Smooth Points on Semi-algebraic Sets

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Abstract

Many algorithms for determining properties of real algebraic or semi-algebraic sets rely upon the ability to compute smooth points. Existing methods to compute smooth points on semi-algebraic sets use symbolic quantifier elimination tools. In this paper, we present a simple algorithm based on computing the critical points of some well-chosen function that guarantees the computation of smooth points in each connected compact component of a real (semi)-algebraic set. Our technique is intuitive in principal, performs well on previously difficult examples, and is straightforward to implement using existing numerical algebraic geometry software. The practical efficiency of our approach is demonstrated by solving a conjecture on the number of equilibria of the Kuramoto model for the $n = 4$ case. We also apply our method to design an efficient algorithm to compute the real dimension of (semi)-algebraic sets, the original motivation for this research.

1 Introduction

Consider the semi-algebraic set

$$S = \{x \in \mathbb{R}^n \ : \ f_1(x) = \cdots = f_s(x) = 0,\ q_1(x) > 0, \ldots, q_m(x) > 0\}$$

for some $f_1, \ldots, f_s, q_1, \ldots, q_m \in \mathbb{R}[x_1, \ldots, x_n]$. When studying real semi-algebraic sets we often first study the complex variety defined by $V = \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_s(x) = 0\}$ and deduce properties of S from the properties of V. In particular, if S contains a smooth point and V is irreducible then S is Zariski dense in V, so all algebraic information of S is contained in V. Thus, deciding the existence of smooth points in real semi-algebraic sets and finding such points is a central problem in real algebraic geometry with many applications. For example, if $\varphi : S \to S'$ is a polynomial map of semi-algebraic sets then smooth points in $\mathrm{Im}(\varphi)$ are points where the Jacobian of φ has maximal rank within its connected component, also called the *typical rank*. In particular, finding real smooth points in each connected component of a semi-algebraic set allows one to compute all typical ranks of real morphisms (see [56] for applications of this property).

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One of the main results of this paper is to give a new technique to compute smooth points on the compact connected components of real semi-algebraic sets. Our method is simple and suggests natural implementation using numerical homotopy methods and deformations. It complements other approaches that compute sample points on real semi-algebraic sets, such as computing the critical points of the distance function, in the sense that our method also guarantees the smoothness of the sample points. In the paper we demonstrate this advantage on examples where the critical points of the distance function are singular from the majority of points, while our method always computes smooth points. The main idea is very simple: Suppose V is irreducible. If a polynomial g vanishes on the singular points of V but does not vanish on all of V then the extreme points of g on S must contain points in every compact connected component of S that are non-singular, if such points exist. We extend this idea to the case when V is not equidimensional by using *infinitesimal* deformations of V and limits. We show that this limiting approach is well-suited for numerical homotopy continuation methods after we translate an infinitesimal real deformation (that may only work for arbitrary small values) into a complex deformation that works along a real arc parameterized by the interval $(0, 1]$.

To demonstrate the practical efficiency of our new approach, we present the solution of a conjecture for the first time: counting the equilibria of the Kuramoto model in the $n = 4$ case [60] (see [40] for the original model and [24] for a detailed historical overview and additional references).

We also apply our method to compute the dimension of real semi-algebraic sets. The difficulty of this problem, compared to its complex counterpart, is that in many cases the real part lies within the singular set of the smallest complex variety containing it, and its real dimension is smaller than the complex one. In terms of worst case complexity bounds of the existing algorithms in the literature, it is an open problem if the real dimension can be computed within the same asymptotic complexity bounds as the complex dimension. The algorithm presented in this paper does not improve the existing complexity bounds in the worst case (see [8] and the references therein). However, in the Conclusions, we give a conjecture which is a slight modification of our main theorem in the limiting case, and if true, it would allow significant improvement in the worst case complexity of computing real dimension of semi-algebraic sets.

1.1 Related Work

Two main approaches to computing at least one real point on every connected component of a semi-algebraic set are quantifier elimination and critical point methods. Quantifier elimination methods go back to Collins [22] and his Cylindrical Algebraic Decomposition (CAD) algorithm, which decomposes a real semi-algebraic set into cells via projections, eliminating variables one after another. This original algorithm is doubly exponential in the number of variables and work has been done in the following decades improve the complexity and provide efficient implementations (e.g., [30, 23, 31, 37]). Current algorithms and implementations of partial CAD and single open cell CAD can be found in [15, 16, 17]. Alternatively, the best current symbolic bound using quantifier elimination is provided by Basu, Pollack, and Roy [9] and the most recent implementation utilizing it can be found in [53].

The use critical point methods to compute real points on every connected component of a semi-algebraic set dates back to Seidenberg [54]. While this method also relies on quantifier elimination, the approach varies from CAD in that it computes critical points of the distance function, and work has been done to present better complexity bounds and implementations ever since. A major leap was made by [48] when they presented a symbolic algorithm to find the critical points of the projection function, introducing only one infinitesimal variable to show improvements over CAD computations. Others have continued to improve on these ideas using algebraic techniques such as triangular decompositions, Rational Univariate Representations, and geometric resolutions (e.g., [1, 49, 27]). A homotopy-based approach is presented in [32] and a similar approach uses critical points with respect to a generic line that can be translated rather than one point is found in [59].

Another line of work has been developed in parallel which specifically focuses on computing critical points while utilizing the tool of *polar varieties*, introduced and developed in [3, 4, 50, 5, 6, 7]. The most recent implementation of such techniques can be found in [51]. It is important to note, however, that these methods, along with the other critical point methods cited above, only guarantee the finding of real points on every connected component of a semi-algebraic set, rather than real *smooth* points.

Prior work on computing the real dimension of semi-algebraic sets begins with the quantifier elimination techniques of CAD [22]. Work done by Koiran [39] and Vorobjov [58] analyzed and introduced ideas for how such an algorithm could be improved, with the former suggesting the use of randomness to create probabilistic algorithms and the latter using stratifications of semi-algebraic sets. In [10, Chap. 14], Basu, Pollack, and Roy provide complexity bounds for a deterministic implementation which improves Vorobjov's results, and Hong and Safey El Din present a variant of quantifier elimination in [38] which works for smooth and compact semi-algebraic sets. Recent work has been presented giving probabilistic algorithms utilizing polar varieties which improve on complexity bounds even further in [52, 8].

One can also compute the real dimension by computing the real radical of a semi-algebraic set, first studied in Becker and Neuhaus [13], with improvement and implementations in [47, 57]. Decompositions of semi-algebraic sest into a union of regular semi-algebraic sets using triangular decompositions, border bases, and moment matrices are detailed in [19, 21, 20]. The most recent implementation can be found in [53]. This approach, however, computes iteratively singularities of singularities, which can increase the complexity significantly in the worst case. An alternative method using semidefinite programming techniques was proposed by Laserre [42] and has been improved upon in [41, 45]. Finally, methods for numerically computing homologies are given in [25, 18], but they only apply to the smooth generic case.

2 Preliminaries

The following collects some basic notions used throughout starting with atomic semi-algebraic sets, semi-algebraic sets, and (real) algebraic sets.

Definition 2.1. A set $S \subset \mathbb{R}^n$ is an *atomic semi-algebraic set* if it is of the form

$$S = \{x \in \mathbb{R}^n \ : \ f_1(x) = \cdots = f_s(x) = 0,\ q_1(x) > 0, \ldots, q_m(x) > 0\}\,. \tag{1}$$

A set $T \subset \mathbb{R}^n$ is a *semi-algebraic set* if it is a finite union of atomic semi-algebraic sets. A set $U \subset \mathbb{R}^n$ is a *(real) algebraic set* if it is defined by polynomial equations only.

Smoothness on atomic semi-algebraic sets is described next.

Definition 2.2. Let $S \subset \mathbb{R}^n$ be an atomic semi-algebraic set as in (1). Then $z \in S$ is smooth in S if z is smooth in the algebraic set $V(f_1, \ldots, f_s) = \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_s(x) = 0\}$, i.e. if

$V \subset \mathbb{C}^n$ is the irreducible component of $V(f_1, \ldots, f_s)$ containing z with

$$\dim T_z(V) = \dim V$$

where $T_z(V)$ is the tangent space of V at z.

An algebraic set $V \subset \mathbb{C}^n$ is *equidimensional* of dimension d if all irreducible components of V has dimension d. The following defines the real dimension of semi-algebraic sets from [9, §5.3]:

Definition 2.3. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. Its real dimension $\dim_{\mathbb{R}} S$ is the largest k such that there exists an injective semi-algebraic map from $(0,1)^k$ to S. Here, a map $\varphi : (0,1)^k \to S$ is semi-algebraic if the graph of φ in $\mathbb{R}^{n+k}$ is semi-algebraic. By convention, the dimension (real or complex) of the empty set is -1.

The main ingredient in our results is the following theorem that was proved in [46, Theorem 12.6.1]:

Theorem 2.4. *Let $V \subset \mathbb{C}^n$ be an irreducible algebraic set and let $V_{\mathbb{R}} := V \cap \mathbb{R}^n$. Then*

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = \dim_{\mathbb{C}} V$$

if and only if there exists $z \in V_{\mathbb{R}}$ that is smooth.

2.1 Semi-algebraic to Algebraic

We next describe a construction of how to get semi-algebraic sets as a union of projections of algebraic sets.

Let $S \subset \mathbb{R}^n$ be a semi-algebraic set defined as the union of atomic semi-algebraic sets $S = \bigcup_{i=1}^k S_i$ where for $i = 1, \ldots, k$

$$S_i = \left\{ x \in \mathbb{R}^n \ : \ f_1^{(i)}(x) = \cdots = f_{s_i}^{(i)}(x) = 0, \ q_1^{(i)}(x) > 0, \ldots, q_{m_i}^{(i)}(x) > 0 \right\}.$$

Then we can define the real algebraic sets for $i = 1, \ldots, k$

$$W_i := \left\{ (x, z) \in \mathbb{R}^n \times \mathbb{R}^{m_i} \ : \ f_1^{(i)}(x) = \cdots = f_{s_i}^{(i)}(x) = \ z_1^2 q_1^{(i)}(x) - 1 = \cdots = z_{m_i}^2 q_{m_i}^{(i)}(x) - 1 = 0 \right\}.$$

Clearly, $S_i = \pi_x(W_i) \subset \mathbb{R}^n$, where $\pi_x : \mathbb{R}^n \times \mathbb{R}^{m_i} \to \mathbb{R}^n$ is the projection to the x coordinates. Moreover, for each $x \in S_i$ there are finitely many preimages in W_i that projects to x. This implies that the dimension of S_i and W_i are the same. Also, for each connected component $C \subset S_i$ there are finitely many connected component of W_i that has projection C. If $C \neq C'$ two connected components of S_i, then the connected components of W_i that project to C and the ones that project to C' are disjoint.

