Numerically computing real points on algebraic sets

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Abstract

Given a polynomial system f, a fundamental question is to determine if f has real roots. Many algorithms involving the use of infinitesimal deformations have been proposed to answer this question. In this article, we transform an approach of Rouillier, Roy, and Safey El Din, which is based on a classical optimization approach of Seidenberg, to develop a homotopy based approach for computing at least one point on each connected component of a real algebraic set. Examples are presented demonstrating the effectiveness of this parallelizable homotopy based approach.

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1 Introduction

Computing real roots of a polynomial system is a difficult and extremely important problem. In many applications in science, engineering, and economics, the real roots are the only ones of interest. Due to the importance of this problem, many approaches have been proposed. Two approaches are the cylindrical algebraic decomposition algorithm [17] and so-called critical point methods, such as Seidenberg's approach of computing critical points of the distance function [42]. The cylindrical algebraic decomposition algorithm has doubly exponential complexity in the number of variables. However, using the idea of Seidenberg and related ideas developed in [5, 16, 21, 22, 27, 39], algorithms with asymptotically optimal complexity estimates for computing at least one real point on each connected component of a real algebraic set were developed. Other related approaches for computing real roots are presented in [1, 2, 3, 40] and the references therein. These symbolic based methods have similar complexity to the best known bounds, but are more efficient in practice compared with [5, 16, 21, 22, 27, 39]. The approach presented here will transform the algorithms presented in [1, 40] into a homotopy based algorithm.

Several homotopy based algorithms have been proposed to compute real roots of a polynomial system. The algorithms in [32] and [13] utilize critical point methods to decompose the real points of a complex curve and a complex surface with finitely many singularities, respectively. An algorithm for directly computing only the real roots that are isolated over the complex numbers is presented in [12]. The complexity of this approach depends upon the fewnomial structure of

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the given polynomial system. The approach presented below is not restricted to low-dimensional cases and the real roots are not assumed to be isolated over the complex numbers.

Two other nonhomotopy based algorithms are presented in [30] and [15]. The approach in [30] (see also [31]) uses semidefinite programming for computing real roots. This algorithm computes every real root assuming the number of real roots is finite. The approach in [15] uses tools related to maximum likelihood estimation in statistics for computing real positive roots of certain types of polynomial systems.

The rest of the article is structured as follows. The remainder of this section describes the needed concepts from complex, real, and numerical algebraic geometry and a brief introduction to Puiseux series. Section 2 describes the homotopy based approach with examples demonstrating the algorithm in Section 3.

1.1 Algebraic sets and genericity

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\}$. The set $\mathcal{V}(f) \subset \mathbb{C}^N$ is called the *algebraic set associated to* f. A set $X \subset \mathbb{C}^N$ is called an *algebraic set* if there exists a polynomial system $g : \mathbb{C}^N \to \mathbb{C}^m$ such that $X = \mathcal{V}(g)$. An algebraic set $X \subset \mathbb{C}^N$ is *reducible* if there exists algebraic sets $Y, Z \subset \mathbb{C}^N$, which are proper subsets of X, such that $X = Y \cup Z$. An algebraic set is *irreducible* if it is not reducible. For an irreducible algebraic set X, the subset of manifold points X_{reg} is dense in X, open, and connected. The *dimension* of an irreducible algebraic set X is the dimension of X_{reg} as a complex manifold.

On irreducible algebraic sets, we can define the notion of genericity.

Definition 1 Let $X \subset \mathbb{C}^N$ be an irreducible algebraic set. A property P is said to hold *generically* on X if the subset of points in X which do not satisfy P are contained in a proper algebraic subset of X. That is, there is a nonempty Zariski open subset U of X such that P holds at every point in U. Each point in U is called a *generic point* of X with respect to P.

Since every proper algebraic subset of $\mathbb{C}$ is a finite set, a property P holds generically on $\mathbb{C}$ if P holds at all but finitely many points in $\mathbb{C}$.

