of §2.1 to provide a complete test for deciding if a given plane quartic lies on $\mathcal{H}$ by tracking at most 54 homotopy paths.

Pseudo-witness set construction: We construct a pseudo-witness set for $\mathcal{H}$ by first identifying the space of plane quartics with $\mathbb{P}^{14}$ so that $\mathcal{H} \subset \mathbb{P}^{14}$. The set $\mathcal{H}$ is the closure of the set of plane quartics Q for which there exists nonzero linear polynomials

j	$\lvert C_{Q_j}\rvert$	$Q_j \in C_{Q_j}$?	$\lvert P_{Q_j}\rvert$	$Q_j \in P_{Q_j}$?	Result from Lemma 1
1	54	No	54	No	$Q_1 \notin \mathcal{H}$
2	54	No	54	No	$Q_2 \notin \mathcal{H}$
3	54	No	54	No	$Q_3 \notin \mathcal{H}$
4	54	No	54	No	$Q_4 \notin \mathcal{H}$
5	54	Yes	54	Yes	$Q_5 \in \mathcal{H}$
6	38	No	39	Yes	$Q_6 \in \mathcal{H}$

Table 3: Summary of membership in the hypersurface of Lüroth quartics

ℓ_j for $j = 1, \ldots, 5$ such that

$$Q = \mathcal{V}\left(\sum_{j=1}^{5} \prod_{\substack{k=1 \\ k\neq j}}^{5} \ell_k\right).$$

This parameterization allows us to use Lemma 3 of [7] to confirm that $\mathcal{H}$ is a hypersurface and compute a pseudo-witness set for $\mathcal{H}$ using Bertini [2]. From this pseudo-witness set, we are able to verify Morley's result that the degree of $\mathcal{H}$ is 54 and, following Remark 2, we chose 54 points from the pseudo-witness point set that correspond to distinct quartics to be used as the starting points for our basic membership test.

Basic membership test: We applied the basic membership test of §2.1 to the quartics $Q_j = \mathcal{V}(q_j)$ defined by the following polynomials:

- $q_1 = (x^2 + y^2 + z^2)^2$;
- (Edge quartic [6, 17]) $q_2 = 25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2)$;
- (Klein quartic [16, §5]) $q_3 = x^3y + y^3z + z^3x$;
- (Vinnikov curve [17, Ex. 4.1]) $q_4 = 2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2$;
- ([16, §5]) $q_5 = xyz(x+y+z) + (x+2y+3z)(xyz + (xy+xz+yz)(x+y+z))$; and
- $q_6 = x^3y + x^2z^2 + xz^3$.

Table 3 summarizes the results of this test. Here, the sets C_{Q_j} and P_{Q_j} are the sets as in §2.1 arising from this basic membership test. We note that since $\mathcal{H} \subset \mathbb{P}^{14}$, compactness yields that P_{Q_j}, as a list, must consist of 54 points. However, in the $j = 6$ case, 16 of these points coincided with Q_6.

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