If the task is to find smooth points on each connected components of $S = \bigcup_{i=1}^k S_i$ where S_i are atomic semi-algebraic sets, we can find smooth points on each connected components of S_i for $i = 1, \ldots, k$. To achieve this, we compute smooth points on each connected component $W_i \subset \mathbb{R}^n \times \mathbb{R}^{m_i}$. The following proposition shows that the projections of these points by π_x will give points on each connected component of S_i that are also smooth.

Proposition 2.5. *Let S be an atomic semi-algebraic set as in* (1) *and let*

$$W := \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^{m_i} \ : \ f_1(x) = \cdots = f_n(x) = \ z_1^2 q_1(x) - 1 = \cdots = z_m^2 q_m(x) - 1 = 0\right\}.$$

If $y \in W$ is smooth then $\pi_x(y) \in S$ is also smooth. Conversely, if $x \in S$ is smooth then for all $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ such that $(x,z) \in W$ we have that (x,z) is smooth.

Proposition 2.6. *Let S be an atomic semi-algebraic set as in* (1) *and*

$$\begin{aligned} W := \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ & f_1(x) = \cdots = f_n(x) = 0 \\ & z_1^2 q_1(x) - 1 = \cdots = z_m^2 q_m(x) - 1 = 0\right\}. \end{aligned}$$

If $y \in W$ is smooth then $\pi_x(y) \in S$ is also smooth. Conversely, if $x \in S$ is smooth, then (x,z) is smooth in W for all $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ such that $(x,z) \in W$.

Proof. Without loss of generaility we can assume that $V(f_1, \ldots, f_s) \subset \mathbb{C}^n$ is irreducible, otherwise we further divide S as a union of smaller atomic semi-algebraic sets. We can also assume that $f_1, \ldots, f_s$ generate a prime ideal, otherwise we could replace these polynomials by the generators of the radical ideal without changing either S or W. The Jacobian matrix of the polynomial system defining W has the structure

$$J(x,z) = \begin{array}{|c|c|} \hline \nabla f(x) & 0 \\ \hline * & \mathrm{diag}(2 z_i g_i(x)) \\ \hline \end{array}$$

Note that for $(x,z) \in W$ we have $z_i g_i(x)) \neq 0$ for $i = 1, \ldots, m$. Thus, the Jacobian $\nabla f(x)$ has full column rank if and only if $J(x,z)$ has full column rank, which proves the claim. $\square$

For the rest of the paper, we assume that we are given a *real algebraic set* and the goal is to compute smooth points on each connected components.

2.2 Boundedness

The next reduction is to replace an arbitrary real algebraic set with a compact one. We use a standard trick used in real algebraic geometry, described in the following proposition.

Proposition 2.7. *Let $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_{n-1}]$ and consider a point $\mathbf{q} = (q_1, \ldots, q_{n-1}) \in \mathbb{R}^{n-1}$. Let $\delta \in \mathbb{R}_+$, introduce a new variable x_n, and consider*

$$f_{s+1} := (x_1 - q_1)^2 + \cdots + (x_{n-1} - q_{n-1})^2 + x_n^2 - \delta$$

Then, $V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n$ is bounded and

$$\pi_{n-1}\left(V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n\right) = V(f_1, \ldots, f_s) \cap \left\{z \in \mathbb{R}^{n-1} \ : \ \|z - \mathbf{q}\|^2 \leq \delta\right\}$$

where $\pi_{n-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$.

Proof. $V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n$ is bounded because it is a subset of the real sphere centered at $(\mathbf{q}, 0)$ of radius δ. To prove the second claim, let $z \in V(f_1, \ldots, f_s) \cap \left\{z \in \mathbb{R}^{n-1} \ : \ \|\mathbf{q} - z\| \leq \delta\right\}$. Then for $z_n := \sqrt{\delta - \|\mathbf{q} - z\|^2}$ we have $z_n \in \mathbb{R}$ and $f_{s+1}(z, z_n) = 0$. Thus $z \in \pi_{n-1}\left(V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n\right)$. Conversely, if $z \in \pi_{n-1}\left(V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n\right)$. Then there exists $z_n \in \mathbb{R}$ such that $(z, z_n) \in V(f_1, \ldots, f_{s+1}) \cap \mathbb{R}^n)$, which implies that $f_j(z) = 0$ for $j = 1, \ldots, s$ and $z_n^2 = \delta - \|\mathbf{q} - z\|^2 \geq 0$, so $\|\mathbf{q} - z\| \leq \delta$ as claimed. $\square$

Remark 2.8. The definition of f_{s+1} above is based on a standard trick used in real algebraic geometry to make an arbitrary real algebraic set bounded (see for example [10]). In general, $V \cap \mathbb{R}^{n-1}$ is embedded into a sphere in $\mathbb{R}^n$ around the origin of radius $1/\zeta$ where ζ is infinitesimal. The introduction of the infinitesimal variable allows to include points in $V \cap \mathbb{R}^{n-1}$ of arbitrary 2-norm. Here, in this paper, we are only interested in computing points with bounded coordinates, so we do not need to embed the entire real variety $V \cap \mathbb{R}^{n-1}$, it is sufficient to embed its intersection with a closed ball around $\mathbf{q}$ of radius δ for some fixed $\delta \in \mathbb{R}_+$. Thus, we will not use infinitesimal variables.

By using Proposition 2.7 as needed, for the rest of the paper we assume that we are given $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is compact.

3 Computation of Real Smooth Points

In this section we present some of the main results of the paper.

Theorem 3.1. *Let $f_1, \ldots, f_{n-d} \in \mathbb{R}[x_1, \ldots, x_n]$ and assume that $V := V(f_1, \ldots, f_{n-d}) \subset \mathbb{C}^n$ is equidimensional of dimension d and $V \cap \mathbb{R}^n$ is compact. Suppose that $g \in \mathbb{R}[x_1, \ldots, x_n]$ satisfies:*

1. $\operatorname{Sing}(V) \cap \mathbb{R}^n \subset V(g)$;
2. $\dim V \cap V(g) \leq d - 1$.

The points in $\mathbb{R}^n$ where $\{g(x) \ : \ x \in V \cap \mathbb{R}^n\} \subset \mathbb{R}$ takes its extreme values intersects each connected component of $(V \setminus \operatorname{Sing}(V)) \cap \mathbb{R}^n$.

We use the following lemma:

Lemma 3.2. *Let V be as in Theorem 3.1. Let $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that $\dim V \cap V(g) \leq d - 1$. Then, either $(V \setminus V(g)) \cap \mathbb{R}^n = \emptyset$ or the set $\{g(x) \ : \ x \in V \cap \mathbb{R}^n\} \subset \mathbb{R}$ attains a non-zero extreme value on each connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$.*

Proof. Assume that $(V \setminus V(g)) \cap \mathbb{R}^n \neq \emptyset$ and let C be a connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$. Since $C \not\subset V(g)$, there exists $x \in C$ with $g(x) \neq 0$. Let $\overline{C}$ be the Euclidean closure of C so that $\overline{C} \subset V \cap \mathbb{R}^n$ is closed and bounded. By the extreme value theorem, g attains both a minimum and a maximum on $\overline{C}$. Since g is not identically zero on $\overline{C}$, either the minimum or the maximum value of g on $\overline{C}$ must be nonzero, so it is in C. □

Proof of Theorem. Assume that $(V \setminus \operatorname{Sing}(V)) \cap \mathbb{R}^n \neq \emptyset$. By Theorem 2.4, $\dim_{\mathbb{R}} V \cap \mathbb{R}^n = d$. By (2), $(V \setminus V(g)) \cap \mathbb{R}^n \neq \emptyset$. By (1), $(V \setminus V(g)) \cap \mathbb{R}^n \subset (V \setminus \operatorname{Sing}(V)) \cap \mathbb{R}^n$ so connected components of $(V \setminus V(g)) \cap \mathbb{R}^n$ are subsets of connected components of $(V \setminus \operatorname{Sing}(V)) \cap \mathbb{R}^n$. By Lemma 3.2, $\{g(x) \ : \ x \in V \cap \mathbb{R}^n\} \subset \mathbb{R}$ attains a non-zero extreme value on each connected component of $(V \setminus V(g)) \cap \mathbb{R}^n$. Thus, these points will give a point in every connected component of $(V \setminus \operatorname{Sing}(V)) \cap \mathbb{R}^n$. □

Example 3.3 ("Thom's lips")**.** An example of a real curve with two singular cusps is often referred to as "Thom's lips," e.g., $f = y^2 - (-x^2 + x^3)^2$ as shown in Figure 1. One straightforward choice for g which satisfies the above conditions is $g = x(x - 1)$. Using Lagrange multipliers to optimize with respect to this g results in the two red points. Alternatively, we could select a g

constructed algorithmically using the polar varieties technique detailed later in the paper, namely $g = 3(2x-1)(x-x^2)^2 + 2y$. Optimization yields the two black points. In either case, this results in a real smooth point on each of the two connected components of $(V \setminus \mathrm{Sing}(V)) \cap \mathbb{R}^n$.

Example 3.4 ("Samosa")**.** An example of a real surface with three singular points coming from semidefinite programming is sometimes referred to as the "Samosa". It's defining equation is found by taking the determinant of a 3×3 matrix, resulting in $f = 2xy - x^2 - y^2 - z^2 + 1$. We note that the surface defined by f is not bounded, but by restricting to the bounded component shown in the figure we can apply the theorem. The most straightforward choice for g which satisfies the above formula is $g = x^2 + y^2 + z^2 - 3$. Using Lagrange multipliers to optimizing with respect to this g results in the red points in the figure. Alternatively, we could select a g which satisfies the theorem using the polar varieties technique detailed later in the paper. In this case, we use $g = 2xy + 2xz + 2yz - 2x - 2y - 2z$ and optimization with Lagrange multipliers results in the two blue points in the figure.