Every algebraic set X can be written uniquely (up to reordering) as the finite union of inclusion maximal irreducible algebraic sets, called the *irreducible decomposition* of X. That is, there are irreducible algebraic sets $A_1, \ldots, A_k$ such that

$$X = \bigcup_{i=1}^{k} A_i \quad \text{and} \quad A_i \not\subset A_j \text{ for } i \neq j.$$

Each A_i is called an *irreducible component* of X.

The dimension of an algebraic set is the maximum dimension of its irreducible components. An algebraic set is called *pure-dimensional* if each irreducible component has the same dimension. The *pure i-dimensional component* of an algebraic set is the union of the irreducible components of dimension i. In summary, the algebraic set $\mathcal{V}(f)$ has an irreducible decomposition of the form

$$\mathcal{V}(f) = \bigcup_{i=0}^{\dim \mathcal{V}(f)} V_i = \bigcup_{i=0}^{\dim \mathcal{V}(f)} \bigcup_{j=1}^{k_i} V_{i,j} \tag{1}$$

where V_i is the pure i-dimensional component of $\mathcal{V}(f)$ and each $V_{i,j}$ is a distinct i-dimensional irreducible component.

1.2 Decomposition of real algebraic sets

Real algebraic sets are subsets of $\mathbb{R}^N$ which arise as the intersection of algebraic sets in $\mathbb{C}^N$ with $\mathbb{R}^N$. That is, a set $X \subset \mathbb{R}^N$ is a real algebraic set if there is an algebraic set $Y \subset \mathbb{C}^N$ such that $X = Y \cap \mathbb{R}^N$. For a polynomial system $f : \mathbb{R}^N \to \mathbb{R}^n$, the *real algebraic set associated to* f is $\mathcal{V}_{\mathbb{R}}(f) = \mathcal{V}(f) \cap \mathbb{R}^N = \{x \in \mathbb{R}^N \mid f(x) = 0\}$.

Consider the algebraic set $X = \mathcal{V}(y^2 - x^2(x-1)) \subset \mathbb{C}^2$. It is easy to verify that X is an irreducible algebraic set and, hence, both X and X_{reg} are connected. However, the real algebraic set $X \cap \mathbb{R}^2$ is not connected. This example suggests that we should consider decomposing real algebraic sets into connected components.

A real algebraic set $X \subset \mathbb{R}^N$ can be written uniquely (up to reordering) as the disjoint union of finitely many path-connected sets $C_1, \ldots, C_\ell \subset \mathbb{R}^N$ such that C_i and $V \setminus C_i$ are both closed in the Euclidean topology on $\mathbb{R}^N$. Each C_i is called a *connected component* of X and one can verify that it is a semi-algebraic set. Expanded details regarding real algebraic sets and decompositions can be found in [4, 14].

To demonstrate this decomposition and contrast it with the irreducible decomposition of algebraic sets, consider the algebraic sets $X = \mathcal{V}(y^2 - x^2(x-1))$, $Y = \mathcal{V}(x-y)$, and $Z = X \cup Y$ with corresponding real algebraic sets $X_{\mathbb{R}} = X \cap \mathbb{R}^2$, $Y_{\mathbb{R}} = Y \cap \mathbb{R}^2$, and $Z_{\mathbb{R}} = Z \cap \mathbb{R}^2$. It is easy to verify that X and Y are irreducible algebraic sets with Z clearly being a reducible algebraic set. The set $X_{\mathbb{R}}$ consists of two connected components, namely $C_1 = \{(0,0)\}$ and a connected curve $C_2 = Y_{\mathbb{R}} \setminus C_1$. Since the real algebraic sets $Y_{\mathbb{R}}$ and $Z_{\mathbb{R}}$ are connected, $Y_{\mathbb{R}}$ and $Z_{\mathbb{R}}$ each have only one connected component.

1.3 Puiseux series

Since we will utilize Puiseux series in Section 2, we will provide a brief review here. For more detailed information, see [4].