We now consider when $V(f_1, \dots, f_{n-d})$ is not equidimensional of dimension d. In this case, we perturb the polynomials by constants. The following lemma was proved in [27, Lemma 1].

Lemma 3.5. *Let $f_1, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$ and fix $l \le s$ and $\mathcal{I} = \{i_1, \dots, i_l\} \subset \{1, \dots, s\}$. Then there exists a Zariski closed sebset $\mathcal{A} \times \mathcal{E} \subset \mathbb{C}^s \times \mathbb{C}$ such that for all $(a_1, \dots, a_s) \in \mathbb{R}^s \setminus \mathcal{A}$ and $e \in \mathbb{R} \setminus \mathcal{E}$, the ideal generated by the polynomials*

$$f_{i_1} - ea_{i_1}, \dots, f_{i_l} - ea_{i_l}$$

is a radical equidimensional ideal and $V(f_{i_1} - ea_{i_1}, \dots, f_{i_l} - ea_{i_l})$ is either empty or smooth of dimension $n - l$.

Definition 3.6. Let $f_1, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$ and consider a point $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{R}^s$. We say that $f_1, \dots f_s$ and $\mathbf{a}$ satisfy Assumption A if

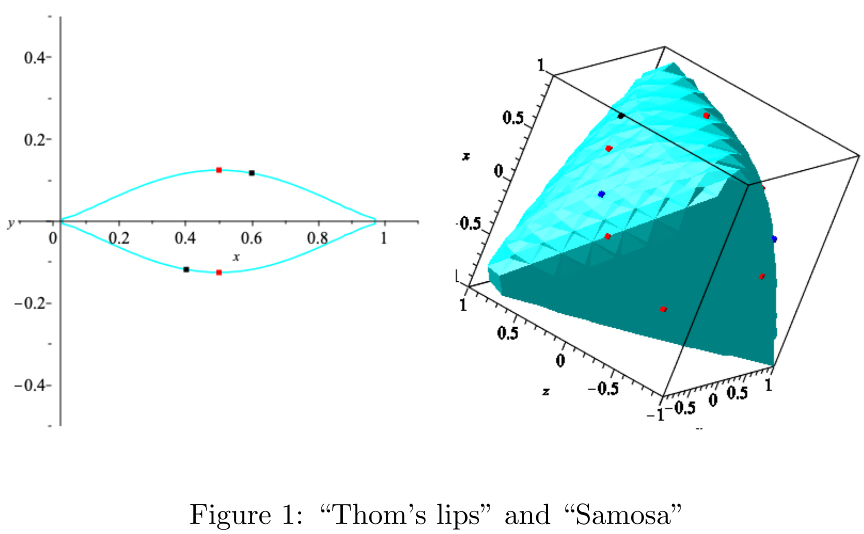


Figure 1: "Thom's lips" and "Samosa"

A: There exists $e_0 > 0$ such that for all $0 < e \le e_0$, the polynomials $f_1 - ea_1, \ldots, f_s - ea_s$ generate a radical equidimensional ideal and $V(f_1 - ea_1, \ldots, f_s - ea_s)$ is smooth and has dimension $n - s$.

Then we have the following theorem:

Theorem 3.7. *Assume that $f_1, \ldots, f_{n-d} \in \mathbb{R}[x_1, \ldots, x_n]$ and point $\mathbf{a} \in \mathbb{Q}^{n-d}$ satisfies Assumption* A. *For $e > 0$, define*

$$V_e^{\mathbf{a}} := V(f_1 - ea_1, \ldots, f_{n-d} - ea_{n-d}) \quad \textit{and} \quad V := \lim_{e \to 0^+} V_e^{\mathbf{a}} \subset \mathbb{C}^n.$$

Suppose that $V \cap \mathbb{R}^n$ is compact. Let $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that

$$\dim V \cap V(g) \le d - 1.$$

Denote by $L_e \subset \mathbb{R}[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-d}]$ the following polynomial system using Lagrange multipliers to compute the critical points of g on $V_\varepsilon^{\mathbf{a}}$:

$$L_e \; := \; \left\{ \frac{\partial g}{\partial x_i} + \sum_{j=1}^{n-d} \lambda_j \frac{\partial f_j}{\partial x_i} \; : \; i = 1, \ldots, n \right\} \cup \{f_1 - ea_1, f_2 - ea_2, \ldots, f_{n-d} - ea_{n-d}\}.$$

Let $\pi_x : \mathbb{C}^n \times \mathbb{C}^{n-d} \to \mathbb{C}^n$ be the projection to the x coordinates and define $\mathcal{S} := \lim_{e \to 0} \pi_x(V(L_e)) \subset V \subset \mathbb{C}^n$. Then, $\mathcal{S}$ is finite and either $(V \setminus V(g)) \cap \mathbb{R}^n = \emptyset$ or for each connected component C of $V \cap \mathbb{R}^n$ where g is not identically zero, there exists $z \in \mathcal{S} \cap C$ such that $g(z) \neq 0$. In particular, if g such that $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$, then $\mathcal{S} \cap \mathbb{R}^n$ contains smooth points in each connected components of $V \cap \mathbb{R}^n$ that has dimension d.

Proof. By Assumption A, we know $V_e^{\mathbf{a}}$ is smooth and equidimensional of dimension d for all sufficiently small $e > 0$. We can apply [10, Prop. 12.38] over $\mathbb{C}$ instead of $\mathbb{R}$ to show $V = \lim_{e \to 0} V_e^{\mathbf{a}} \subset \mathbb{C}^n$ is a Zariski closed set and is equidimensional of dimension d or empty if all limits go to infinity. Assumption A also implies that $V(L_e) \subset \mathbb{C}^n \times \mathbb{C}^{n-d}$ is zero dimensional for all sufficiently small $e > 0$, and so $\mathcal{S}$ is finite. Suppose $(V \setminus V(g)) \cap \mathbb{R}^n \neq \emptyset$. Let $C_1, \ldots, C_r \subset V \cap \mathbb{R}^n$ be the connected components of $V \cap \mathbb{R}^n$ where g is not identically zero. Since $V \cap \mathbb{R}^n$ is compact, each C_i is compact, which implies that the distance from C_i to C_j is positive for each $i \neq j$. Also, for all sufficiently small e, $V_e^{\mathbf{a}} \cap \mathbb{R}^n$ is also compact. Fix $i \in \{1, \ldots, r\}$. In [52, Prop. 5], it is proved that there exist connected components $C_{i,1}^{(e)}, \ldots, C_{i,s_i}^{(e)}$ of $V_e^{\mathbf{a}} \cap \mathbb{R}^n$ for all sufficiently small $e > 0$ such that $C_i = \bigcup_{k=1}^{s_i} \lim_{e \to 0^+} C_{i,k}^{(e)}$. Moreover, since C_i and C_j has positive distance for $i \neq j$, also by [52, Prop. 5] we have that $\bigcup_{k=1}^{s_i} C_{i,k}^{(e)}$ is disjoint from $\bigcup_{k=1}^{s_j} C_{j,k}^{(e)}$. For each $k = 1, \ldots, s_i$ let $\mathcal{S}_{i,k}^{(e)} := \pi_x(V(L_e)) \cap C_{i,k}^{(e)}$. Then, by Theorem 3.1, $\mathcal{S}_{i,k}^{(e)} \neq \emptyset$, and it contains all points in $C_{i,k}^{(e)}$ where g takes its extreme values. Let $\mathcal{S}_i := \bigcup_{k=1}^{s_i} \lim_{e \to 0} \mathcal{S}_{i,k}^{(e)}$. Since $\mathcal{S}_{i,k}^{(e)}$ is bounded for all sufficiently small e, none of the limit points escape to infinity. Suppose that for all $z \in \mathcal{S}_i$ we have $g(z) = 0$. Since C_i is compact, by the extreme value theorem, g attains both a minimum and a maximum on C_i. Since g is not identically zero on C_i, either the minimum or the maximum value of g on C_i must be nonzero. Let $z^* \in C_i$ such that $|g(z^*)| > 0$. Let $z_e^* \in C_i^{(e)}$ such that $\lim_{e \to 0} z_e^* = z^*$. Then for any $z \in \mathcal{S}_i$, if $z_e \in \mathcal{S}_i^{(e)}$ such that $\lim_{e \to 0} z_e = z$, then for sufficiently small e we have that $|g(z_e^*)| > |g(z_e)|$. Since $\mathcal{S}_i$ is finite, we can choose a common e_0 value for all $z \in \mathcal{S}_i$ so that if $0 < e < e_0$ then $|g(z_e^*)| > |g(z_e)|$ for all $z_e \in \mathcal{S}_i^{(e)}$. Thus, $\mathcal{S}_i^{(e)}$ could not contain all points of $C_i^{(e)}$ where g takes its extreme values, a contradiction. So this proves that for each $i = 1, \ldots r$, $\mathcal{S} \cap \mathbb{R}^n$ contains a point $z \in C_i$ such that $g(z) \neq 0$.

To prove the last claim, assume that $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$. If C_i is a connected component of $V \cap \mathbb{R}^n$ where g is not identically zero, then C_i has non-singular points, so by Theorem 2.4 $\dim C_i = d$. Conversely, if C_i a connected component of $V \cap \mathbb{R}^n$ of dimension d then g cannot identically vanish on C_i by the assumption that $\dim V \cap V(g) \leq d - 1$. Thus $\mathcal{S} \cap \mathbb{R}^n$ contains a point $z \in C_i \setminus \mathrm{Sing}(V)$, so it is smooth.

□

Remark 3.8. We assumed in Theorems 3.1 and 3.7 that g is a polynomial, but we can straightforwardly extend the results to $g : \mathbb{R}^n \to \mathbb{R}$ differentiable functions as long as $\frac{\partial g}{\partial x_i}$ for $i = 1, \ldots, n$ are rational functions.

4 Application to Real Dimension

This section applies Theorem 3.7 to compute the real dimension of real algebraic (and semi-algebraic) sets.