The field of algebraic Puiseux series over $\mathbb{C}$ is

$$\mathbb{C}\langle\epsilon\rangle = \left\{ \sum_{j \geq j_0} a_j \epsilon^{j/q} \;\middle|\; j_0 \in \mathbb{Z}, q \in \mathbb{N}, a_j \in \mathbb{C} \text{ with } a_{j_0} \neq 0 \right\}.$$

To simplify the notation, we shall define $a_j = 0$ for all $j < j_0$. An element in $\mathbb{C}\langle\epsilon\rangle$ is *bounded* if $j_0 \geq 0$ and *infinitesimal* if $j_0 > 0$. The subset consisting of bounded elements, denoted $\mathbb{C}_b\langle\epsilon\rangle$, is a ring which is naturally mapped to $\mathbb{C}$ by the ring homomorphism $\lim_0$ defined by

$$\lim\nolimits_0 \sum_{j \geq j_0} a_j \epsilon^{j/q} = a_0.$$

1.4 Numerical irreducible decomposition and witness sets

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system. A numerical irreducible decomposition of $\mathcal{V}(f)$, first presented in [49], is a numerical decomposition analogous to (1) using witness sets (see [50, Chaps. 12-15] for more expanded details). Suppose that V is the pure i-dimensional component of $\mathcal{V}(f)$ with $d = \deg V$. For a fixed generic i-codimensional linear space $H \subset \mathbb{C}^N$, we have that $V \cap H$ consists of d points. Let $L : \mathbb{C}^N \to \mathbb{C}^i$ be a system of linear polynomials such that $\mathcal{V}(L) = H$. The set $V \cap \mathcal{V}(L) = V \cap H$ is called a *witness point set* for V with the triple

$\mathcal{W} = \{f, L, V \cap \mathcal{V}(L)\}$ called a *witness set* for V. A *numerical irreducible decomposition* of $\mathcal{V}(f)$ is of the form

$$\bigcup_{i=0}^{\dim \mathcal{V}(f)} \bigcup_{j=1}^{k_i} \mathcal{W}_{i,j} \tag{2}$$

where $\mathcal{W}_{i,j}$ is a witness set for a distinct i-dimensional irreducible component of $\mathcal{V}(f)$. We note that the union of witness sets in (2) should be considered as a formal union. Numerical irreducible decompositions can be computed using the algorithms presented in [6, 26, 43, 45, 46, 47, 48, 49].

1.5 Trackable paths

Numerical homotopy methods rely on the ability to construct homotopies with solution paths that are *trackable*. The following is the definition of a trackable solution path starting at a nonsingular point adapted from [25].

Definition 2 Let $H(x,t) : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ be polynomial in x and complex analytic in t and let x^* be a nonsingular isolated solution of $H(x,1) = 0$. We say that x^* is *trackable* for $t \in (0,1]$ from $t = 1$ to $t = 0$ using $H(x,t)$ if there is a smooth map $\xi_{x^*} : (0,1] \to \mathbb{C}^N$ such that $\xi_{x^*}(1) = x^*$ and, for $t \in (0,1]$, $\xi_{x^*}(t)$ is a nonsingular isolated solution of $H(x,t) = 0$.

The solution path starting at x^* is said to *converge* if $\lim_{t\to 0^+} \xi_{x^*}(t) \in \mathbb{C}^N$, where $\lim_{t\to 0^+} \xi_{x^*}(t)$ is called the *endpoint* (or *limit point*) of the path.

2 Real points on an algebraic set

Let $f : \mathbb{R}^N \to \mathbb{R}^n$ be a polynomial system and $V \subset \mathcal{V}_{\mathbb{C}}(f)$ be a pure d-dimensional algebraic set. The main problem we consider is, given a witness set $\{f, \mathcal{L}, W\}$ for V, compute a finite set of points which contains at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ contained in V. We note that if $d = 0$, then $V = W$ so that one can compute the real points in V simply by considering the finitely many points in W. Hence, we will assume that $d > 0$.

We will also reduce to the case $n = N - d$. One way to always reduce down to this case is to consider the polynomial g which is the sum of squares of f, that is, $g = f_1^2 + \cdots + f_n^2$, with $V_g = \mathcal{V}(g)$. If T is a finite set of points which contains at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(g)$, then $T \cap V$ contains at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f) = \mathcal{V}_{\mathbb{R}}(g)$ contained in V. The set $T \cap V$ can be computed from T and a witness set for V using the homotopy membership test [46].

We summarize the assumptions in the following statement.