The main idea of our algorithm is as follows. We can apply Theorem 3.7 to try to compute real smooth points on the algebraic variety. Using Theorem 2.4, if we find a real smooth point, we find the real dimension to be the same as the complex one. If there are no real smooth points, we conclude that the real dimension is smaller than the complex dimension. In that case, we need to lower the complex dimension in a way that we do not lose any real points in the variety. To do this we use the following notion of *polar varieties*, using the notation in [52]:

Definition 4.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $V = V(f) \subset \mathbb{C}^n$. Consider the projections $\pi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i)$ for $i = 1, \ldots, n$. The *polar variety associated to* π_i of V is defined as

$$\mathrm{crit}(V, \pi_i) := V\left(f, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset \mathbb{C}^n \quad i = 1, \ldots, n.$$

Remark 4.2. There is extensive literature about different notions of polar varieties (see for example [7] for a survey). Here we use the simplest kind of polar varieties, following [52], using a single polynomial f that we think of the sum of squares of the input polynomials $f_1, \ldots, f_s$. If V is singular then $\mathrm{crit}(V, \pi_i)$ contains all singular points of V, as well as the critical points of the projection π_i. In the usual definition of polar varieties in the literature they usually exclude the singular points of V and then take Zariski closure, the definition above is different. See the details on the results we need about these polar varieties in Section 5. In practice, other notions of polar varieties may work better, but their presentation require more space.

The other ingredient in our algorithm is to compute points on varieties that are given as limits. This is to avoid higher dimensional components that appear in the polar varieties, but disappear in perturbations, so also disappear in the limits. Numerical homotopy continuation methods are especially well suited for taking limits. Section 7 describes how to weed out extraneous components in the limit embedded in those higher dimensional components that we want to avoid.

The outline of our real dimension algorithm is as follows:

ALGORITHM: Numerical Real Dimension

Input: $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$ such that $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$ is compact where $n \geq 2$.
Output: The real dimension of $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$.

---

(1) Choose random $A \in \mathrm{GL}_n(\mathbb{R})$ and define

$$f(x) := \sum_{i=1}^{s} f_i(Ax)^2 \in \mathbb{R}[x_1, \ldots, x_n].$$

Choose a random $\xi \in \mathbb{C}$ with $|\xi| = 1$. Let $i := n$.

(2) For simplicity, suppose here that

$$V := \lim_{t \to 0} \operatorname{crit}\left(V(f - t\xi), \pi_i\right)$$

is irreducible and compute g with the following properties:

(a) $\mathrm{Sing}(V) \cap \mathbb{R}^n \subset V(g)$

(b) $\dim V \cap V(g) = i - 2$.

See Section 6 for details and also for how to handle the case when V is not irreducible.

(3) Define

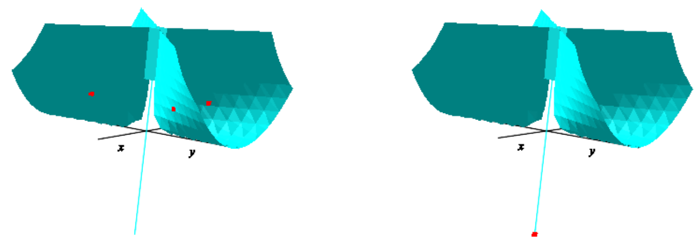
$$L_i^{t\xi} := \left\{ \frac{\partial g}{\partial x_k} + \sum_{j=1}^{n-d} \lambda_j \frac{\partial f_j}{\partial x_k} \ : \ k = 1, \ldots, n \right\} \cup \left\{ f - t\xi, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n} \right\}$$

and compute $\mathcal{S}_i := \lim_{t \to 0} \pi_x \left(V(L_{i,j}^{t\xi}) \right) \cap \mathbb{R}^n \setminus V(g)$. If $\mathcal{S}_i \neq \emptyset$ then RETURN i-1.

(4) Set $i := i - 1$. If $i = 0$ RETURN(-1). If $i > 0$ go to Step 2.

Example 4.3. The Whitney umbrella is a real surface that consists of a 2-dimensional umbrella-like surface with a 1-dimensional handle along the z-axis and defined by $f_1 = x^2 - y^2 z$. Since the surface is not compact, we can add $f_2 = x^2 + y^2 + z^2 + w^2 - 4$ and only consider the bounded part shown in Figure 2. The polynomial $g = x$ satisfies the requirements and results in the red smooth points shown in Figure 2(a) confirming the real dimension is two.

However, suppose instead that we wish to determine the local real dimension of the handle of the umbrella. To do this, we localize our computations by taking $f_2 = x^2 + y^2 + (z+1)^2 + w^2 - \frac{1}{4}$. As expected, the optimization using $g = z + 1$ results in no smooth real points. So, we compute the equation defining our next polar variety, namely $f_3 = 2y^3 - 4yz^2 - 4yz$. Optimizing with respect to $g = z + 1$ which satisfies the requirements yields the smooth real point on the handle shown in Figure 2(b). This computation confirms that the real dimension of the handle is one.



(a) Dim. 2 Smooth Points (b) Dim. 1 Smooth Point

Figure 2: Whitney umbrella

Remark 4.4. In Step 2 of the Algorithm, we could replace g with a polynomial satisfying the following conditions:

1. $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_{i-1}) \cap \mathbb{R}^n \subset V(g)$
2. $\dim \lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i) \cap V(g) = i-2$.

Note that these conditions imply the conditions in Step 2 for all irreducible component of V. The reason we chose the conditions in Step 2 is because it allows g's that may have lower degree or easier to compute.

Remark 4.5. Regarding certification of Step 3 of the Algorithm, suppose our input polynomials $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ and we want to certify the correctness of the elements of $\mathcal{S}_i$ computed numerically. We can certify numerical approximations to the points in the finite set $\lim_{t\to 0} \pi_x \left(V(L_{i,j}^{t\xi})\right)$ using the symbolic-numeric certification algorithm of [2], which first computes a rational univariate representation for this set. Then we use α-theory as in [34] to certify that the computed points are approximate roots of the rational univariate representation, are real and also give upper bounds for their approximation errors. Using these error bounds, we can also certify that g does not vanish at the real roots. Conversely, to certify that a point approximates a root of g, again we can use [2].

5 Polar varieties

In the above algorithm we take a random coordinate transformation to obtain the polynomial f and compute points on the limit of the polar varieties of $V(f-t\xi)$ as $t \to 0$, where $\xi \in \mathbb{C}$, $|\xi| = 1$ is random and $t \in (0,1]$. In this section we prove that for almost all linear transformation and ξ, the complex dimension of the limit of the polar variety $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ is $i-1$ and it contains all real points of $V(f)$ as long as $\dim(V(f) \cap \mathbb{R}^n) \leq i-1$. The proof is based on the analogous results of [52, Props. 2-3] where they use a perturbation $f - \varepsilon$ by an infinitesimal variable ε.

In the rest of this section, we use the following notation from [52]: For $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$ and $\mathbf{x} = (x_1, \ldots, x_n)$

$$U := \{x \in \mathbb{R}^n \ : \ g_1(x) > 0, \ldots, g_m(x) > 0\} \subset \mathbb{R}^n,$$

and

$$S := V(f) \cap U \subset \mathbb{R}^n.$$

Definition 5.1. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$, $V = V(f) \subset \mathbb{C}^n$, and $A \in \mathrm{GL}_n(\mathbb{R})$. Then, we denote $f^A(\mathbf{x}) := f(A\mathbf{x})$, i.e., $V(f^A)$ is the image of V via the map $\mathbf{x} \mapsto A^{-1}\mathbf{x}$.

Theorem 5.2. *Let $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$, and U and S as above. Suppose $f \geq 0$ on $\mathbb{R}^n$ and $V(f) \cap \mathbb{R}^n$ is bounded. There is an open dense subset $O \subset \mathrm{GL}_n(\mathbb{R})$ and finite set $Z \subset \mathcal{S}_1 := \{\xi \in \mathbb{C} \ : \ |\xi| = 1\}$ such that for all $A \in O$ and $\xi \in \mathcal{S}_1 \setminus Z$, $i = 1, \ldots, n$ and $t \in (0, 1]$*

$$\mathrm{crit}(V(f^A - t\xi), \pi_i) \text{ empty or smooth and equidimensional of dim } i-1 \tag{2}$$

$$\left(\lim_{t\to 0} \mathrm{crit}(V(f^A - t\xi), \pi_i)\right) \cap U = S \quad \Leftrightarrow \quad \dim_{\mathbb{R}}(S) \leq i - 1. \tag{3}$$

Proof. First, we show that for all but a finite number of choices of $\xi \in \mathbb{C}$, $V(f - \xi)$ is smooth. This follows from Sard's theorem [14, Thm. 9.6.2] since $V(f - \xi, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \subset \mathbb{C}^n$ is empty for all but finitely many $\xi \in \mathbb{C}$. This also implies that for all but finitely many $\xi \in \mathbb{C}$ with $|\xi| = 1$ and for all $t \in (0, 1]$ we have that $V(f - t\xi)$ is smooth. Fix $\xi \in \mathbb{C}$ such that $V(f - t\xi)$ is smooth for all $t \in (0, 1]$. Fix $i \in \{1, \ldots, n\}$. Next we show that there is an open dense subset $O_i \subset \mathrm{GL}_n(\mathbb{R})$ such that if $A = [a_{k,l}]_{k,l=1}^n \in O_i$ then $\mathrm{crit}(V(f^A - t\xi)), \pi_i) \in \mathbb{C}^n$ is empty or smooth and equidimensional of dimension $i - 1$. The proof is a slight modification of the proof of [3, Prop. 3] using the map

$$\Phi_i : \mathbb{C}^n \times \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}^{n-i+1}$$

mapping $(y_1, \ldots, y_n, A = [a_{k,l}]_{k,l=1}^n) \in \mathbb{C}^n \times \mathrm{GL}_n(\mathbb{R})$ to

$$\left(f(\mathbf{y}) - t\xi, \sum_{k=1}^n a_{k,i+1} \frac{\partial f(\mathbf{y})}{\partial y_k}, \ldots, \sum_{k=1}^n a_{k,n} \frac{\partial f(\mathbf{y})}{\partial y_k} \right).$$

Here $\mathbf{y} = A\mathbf{x}$. Since $(f - t\xi)^A = f(\mathbf{y}) - t\xi$, our assumption on t and ξ implies that $V(f(\mathbf{y}) - t\xi)$ is smooth. This implies that for every point in $\Phi_i^{-1}(0)$ the Jacobian matrix of Φ_i has maximal rank of $n - i + 1$. If $\Phi_i^{-1}(0) = \emptyset$ then for all $A \in \mathrm{GL}_n(\mathbb{R})$ we have $\mathrm{crit}(V(f^A - t\xi)), \pi_i) = \emptyset$. If $\Phi_i^{-1}(0) \neq \emptyset$, then just as in the proof of [3, Prop. 3], we can apply weak transversality due to Thom/Sard to the surjective projection $\Phi_i^{-1}(0) \to \mathrm{GL}_n(\mathbb{R})$ to prove that for a dense subset of $O_i \subset \mathrm{GL}_n(\mathbb{R})$, for any fixed $A \in O_i$ the Jacobian of the map $\Phi_i|_A$ has full row rank at every point in $(\Phi_i|_A)^{-1}(0)$. This is equivalent to $\mathrm{crit}(V(f^A - t\xi)), \pi_i) \in \mathbb{C}^n$ being smooth and equidimensional of dimension $i - 1$ for every $A \in O_i$ as above. Setting $O' := \bigcap_{i=1}^n O_i$, O' is still dense and open $\mathrm{GL}_n(\mathbb{R})$, which proves the first claim.

The second claim follows from Propositions 5.3 and 5.4 and Lemma 5.5 as follows. Let $Z \subset \mathbb{C}$ be a finite set such that for all $\xi \in S_1 \setminus Z$ we have that $V(f - t\xi)$ is smooth for all $t \in (0, 1]$, as was proved above. Let $O' \subset \mathrm{GL}_n(\mathbb{R})$ defined in part (1). Let $O'' \subset \mathrm{GL}_n(\mathbb{C})$ Zariski open defined in Proposition 5.3 such that for all $A \in O'' \cap \mathrm{GL}_n(\mathbb{R})$, f^A satisfies properties $\mathtt{N}$. Let $O := O' \cap O''$. Then O is open and dense in $\mathrm{GL}_n(\mathbb{R})$ and for all $A \in O$ f^A satisfies the assumptions of both Proposition

5.4 and Lemma 5.5. Thus for all $\xi \in S_1 \setminus Z$, $t \in (0,1]$ and $A \in O$ we have $\dim_{\mathbb{R}}(S) \leq i-1$ if and only if

$$\left(\lim_{t \to 0} \operatorname{crit}(V(f^A - t\xi), \pi_i)\right) \cap U = \left(\lim_{\varepsilon \to 0} \operatorname{crit}(V(f^A - \varepsilon), \pi_i)\right) \cap U = S.$$

□

To prove the second claim of Theorem 5.2, we use Puiseux series in an infinitesimal variable ε. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and denote by $\mathbb{K}\langle\varepsilon\rangle$ the field of Puiseux series over $\mathbb{K}$, i.e.

$$\mathbb{K}\langle\varepsilon\rangle := \left\{ \sum_{i \geq i_0} a_i \varepsilon^{i/q} \ : \ i_0 \in \mathbb{Z}, q \in \mathbb{Z}_{>0}, a_i \in \mathbb{K} \right\}.$$

A Puiseux series $z = \sum_{i \geq i_0} a_i \varepsilon^{i/q} \in \mathbb{K}\langle\varepsilon\rangle$ is called bounded if $i_0 \geq 0$.

The following two propositions are proved in [52, Props. 2-3]:

Proposition 5.3. *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$. There exists a non-empty Zariski open set $O \in \mathrm{GL_n}(\mathbb{C})$ such that for $A \in O \cap GL_n(\mathbb{R})$, if $V^A = V(f^A)$ and $V_\varepsilon^A := V(f^A - \varepsilon) \subset \mathbb{C}\langle\varepsilon\rangle^n$ for ε infinitesimal then*

- $\mathtt{N}_1$: *For all $1 \leq i \leq n$, $\operatorname{crit}(V_\varepsilon^A, \pi_i)$ is either empty or is smooth and equidimensional with complex dimension $i-1$.*
- $\mathtt{N}_2$: *For all $\mathbf{p} \in V^A \cap \mathbb{R}^n$ we have $\pi_d^{-1}(\pi_d(\mathbf{p})) \cap V^A \cap \mathbb{R}^n$ is finite, where d is larger than or equal to the local real dimension of V^A at $\mathbf{p}$.*

Proposition 5.4. *Let $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$ and let ε be infinitesimal. Suppose $f \geq 0$ on $\mathbb{R}^n$, f satisfies property $\mathtt{N}_1$ and $\mathtt{N}_2$, and $V(f) \cap \mathbb{R}^n$ is bounded. Then for $i = 1, \ldots, n$, $\lim_{\varepsilon \to 0} \operatorname{crit}(V_\varepsilon, \pi_i)$ is equidimensional of dimension $i-1$ and for U and S as above we have*

$$\left(\lim_{\varepsilon \to 0} \operatorname{crit}(V_\varepsilon, \pi_i)\right) \cap U = S \quad \Leftrightarrow \quad \dim_{\mathbb{R}}(S) \leq i-1.$$

Finally, the next lemma is the last missing piece in the proof of (2) in Theorem 5.2.

Lemma 5.5. *Let $f, g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_n]$, let ε be infinitesimal. Assume that $\mathcal{V}_\varepsilon := V(f - \varepsilon, g_1, \ldots, g_m) \subset \mathbb{C}\langle\varepsilon\rangle^n$ is smooth and equidimensional of dimension $n-m-1$. Suppose $\xi \in \mathbb{C}$ with $|\xi| = 1$ and assume that for $\mathcal{V}_{t\xi} := V(f - t\xi, g_1, \ldots, g_m) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n-m-1$ for all $t \in (0,1]$. Then we have that*

$$\lim_{\varepsilon \to 0} \mathcal{V}_\varepsilon = \lim_{t \to 0} \mathcal{V}_{t\xi}.$$

Proof. Let $L_1, \ldots, L_{n-m-1} \in \mathbb{C}[x_1, \ldots, x_n]$ be linear polynomials such that $\mathcal{L} = \{L_1, \ldots, L_{n-m-1}\}$ is a generic linear space of codimension $m+1$ which intersects both $\lim\limits_{\varepsilon \to 0} \mathcal{V}_\varepsilon$ and $\lim\limits_{t \to 0} \mathcal{V}_{t\xi}$ transversely. By our assumptions, both $\mathcal{V}_\varepsilon \cap \mathcal{L}$ and $\mathcal{V}_{t\xi} \cap \mathcal{L}$ are finite. Then since $\mathcal{L}$ does not depend on either ε or t, one can see that is suffice to show that

$$\lim_{\varepsilon \to 0} \left(\mathcal{V}_\varepsilon \cap \mathcal{L}\right) = \lim_{t \to 0} \left(\mathcal{V}_{t\xi} \cap \mathcal{L}\right).$$

Let $H \subset \mathbb{R}[x_1, \ldots, x_n, \varepsilon]$ be the system

$$H := H(x, \varepsilon) = [f(x) - \varepsilon, g_1, \ldots, g_m, L_1, \ldots, L_{n-m-1}].$$

Let $S \subset \mathbb{C}\langle\varepsilon\rangle^n$ be the finite set of bounded solutions of $H = 0$, where bounded is as defined for Puiseux series above. Then for all $x(\varepsilon) \in S$, let $x_0 = \lim_{\varepsilon\to 0} x(\varepsilon) = x_0 \in \mathbb{C}^n$. Furthermore, by the definition of H, $\lim_{\varepsilon\to 0} S = \lim_{\varepsilon\to 0}\left(\mathcal{V}_\varepsilon \cap \mathcal{L}\right)$.

Since $\varepsilon > 0$ is a real infinitesimal, each $x(\varepsilon)$ has an interval of convergence $(0, \varepsilon_x) \subset \mathbb{R}$ for some $\varepsilon_x > 0$. Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < \min_{x\in S} \varepsilon_x$. Then, for $z \in \mathbb{C}$ with $|z| \le \varepsilon_0$, $x(z) \in \mathbb{C}^n$ for $x \in S$.

We consider the branch points of $x(z)$ for all $x \in S$. In particular, the critical points $\mathcal{C}$ associated to these branch points are all $z \in \mathbb{C}$ such that there exists an $x \in \mathbb{C}^n$ where $H(x, z) = 0$ and $\det JH(x, z) = 0$, where JH is the Jacobian matrix of H with respect to x. Then, since $|S| < \infty$, we know $|\mathcal{C}| < \infty$,

Now let $z \in \mathcal{C}$. Then there exists some $\xi_z \in \mathcal{S}_1$ such that for $t \in \mathbb{R}$, the path $\xi_z t$ passes through z, so that $x(t\xi_z) \in \mathbb{C}^n$ has some branching point. Let $Z = \{\xi_z : z \in \mathcal{C}\} \subset \mathcal{S}_1$, since $|\mathcal{C}| < \infty$, $|Z| < \infty$. Then, for $\xi \in S_1 \setminus Z$, $x(t\xi) \in \mathbb{C}^n$ for $t \in (0, 1]$ does not pass through branching points. Since $S_1 \setminus Z$ is Zariski dense in S_1, the same holds for generic $\xi \in S_1$.

So let $\xi \in S_1$ be generic and $H_\xi \subset \mathbb{C}^{n+1}$ be the homotopy defined by the system

$$H_\xi := H_\xi(x, t) = [f(x) - t\xi, g_1, \ldots, g_m, L_1 \ldots, L_{n-m-1}]. \tag{4}$$

The limit points of the solutions of H_ξ are $\lim_{t\to 0}\left(\mathcal{V}_{t\xi} \cap \mathcal{L}\right)$. Let $T \subset \mathbb{C}^n$ be the roots of $H_\xi(x, 1)$. Then $|T| = |\mathcal{V}_\varepsilon \cap \mathcal{L}| < \infty$. Furthermore, by the above argument the homotopy paths for H_ξ are exactly described by the points in $\mathcal{V}_\varepsilon \cap \mathcal{L} \subset \mathbb{C}\langle\varepsilon\rangle^n$ by replacing ε with $t\xi$. Hence,

$$\lim_{\varepsilon\to 0}\left(\mathcal{V}_\varepsilon \cap \mathcal{L}\right) = \lim_{t\to 0}\left(\mathcal{V}_{t\xi} \cap \mathcal{L}\right).$$

□

6 Computation of g

In this section we present a method to compute g that satisfy the conditions of Step 2 in our Algorithm using *isoingular deflations* (see [35] and Section 7 below).