Assumption 3 *Let $N > d > 0$, $f : \mathbb{R}^N \to \mathbb{R}^{N-d}$ be a polynomial system, and $V \subset \mathcal{V}(f)$ be a pure d-dimensional algebraic set with witness set $\{f, \mathcal{L}, W\}$.*

The following lemma considers the solutions of $f(x) = z$ for $z \in \mathbb{C}^{N-d}$.

Lemma 4 *With Assumption 3, there is a nonempty Zariski open set $Z \subset \mathbb{C}^{N-d}$ such that, for every $z \in Z$, $\mathcal{V}(f - z)$ is a smooth algebraic set of dimension d.*

Proof. Let $r = \dim \overline{f(\mathbb{C}^N)}$ and $c = N - r$, which are called *rank of* f and the *corank of* f respectively [50, §13.4]. Since $\overline{f(\mathbb{C}^N)} \subset \mathbb{C}^{N-d}$, we have $r \leq N - d$ and hence $d \leq N - r = c$. Since $\mathcal{V}(f)$ has a component of dimension d, Theorem 13.4.2 of [50] yields that $d \geq c$. Therefore, $c = d$ and $r = N - d$. The lemma now follows immediately from Lemma 13.4.1 of [50]. □

Lemma 4 permits the use of continuation techniques as stated in the following theorem.

Theorem 5 *Suppose that Assumption 3 holds. Let* $z \in \mathbb{R}^{N-d}$, $\gamma \in \mathbb{C}$, $y \in \mathbb{R}^N \setminus \mathcal{V}_{\mathbb{R}}(f)$, $\alpha \in \mathbb{C}^{N-d+1}$, *and* $H : \mathbb{C}^N \times \mathbb{C}^{N-d+1} \times \mathbb{C} \to \mathbb{C}^{2N-d+1}$ *be the homotopy defined by*

$$H(x, \lambda, t) = \begin{bmatrix} f(x) - t\gamma z \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots + \lambda_{N-d} \nabla f_{N-d}(x)^T \\ \alpha_0 \lambda_0 + \cdots + \alpha_{N-d} \lambda_{N-d} - 1 \end{bmatrix} \tag{3}$$

where $f(x) = [f_1(x), \ldots, f_{N-d}(x)]^T$ *such that following statements hold.*

1. *The set* $S \subset \mathbb{C}^N \times \mathbb{C}^{N-d+1}$ *of roots of* $H(x, \lambda, 1)$ *is finite and each is a nonsingular solution of* $H(x, \lambda, 1) = 0$.
2. *The number of points in* S *is equal to the maximum number of isolated solutions of* $H(x, \lambda, 1) = 0$ *as* z, γ, y, *and* α *vary over sets* $\mathbb{C}^{N-d}$, $\mathbb{C}$, $\mathbb{C}^N$, *and* $\mathbb{C}^{N-d+1}$, *respectively.*
3. *The solution paths defined by* H *starting, with* $t = 1$, *at the points in* S *are trackable.*
4. *If* $\pi(x, \lambda) = x$,

$$\begin{aligned} E &= \{\lim_{t \to 0^+} \xi_s(t) \mid s \in S \text{ and the solution path } \xi_s \text{ converges}\}, \quad \text{and} \\ E_1 &= \{\lim_{t \to 0^+} \pi(\xi_s(t)) \mid s \in S \text{ and the path } \pi(\xi_s) \text{ converges}\}, \end{aligned}$$

   we have $E_1 = \pi(E)$.

Then, $E_1 \cap V \cap \mathbb{R}^N$ *contains a point on each connected component of* $\mathcal{V}_{\mathbb{R}}(f)$ *contained in* V.