Proposition 6.1. *Suppose $f \in \mathbb{R}[x_1, \ldots, x_n]$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$ satisfy (2) in Theorem 5.2 with $U = \mathbb{R}^n$ and $A = I$. Let $V \subset \mathbb{C}^n$ be an irreducible component of $\lim_{t\to 0} \operatorname{crit}(V(f - t\xi), \pi_i)$ and $z \in V$ be a non-singular point on V. Let $F = \{F_1, \ldots, F_N\} \subset \mathbb{R}[x_1, \ldots, x_n]$ be the* isosingular deflation sequence *satisfying*

$$\left\{f, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n}\right\} \subset F,$$

$F(z) = 0$ and $\operatorname{rank}(JF(z)) = n - i + 1$. Here $JF(x) \in \mathbb{R}[x_1, \ldots, x_n]^{N\times n}$ is the Jacobian matrix of F. Let $M(x)$ be an $(n - i + 1) \times (n - i + 1)$ submatrix of $JF(x)$ such that $M(z)$ is non-singular. Then for $g(x) := \det(M(x))$ we have

1. $\operatorname{Sing}(V) \subset V(g)$

2. $\dim V \cap V(g) \leq i-2$.

Proof. If $y \in \operatorname{Sing}(V)$ then $\operatorname{rank}(JF(y)) < n-i+1$ so all $(n-i+1)\times(n-i+1)$ subdeterminants of $JF(y)$ are singular, so $g(y) = \det(M(y)) = 0$, which proves (1). By Theorem 5.2 (2), $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ is equidimensional of dimension $i-1$, so $\dim V = i-1$. Since V is irreducible and $V \not\subset V(g)$, we have (2). $\square$

Section 7 explains how to compute an isosingular deflation sequence F for one or more irreducible components of the limit variety $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ by first computing a witness set as in Definition 7.1.

The next theorem proves the correctness of our Algorithm even in the case when the irreducible components of the limit variety $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ require several isosingular deflation sequences. In the Algorithm, for the simplicity of the presentation, we assumed that $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ is irreducible.

Theorem 6.2. *Suppose $f \in \mathbb{R}[x_1, \dots, x_n]$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$ satisfy (2) and (3) in Theorem 5.2 with $U = \mathbb{R}^n$ and $A = I$. Fix $i \in \{1, \dots, n\}$ and assume that*

$$\dim_{\mathbb{R}} V(f) \cap \mathbb{R}^n \leq i-1.$$

Let $g_1, \dots, g_r \subset \mathbb{R}[x_1, \dots, x_n]$ be polynomials such that for each $j \in \{1, \dots, r\}$ there exists a nonempty set of irreducible components $\mathcal{I}_j := \{V_{j,1}, \dots, V_{j,s_j}\}$ of $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ such that $\bigcup_{j=1}^r \bigcup_{V\in\mathcal{I}_j} V = \lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ and for $k = 1, \dots, s_j$

1. $\operatorname{Sing}(V_{j,k}) \cap \mathbb{R}^n \subset V(g_j)$

2. $\dim V_{j,k} \cap V(g_j) \leq i-2$.

Define for $j = 1, \dots, r$,

$$L_{i,j}^{t\xi} := \left\{ \frac{\partial g_j}{\partial x_k} + \sum_{s=1}^{n-d} \lambda_s \frac{\partial f_s}{\partial x_k} \ : \ k = 1, \dots, n \right\} \cup \left\{ f - t\xi, \frac{\partial f}{\partial x_{i+1}}, \dots, \frac{\partial f}{\partial x_n} \right\}$$

and let

$$\mathcal{S}_{i,j} := \left(\lim_{t\to 0} \pi_x \left(V(L_{i,j}^{t\xi}) \right) \setminus V(g_j) \right) \cap \bigcup_{V\in\mathcal{I}_j} V \cap \mathbb{R}^n.$$

Define $\mathcal{S}_i := \bigcup_{j=1}^r \mathcal{S}_{i,j}$. Then $\mathcal{S}_i \neq \emptyset$ if and only if $\dim_{\mathbb{R}} V(f) \cap \mathbb{R}^n = i-1$.

Proof. Suppose $\dim_{\mathbb{R}} V(f) \cap \mathbb{R}^n = i-1$. Then by Theorem 5.2 we have $(\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)) \cap \mathbb{R}^n = V(f) \cap \mathbb{R}^n$. Since $\dim_{\mathbb{C}} (\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)) = i-1$, its complex and real dimensions are the the same, so it must contain a real smooth point by Theorem 2.4. Suppose the irreducible component $V \subset \mathbb{C}^n$ of $\lim_{t\to 0} \operatorname{crit}(V(f-t\xi), \pi_i)$ contains such smooth real points. Let g_j be the polynomial satisfying (1) and (2) of the claim for V. Then by Theorem 3.7 $\lim_{t\to 0} \pi_x \left(V(L_{i,j}^{t\xi}) \right)$ contains a smooth point on every component of $V \cap \mathbb{R}^n$. Thus, $\mathcal{S}_i \neq \emptyset$. Suppose now that $\mathcal{S}_i \neq \emptyset$. Then there exists $j \in \{1, \dots, r\}$ such that $\mathcal{S}_{i,j} \neq \emptyset$, so let $z \in \mathbb{R}^n$ be an element of it. Since $\mathcal{S}_{i,j} \subset \bigcup_{V\in\mathcal{I}_j} V$, there exists $k \in \{1, \dots, s_j\}$ such that $z \in V_{j,k}$. Since $\operatorname{Sing}(V_{j,k}) \cap \mathbb{R}^n \subset V(g_j)$ and $g_j(z) \neq 0$, z is a smooth point of $V_{j,k} \cap \mathbb{R}^n$, so $\dim_{\mathbb{R}} V_{j,k} \cap \mathbb{R}^n = \dim_{\mathbb{C}} V_{j,k} = i-1$. $\square$

To compute elements in $\left(\lim_{t\to 0}\pi_x\left(V(L_{i,j}^{t\xi})\right)\right)\cap\bigcup_{V\in\mathcal{I}_j}V$ as in Theorem 6.2, we need to exclude points from $\lim_{t\to 0}\pi_x(V(L_{i,j}^{t\xi}))$ that do not correspond to components with the prescribed isosingular deflation sequence. Once we computed the isosingular deflation sequence, this can be done by a membership test, e.g., [12, Chap. 8].

7 Witness sets and isosingular deflation

In this section, we show how to construct witness sets, which is a numerical algebraic geometry data structure for representing algebraic sets, for the limits of polar varieties.

Definition 7.1. If $V\subset\mathbb{C}^n$ is equidimensional with $\dim V = k$, a *witness set* for V is the triple $\{F, L, W\}$ such that

- F is a *witness system* for V in that each irreducible component of V is an irreducible component of $V(F)$;
- L is a *linear system* where $V(L)$ is a linear space of codimension k that intersects V transversely;
- W is a *witness point set* which is equal to $V\cap V(L)$.

See the books [12, 55] for more details on witness sets.

The proof of Lemma 5.5 describe computing witness point sets in the limits of polar varieties. Thus, all that remains is to compute a witness system in order to perform computations on this limit variety. One difficulty is that a witness point could lie on some irreducible component of $\mathrm{crit}(V,\pi_i)$ of dimension greater than $i-1$. Another difficulty is that the limit points may be singular arising from multiple paths converging to the same limit point. Isosingular deflation [35] with [36, Thm. 6.2] yields a procedure to compute a witness system for each irreducible component of the limit.

Let H_ξ be as (4) and $\mathbf{p}$ a limit point of the solutions of H_ξ. Therefore, $\mathbf{p}$ is a general point on an irreducible component $V_{\mathbf{p}}$ of $V=\lim_{\varepsilon\to 0}\mathcal{V}_\varepsilon$. The following describes constructing a witness system for $V_{\mathbf{p}}$. Define $F_0(x,t,s)=\{H_\xi(x,t),s\}$ and $\mathbf{q}=(\mathbf{p},0,0)\in V(F_0)$. Define the *isosingular deflation operator* $\mathcal{D}$ via

$$(F_1,\mathbf{q}) := \mathcal{D}(F_0,\mathbf{q})$$

where F_1 consists of F_0 and all $(r+1)\times(r+1)$ minors of the Jacobian matrix JF_0 for F_0 where $r=\mathrm{rank}\, JF_0(\mathbf{q})$. Thus, $\mathbf{q}\in V(F_1)$ meaning that we can iterate this operator to construct a sequence of systems $F_j(x,t,s)$ with $(F_j,\mathbf{q})=\mathcal{D}(F_{j-1},\mathbf{q})$ for $j\geq 1$.

Theorem 7.2. *With the setup described above, there exists $j^*\geq 0$ such that, for all $j\geq j^*$, the system $F(x):=F_j(x,0,0)$ is a witness system for $V_{\mathbf{p}}\subset V$.*

Proof. Since $V_{\mathbf{p}}$ is an irreducible component of V, we know that $V_{\mathbf{p}}\times\{0\}$ is an irreducible component of $V\cap V(t)$. Hence, it follows from the isosingular deflation approach applied to intersections in [36, Thm. 6.2] that there exists $j^*\geq 0$ such that, for all $j\geq j^*$, $G_j(x,t):=F_j(x,t,t)$ is a witness system for $V_{\mathbf{p}}\times\{0\}$. Since s is contained in $F_0(x,t,s)$, we know $t\in G_j(x,t)$ yielding $V(G_j)\subset\mathbb{C}^n\times\{0\}$. Thus, it immediately follows that, for every $j\geq j^*$, $V_{\mathbf{p}}$ must be an irreducible component of $\pi_x(V(G_j))=V(F)$ where $F(x)=G_j(x,0)=F_j(x,0,0)$. $\square$

The number j^* can be determined algorithmically [35, Alg. 6.3].

Example 7.3. Consider $H_1(x,t) = [x_1x_2 - t, x_1x_2 - x_1]$ with $\mathbf{p} = (0,1)$ and $V_\mathbf{p} = V = \{(0,1)\}$. The system $H_1(x,0)$ is not a witness system for $V_\mathbf{p}$ since $V(H_1(x,0)) = V(x_1)$ is one-dimensional. Take $\mathbf{q} = (0,1,0,0)$ and $F_0(x,t,s) = \{H_1(x,t), s\}$. Since rank $JF_0(\mathbf{q}) = 2$,

$$F_1(x,t,s) = \{F_0(x,t,s), x_1, x_2 - 1\}$$

Hence, $F(x) := F_1(x,0,0)$ is clearly a witness system for $V_\mathbf{p}$.

We note that Theorem 7.2 only guarantees that $V_\mathbf{p}$ is an irreducible component of $V(F)$, but it may have multiplicity larger than one. Hence, once a witness system $F(x)$ has been constructed for $V_\mathbf{p}$, one can construct another sequence of polynomials, say $G_0(x) := F(x)$ and $(G_j, \mathbf{p}) = \mathcal{D}(G_{j-1}, \mathbf{p})$ for $j \geq 1$. Once $\dim \operatorname{null} JG_j(\mathbf{p}) = k = \dim V_\mathbf{p}$, then $G_j(x)$ is a witness system for $V_\mathbf{p}$ which has multiplicity one. This is the system that we use in Proposition 6.1.

Example 7.4. For $H_1(x,t) = \{x_1^2 - x_2 - t, x_2\}$ with $\mathbf{p} = (0,0)$ and $V_\mathbf{p} = V = \{(0,0)\}$, the system $H_1(x,0) = \{x_1^2 - x_2, x_2\}$ is a witness system for $V_\mathbf{p}$, but has multiplicity 2. Moreover, $F_j(x,t,s) = F_0(x,t,s) = \{H_1(x,t), s\}$ for all $j \geq 1$ showing that the witness system generated by Theorem 7.2 may have multiplicity greater than 1. For $G_0(x) = \{x_1^2 - x_2, x_2\}$, isosingular deflation yields $G_1(x) = \{G_0(x), 2x_1\}$ with $\dim \operatorname{null} JG_1(\mathbf{p}) = 0 = \dim V_\mathbf{p}$.

We note that the isosingular deflation operator presented above utilized all appropriate minors for constructing the two sequences of polynomial systems. One could utilize alternative deflation approaches such as [26, 28, 33, 43, 44] to possibly simplify the construction of the witness system.

8 Application to Kuramoto model

The Kuramoto model [40] is a dynamical system used to model synchronization amongst n coupled oscillators. The maximum number of equilibria (i.e., real solutions to steady-state equations) for $n \geq 4$ remains an open problem [24]. The following confirms the conjecture in [60] for $n = 4$ with the rest of the section describing its proof.

Theorem 8.1. *The maximum number of equilibria for the Kuramoto model with $n = 4$ oscillators is 10.*

The steady-state equations for the $n = 4$ Kuramoto model are

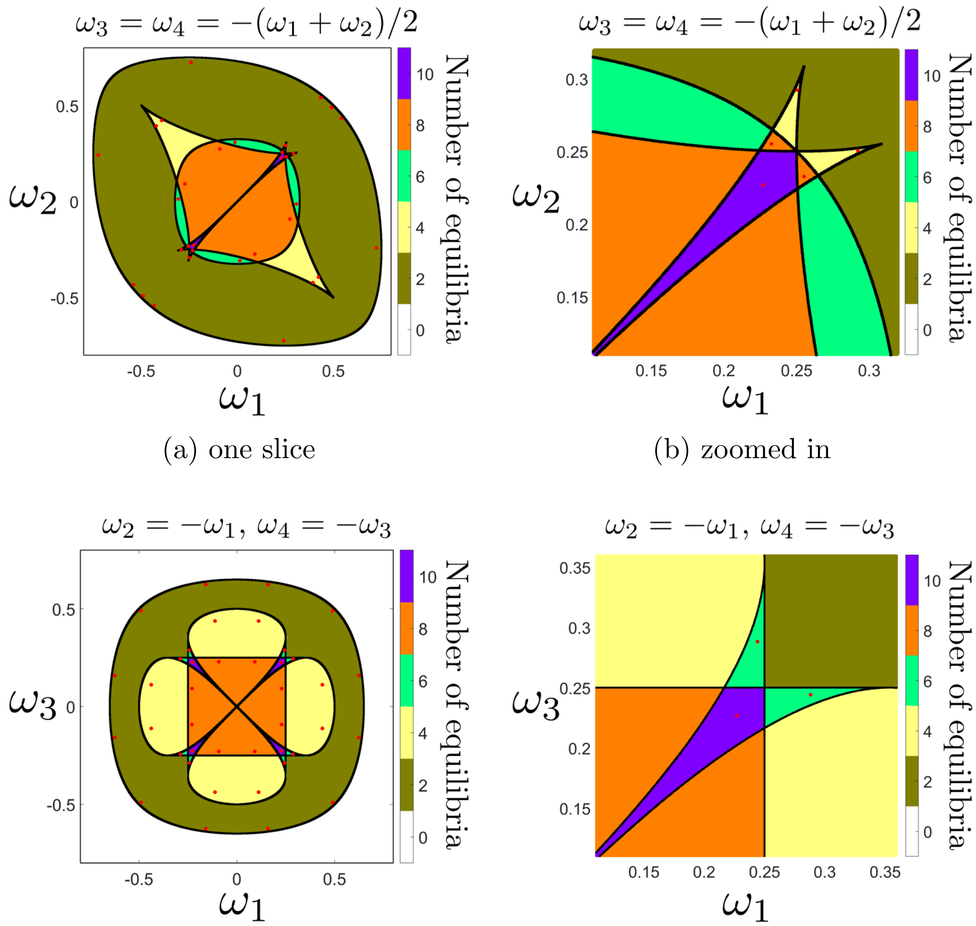
$$f_i(\theta;\omega) = \omega_i - \frac{1}{4}\sum_{j=1}^{4} \sin(\theta_i - \theta_j) = 0, i = 1, \ldots, 4$$

parameterized by the natural frequencies $\omega_i \in \mathbb{R}$. Since only the angle differences matter, one can assume $\theta_4 = 0$ and observe a necessary condition for equilibria is

$$0 = f_1 + f_2 + f_3 + f_4 = \omega_1 + \omega_2 + \omega_3 + \omega_4,$$

i.e., assume $\omega_4 = -(\omega_1 + \omega_2 + \omega_3)$. Replacing θ_i by $s_i = \sin(\theta_i)$ and $c_i = \cos(\theta_i)$ yields the polynomial system

$$F(s,c;\omega) = \left\{ \omega_i - \frac{1}{4}\sum_{j=1}^{4}(s_ic_j - s_jc_i), s_i^2 + c_i^2 - 1, \text{ for } i = 1,2,3 \right\}$$



(a) one slice

(b) zoomed in

(c) other slice

(d) zoomed in

Figure 3: Compact connected regions and critical points for the Kuramoto model with $n = 4$

with variables $s = (s_1, s_2, s_3)$ and $c = (c_1, c_2, c_3)$, parameters $\omega = (\omega_1, \omega_2, \omega_3)$, and constants $s_4 = 0$ and $c_4 = 1$.

The goal is to compute the maximum number of isolated real solutions of $F = 0$ as ω varies over $\mathbb{R}^3$. Clearly, a necessary condition for having real solutions is for $|\omega_i| \leq 0.75$ showing that we are only interested in computing at least one point in each compact connected component of $\mathbb{R}^3 \setminus V(D)$ where $D(\omega)$ is the discriminant, a polynomial of degree 48. Since D attains a non-zero extreme value on each compact connected component of $\mathbb{R}^3 \setminus V(D)$, one simply needs to compute all real solutions of $\nabla D = 0$ where $D \neq 0$. This was accomplished by exploiting symmetry and utilizing `Bertini` [11], `alphaCertified` [34], and `Macaulay2` [29] certifying all solutions have been found. In fact, this computation showed that all real critical points arose, up to symmetry, along two slices shown in Figure 3. A similar computation then counted the number of real solutions to $F = 0$ showing that the maximum number of equilibria is 10. All code used in these computations is available at `dx.doi.org/10.7274/r0-5c1t-jw53`.

9 Conclusion

This paper describes an algorithm that computes real smooth points on a given algebraic set V via a polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ that vanishes on $\mathrm{Sing}(V)$ but transversely intersects V. We applied this result to compute the dimension of real varieties.

The bottleneck of our algorithm is that the polynomial g satisfying the above conditions may have high degree. Since the Lagrange multiplier system uses the partial derivatives of g to construct a zero dimensional system, the *a priori* bound on the number of roots of this system may be very large.

We conjecture that one can replace the condition $\dim V \cap V(g) \leq d-1$ in Theorem 3.7 with the weaker condition $\dim W_g \leq d-1$ for $W_g := \lim_{e \to 0}(V_e^{\mathbf{a}} \cap V(g))$. This would allow using polynomials g that have degrees similar to the degrees of the input polynomials.

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References

[1] P. Aubry, F. Rouillier, and M. Safey El Din. Real solving for positive dimensional systems. *J. Symbolic Comput.*, 34(6):543–560, 2002.

[2] T. Ayyildiz Akoglu, J. D. Hauenstein, and A. Szanto. Certifying solutions to overdetermined and singular polynomial systems over $\mathbb{Q}$. *J. Symbolic Comput.*, 84:147–171, 2018.

[3] B. Bank, M. Giusti, J. Heintz, R. Mandel, and G. M. Mbakop. Polar varieties and efficient real equation solving: the hypersurface case. In *Proceedings of the 3rd International Conference on Approximation and Optimization in the Caribbean (Puebla, 1995)*, page 13. Benemérita Univ. Autón. Puebla, Puebla, 1997.