The homotopy H defined in (3) is based on the classical approach of Seidenberg [42]. If $y \in \mathbb{R}^N \setminus \mathcal{V}_{\mathbb{R}}(f)$, consider the quadratic polynomial

$$d_y(x) = (x - y)^T (x - y) = \sum_{i=1}^{N} (x_i - y_i)^2$$

and the optimization problem

$$\text{(P)} \qquad \min \{d_y(x) \mid x \in \mathcal{V}_{\mathbb{R}}(f)\}.$$

We want to compute the points on $\mathcal{V}(f)$ for which $\nabla d_y(x) = 2(x - y)^T$ and $\nabla g(x)$ are linearly dependent. The approach in [40] for hypersurfaces uses determinants to describe this linear dependence condition, while the approach in both Theorem 5 and [41] use auxiliary variables. In particular, the polynomial system $\mathcal{G}_{f,y} : \mathbb{C}^N \times \mathbb{P}^{N-d} \to \mathbb{C}^{2N-d}$ defined by

$$G_{f,y}(x, \lambda) = \begin{bmatrix} f(x) \\ \lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots + \lambda_{N-d} \nabla f_{N-d}(x)^T \end{bmatrix} \tag{4}$$

comprises the Fritz John conditions [29] for problem (P) and provides necessary conditions for optimality. That is, if ξ is a local minimizer for problem (P), then there exists $\lambda \in \mathbb{P}^{N-d}$ such that $(\xi, \lambda) \in \mathcal{V}(\mathcal{G}_{f,y})$.

Clearly, $x \in \mathcal{V}(f)$ such that

$$\text{rank} \begin{bmatrix} x-y & \nabla f_1(x)^T & \cdots & \nabla f_{N-d}(x)^T \end{bmatrix} \le N-d$$

if and only if there exists $\lambda \in \mathbb{P}^{N-d}$ such that $(x,\lambda) \in \mathcal{V}(\mathcal{G}_{f,y})$. A point $x \in \pi(\mathcal{V}(\mathcal{G}_{f,y}))$ is called a *critical point* of the distance function with respect to f, where $\pi(x,\lambda) = x$.

Consider $\text{Sing}(f) = \{x \in \mathbb{C}^N \mid \text{rank } Jf(x) < N-d\}$, where $Jf(x)$ is the Jacobian matrix of f evaluated at x. If $\text{Sing}(f)$ is positive dimensional, then $\mathcal{V}(\mathcal{G}_{f,y})$ is also positive dimensional. By using Lemma 4, we can consider smooth algebraic sets thereby allowing the computation of finitely many points in $\mathcal{V}_{\mathbb{R}}(f)$ containing the points of interest.

The following lemma will be used to complete the proof of Theorem 5

Lemma 6 *Suppose that Assumption 3 holds. Let ϵ be an infinitesimal, $y \in \mathbb{R}^N \setminus \mathcal{V}_{\mathbb{R}}(f)$, $z \in \mathbb{R}^{N-d}$ with $z_i \neq 0$, and $f_\epsilon(x) = f(x) - \epsilon z$ be such that $\mathcal{V}(\mathcal{G}_{f_\epsilon,y})$ is finite and $|\mathcal{V}(\mathcal{G}_{f_\epsilon,y})|$ is equal to the maximum number of isolated solutions as y and z varies over the sets $\mathbb{C}^N$ and $\mathbb{C}^{N-d}$, respectively. Then,*

1. *$V \subset \lim_0 \left(\mathcal{V}(f_\epsilon) \cap \mathbb{C}_b\langle\epsilon\rangle^N\right)$,*
2. *$\lim_0 \left(\mathcal{V}(f_\epsilon) \cap \mathbb{C}_b\langle\epsilon\rangle^N\right) \cap \mathbb{R}^N = \lim_0 \left(\mathcal{V}(f_1^2 - \epsilon^2 z_1^2, \ldots, f_{N-d}^2 - \epsilon^2 z_{N-d}^2) \cap \mathbb{R}_b\langle\epsilon\rangle^N\right)$, and*
3. *$\lim_0 \left(\pi(\mathcal{V}(\mathcal{G}_{f_\epsilon,y})) \cap \mathbb{C}_b\langle\epsilon\rangle^N\right) \cap \mathbb{R}^N$ contains a point in each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ contained in V where $\pi(x,\lambda) = x$.*

Proof. This setup implies that $\mathcal{V}(f_\epsilon)$ is a d-dimensional smooth algebraic set for which we clearly have $V \subset \lim_0(\mathcal{V}(f_\epsilon) \cap \mathbb{C}_b\langle\epsilon\rangle^N)$ yielding Item 1. Item 2 follows from the fact that $z_i \in \mathbb{R}$ and

$$\lim_0 \left(\mathcal{V}(g-\epsilon) \cap \mathbb{C}_b\langle\epsilon\rangle^N\right) \cap \mathbb{R}^N = \lim_0 \left(\mathcal{V}(g^2-\epsilon^2) \cap \mathbb{R}_b\langle\epsilon\rangle^N\right)$$

for any polynomial $g : \mathbb{R}^N \to \mathbb{R}$. Item 3 follows by using the same proof as Lemma 3.7 in [40] with the replacement of Lemma 3.6 of [40] with Items 1 and 2. □

Before we prove Theorem 5, we note that the polynomial system $\mathcal{G}_{g,y}(x,\lambda)$ defined in (4), has $\lambda \in \mathbb{P}^{N-d}$. The polynomial system $H(x,\lambda,0)$ defined in (3) has $\lambda \in \mathbb{C}^{N-d+1}$ restricted to the Euclidean patch defined by $\alpha_0\lambda_0 + \cdots + \alpha_{N-d}\lambda_{N-d} = 1$. Item 3 in Theorem 5 enforces that this Euclidean patch is in general position with respect to the finitely many solution paths. Therefore, we can use the results of Lemma 6 in the following proof of Theorem 5.

Proof of Theorem 5. Let ϵ be an infinitesimal and $f_\epsilon = f - \epsilon z$. Item 2 yields that $|S| = |\mathcal{V}(\mathcal{G}_{f_\epsilon,y})| < \infty$. The result will follow from Lemma 6 upon showing

$$E_1 = \lim_0 \left(\pi(\mathcal{V}(\mathcal{G}_{f_\epsilon,y})) \cap \mathbb{C}_b\langle\epsilon\rangle^N\right). \tag{5}$$

We will deduce (5) by comparing the polynomial systems $\mathcal{G}_{f_\epsilon,y}$ and

$$\mathcal{G}^a_{f_\epsilon,y}(x,\lambda) = \begin{bmatrix} f(x) - \epsilon z \\ \lambda_0(x-y) + \lambda_1 \nabla f_1(x)^T + \cdots + \lambda_{N-d}\nabla f_{N-d}(x)^T \\ \alpha_0\lambda_0 + \cdots + \alpha_{N-d}\lambda_{N-d} - 1 \end{bmatrix}.$$

Item 2 also yields that $|S| = |\mathcal{V}(\mathcal{G}_{f_\epsilon,y})| = |\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y})|$. In particular, by abuse of notation regarding π, we have

$$\pi(\mathcal{V}(\mathcal{G}_{f_\epsilon,y})) = \pi(\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y})). \tag{6}$$

Since there are finitely many homotopy paths, there exists $0 < t_0 < 1$ such that all of the homotopy paths for H with $0 < t < 2t_0$ are described by the points in $\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y}) \subset \mathbb{C}\langle\epsilon\rangle^{2N-d}$ by replacing ϵ with $t\gamma$. This yields that the set of limit points of the homotopy $H_0(x,\lambda,t) := H(x,\lambda,(1-t)\cdot t_0)$ starting at the roots of $H(x,\lambda,t_0)$ is

$$T = \lim\nolimits_0 \left(\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y}) \cap \mathbb{C}_b\langle\epsilon\rangle^{N+2}\right).$$

Since, by Items 1 and 3, the homotopy paths of H are nonsingular for $t \in (0,1]$, coefficient-parameter continuation [35] yields that $T = E$. Item 4 yields

$$\pi(E) = \pi\left(\lim\nolimits_0 \left(\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y}) \cap \mathbb{C}_b\langle\epsilon\rangle^{N+2}\right)\right) = \lim\nolimits_0 \left(\pi(\mathcal{V}(\mathcal{G}^a_{f_\epsilon,y})) \cap \mathbb{C}_b\langle\epsilon\rangle^{N}\right) = E_1.$$

This equation together with (6) yields (5). □

We note that in the hypersurface case, that is $n = N - d = 1$, if f has degree $2k$, the 2-homogeneous Bézout count yields that

$$|S| \le K(N,2k) := N \cdot 2k \cdot (2k-1)^{N-1}.$$

In particular, $\mathcal{V}_\mathbb{R}(f)$ can have at most $K(N,2k)$ connected components and hence $K(N,2k)$ bounds the number of real roots of f that are isolated over $\mathbb{R}$. This bound is only N times larger than the bound obtained in [14, Prop. 11.5.2].