[4] B. Bank, M. Giusti, J. Heintz, and G. M. Mbakop. Polar varieties and efficient real elimination. *Math. Z.*, 238(1):115–144, 2001.

[5] B. Bank, M. Giusti, J. Heintz, and L. M. Pardo. Generalized polar varieties and an efficient real elimination procedure. *Kybernetika (Prague)*, 40(5):519–550, 2004.

[6] B. Bank, M. Giusti, J. Heintz, and L. M. Pardo. On the intrinsic complexity of point finding in real singular hypersurfaces. *Inform. Process. Lett.*, 109(19):1141–1144, 2009.

[7] B. Bank, M. Giusti, J. Heintz, M. Safey El Din, and E. Schost. On the geometry of polar varieties. *Applicable Algebra in Engineering, Communication and Computing*, 21(1):33–83, Jan 2010.

[8] I. Bannwarth and M. Safey El Din. Probabilistic algorithm for computing the dimension of real algebraic sets. In *ISSAC'15—Proceedings of the 2015 ACM International Symposium on Symbolic and Algebraic Computation*, pages 37–44. ACM, New York, 2015.

[9] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006.

[10] S. Basu, R. Pollack, and M.-F. Roy. Computing the dimension of a semi-algebraic set. *Journal of Mathematical Sciences*, 134(5):2346–2353, May 2006.

[11] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. Bertini: Software for numerical algebraic geometry. Available at `bertini.nd.edu`.

[12] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. *Numerically solving polynomial systems with Bertini*, volume 25 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[13] E. Becker and R. Neuhaus. Computation of real radicals of polynomial ideals. In *Computational algebraic geometry (Nice, 1992)*, volume 109 of *Progr. Math.*, pages 1–20. Birkhäuser Boston, Boston, MA, 1993.

[14] J. Bochnak, M. Coste, and M.-F. Roy. *Géométrie algébrique réelle*, volume 12 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[15] C. W. Brown. Qepcad b: A program for computing with semi-algebraic sets using cads. *SIGSAM Bull.*, 37(4):97–108, Dec. 2003.

[16] C. W. Brown. Constructing a single open cell in a cylindrical algebraic decomposition. In *ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*, pages 133–140. ACM, New York, 2013.

[17] C. W. Brown and M. Košta. Constructing a single cell in cylindrical algebraic decomposition. *J. Symbolic Comput.*, 70:14–48, 2015.

[18] P. Bürgisser, F. Cucker, and P. Lairez. Computing the homology of basic semialgebraic sets in weak exponential time. *J. ACM*, 66(1):Art. 5, 30, 2019. [Publication date initially given as 2018].

[19] C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. In *ISSAC 2010—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 187–194. ACM, New York, 2010.

[20] C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. *J. Symbolic Comput.*, 49:3–26, 2013.

[21] C. Chen, J. H. Davenport, M. Moreno Maza, B. Xia, and R. Xiao. Computing with semi-algebraic sets represented by triangular decomposition. In *ISSAC 2011—Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, pages 75–82. ACM, New York, 2011.

[22] G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decompostion. In H. Brakhage, editor, *Automata Theory and Formal Languages*, pages 134–183, Berlin, Heidelberg, 1975. Springer Berlin Heidelberg.

[23] G. E. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *J. Symbolic Comput.*, 12(3):299–328, 1991.

[24] O. Coss, J. D. Hauenstein, H. Hong, and D. K. Molzahn. Locating and counting equilibria of the Kuramoto model with rank-one coupling. *SIAM J. Appl. Algebra Geom.*, 2(1):45–71, 2018.

[25] F. Cucker, T. Krick, and M. Shub. Computing the homology of real projective sets. *Found. Comput. Math.*, 18(4):929–970, 2018.

[26] B. H. Dayton and Z. Zeng. Computing the multiplicity structure in solving polynomial systems. In *ISSAC'05*, pages 116–123. ACM, New York, 2005.

[27] J.-C. Faugère, G. Moroz, F. Rouillier, and M. S. El Din. Classification of the perspective-three-point problem, discriminant variety and real solving polynomial systems of inequalities. In *Proceedings of the Twenty-first International Symposium on Symbolic and Algebraic Computation*, ISSAC '08, pages 79–86, New York, NY, USA, 2008. ACM.

[28] M. Giusti and J.-C. Yakoubsohn. Multiplicity hunting and approximating multiple roots of polynomial systems. In *Recent advances in real complexity and computation*, volume 604 of *Contemp. Math.*, pages 105–128. Amer. Math. Soc., Providence, RI, 2013.

[29] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[30] D. Y. Grigor′ev and N. N. Vorobjov, Jr. Solving systems of polynomial inequalities in subexponential time. *J. Symbolic Comput.*, 5(1-2):37–64, 1988.

[31] D. Y. Grigor′ev and N. N. Vorobjov, Jr. Counting connected components of a semialgebraic set in subexponential time. *Comput. Complexity*, 2(2):133–186, 1992.

[32] J. D. Hauenstein. Numerically computing real points on algebraic sets. *Acta Applicandae Mathematicae*, 125(1):105–119, Jun 2013.

[33] J. D. Hauenstein, B. Mourrain, and A. Szanto. On deflation and multiplicity structure. *J. Symbolic Comput.*, 83:228–253, 2017.

[34] J. D. Hauenstein and F. Sottile. Algorithm 921: alphaCertified: certifying solutions to polynomial systems. *ACM Trans. Math. Software*, 38(4):Art. 28, 20, 2012.

[35] J. D. Hauenstein and C. W. Wampler. Isosingular sets and deflation. *Found. Comput. Math.*, 13(3):371–403, 2013.

[36] J. D. Hauenstein and C. W. Wampler. Unification and extension of intersection algorithms in numerical algebraic geometry. *Appl. Math. Comput.*, 293:226–243, 2017.

[37] J. Heintz, M.-F. Roy, and P. Solernó. Description of the connected components of a semialgebraic set in single exponential time. *Discrete Comput. Geom.*, 11(2):121–140, 1994.

[38] H. Hong and M. Safey El Din. Variant real quantifier elimination: algorithm and application. In *ISSAC 2009—Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation*, pages 183–190. ACM, New York, 2009.

[39] P. Koiran. The real dimension problem is $\mathrm{NP}_{\mathbf{R}}$-complete. *J. Complexity*, 15(2):227–238, 1999.

[40] Y. Kuramoto. Self-entrainment of a population of coupled non-linear oscillators. *Lect. Notes Phys.*, 39:420–422, 1975.

[41] J.-B. Lasserre, M. Laurent, B. Mourrain, P. Rostalski, and P. Trébuchet. Moment matrices, border bases and real radical computation. *J. Symbolic Comput.*, 51:63–85, 2013.

[42] J. B. Lasserre, M. Laurent, and P. Rostalski. Semidefinite characterization and computation of zero-dimensional real radical ideals. *Found. Comput. Math.*, 8(5):607–647, 2008.

[43] A. Leykin, J. Verschelde, and A. Zhao. Newton's method with deflation for isolated singularities of polynomial systems. *Theoret. Comput. Sci.*, 359(1-3):111–122, 2006.

[44] A. Leykin, J. Verschelde, and A. Zhao. Higher-order deflation for polynomial systems with isolated singular solutions. In *Algorithms in algebraic geometry*, volume 146 of *IMA Vol. Math. Appl.*, pages 79–97. Springer, New York, 2008.

[45] Y. Ma, C. Wang, and L. Zhi. A certificate for semidefinite relaxations in computing positive-dimensional real radical ideals. *J. Symbolic Comput.*, 72:1–20, 2016.

[46] M. Marshall. *Positive polynomials and sums of squares*, volume 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[47] R. Neuhaus. Computation of real radicals of polynomial ideals. II. *J. Pure Appl. Algebra*, 124(1-3):261–280, 1998.

[48] F. Rouillier, M.-F. Roy, and M. S. E. Din. Finding at least one point in each connected component of a real algebraic set defined by a single equation. *Journal of Complexity*, 16(4):716 – 750, 2000.

[49] M. Safey El Din. Testing sign conditions on a multivariate polynomial and applications. *Math. Comput. Sci.*, 1(1):177–207, 2007.

[50] M. Safey El Din and E. Schost. Polar varieties and computation of one point in each connected component of a smooth algebraic set. In *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation*, pages 224–231. ACM, New York, 2003.

[51] M. Safey El Din and P.-J. Spaenlehauer. Critical point computations on smooth varieties: degree and complexity bounds. In *Proceedings of the 2016 ACM International Symposium on Symbolic and Algebraic Computation*, pages 183–190. ACM, New York, 2016.

[52] M. Safey El Din and E. Tsigaridas. A probabilistic algorithm to compute the real dimension of a semi-algebraic set. *CoRR*, abs/1304.1928, 2013.

[53] M. Safey El Din, Z.-H. Yang, and L. Zhi. On the complexity of computing real radicals of polynomial systems. In *ISSAC'18—Proceedings of the 2018 ACM International Symposium on Symbolic and Algebraic Computation*, pages 351–358. ACM, New York, 2018.

[54] A. Seidenberg. A new decision method for elementary algebra. *Ann. of Math. (2)*, 60:365–374, 1954.

[55] A. J. Sommese and C. W. Wampler, II. *The numerical solution of systems of polynomials*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. Arising in engineering and science.

[56] F. Sottile. Real algebraic geometry for geometric constraints. In *Handbook of geometric constraint systems principles*, Discrete Math. Appl. (Boca Raton), pages 273–285. CRC Press, Boca Raton, FL, 2019.

[57] S. J. Spang. A zero-dimensional approach to compute real radicals. *Comput. Sci. J. Moldova*, 16(1):64–92, 2008.

[58] N. Vorobjov. Complexity of computing the local dimension of a semialgebraic set. *J. Symbolic Comput.*, 27(6):565–579, 1999.

[59] W. Wu and G. Reid. Finding points on real solution components and applications to differential polynomial systems. In *ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*, pages 339–346. ACM, New York, 2013.

[60] X. Xin, T. Kikkawa, and Y. Liu. Analytical solutions of equilibrium points of the standard kuramoto model: 3 and 4 oscillators. In *2016 American Control Conference (ACC)*, pages 2447–2452, 2016.