2.1 An algorithm

Theorem 5 yields an approach for computing a point on each connected component of $\mathcal{V}_\mathbb{R}(f)$. Before presenting an algorithm which implements the ideas of this theorem, we state two remarks. First, Item 2 of Theorem 5 holds for a nonempty Zariski open set of $\mathbb{C}^{N-d} \times \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{N-d+1}$. The following algorithm assumes that the given point (z,γ,y,α) lies in this Zariski open set. As part of the procedure, it computationally verifies Items 1, 3, and 4 of Theorem 5 hold. Second, the use of γ is based on the "Gamma Trick" [50, Lemma 7.1.3] first introduced by Morgan and Sommese [34].

Second, since there exist many suitable methods to compute the start points S, the following algorithm does not directly specify which one to utilize. Nonetheless, to improve efficiency in this computation, the method should, in some way, utilize the natural 2-homogeneous structure.

Procedure $[v,R] =$ **RealPoints**$(f,\mathcal{W},z,\gamma,y,\alpha)$

Input A polynomial system $f : \mathbb{R}^N \to \mathbb{R}^{N-d}$, a witness set $\mathcal{W}$ for a pure d-dimensional algebraic set $V \subset \mathcal{V}(f)$, $z \in \mathbb{R}^{N-d}$, $\gamma \in \mathbb{C}$, $y \in \mathbb{R}^N \setminus \mathcal{V}_\mathbb{R}(f)$, and $\alpha \in \mathbb{C}^{N-d+1}$ such that Item 2 of Theorem 5 holds.

Output A boolean v which is *true* if Items 1, 3, and 4 in Theorem 5 have been computationally verified, otherwise *false*. If v is *true*, R is a finite subset of $\mathbb{R}^N$ containing a point on each connected component of the real algebraic set $\mathcal{V}_\mathbb{R}(f)$ contained in V.

Begin

1. Construct the homotopy H defined in (3).
2. Compute the solutions S of $H(x,\lambda,1) = 0$.

(a) Use S to verify that Item 1 of Theorem 5 holds. If it does not hold, **Return** $[false, \emptyset]$.

3. Track the solution paths of H starting at each point in S to compute the sets E and E_1 defined in Theorem 5.

   (a) If the tracking fails for a path or $\pi(E) \neq E_1$ where $\pi(x,\lambda) = x$, **Return** $[false, \emptyset]$.

4. Use the homotopy membership test to compute the set R consisting of the points in $E_1 \cap \mathbb{R}^N$ contained in V.

Return $[true, R]$.

Since the endpoints E computed in Step 3 may be singular solutions of $H(x,\lambda,0) = 0$, the use of an endgame, e.g., [8, 28, 37, 36], together with adaptive precision tracking [7, 9, 11] may be required to accurately compute them. Also, Step 3 should use the method of [33] to avoid infinite length paths.

Example 7 To illustrate the algorithm for a hypersurface, consider the polynomial $f(x_1,x_2,x_3) = (x_1+x_3)^2 + x_2^2$ with $V = \mathcal{V}(f)$. Clearly, $\mathcal{V}_{\mathbb{R}}(f) = \{(a,0,-a) \mid a \in \mathbb{R}\} \subset \mathrm{Sing}(f)$. Item 2 holds with $z = 1$, $\gamma = 2+3i$,

$$y = \begin{bmatrix} 3/8 \\ 5/9 \\ 1/3 \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} 1/2 - i/5 \\ 6/7 + 2i/3 \end{bmatrix}, \quad \text{where} \quad i = \sqrt{-1}.$$

Let H be the homotopy defined by (3).

- For Step 2, we used a standard 2-homogeneous homotopy, which required tracking $K(3,2) = 6$ paths, to compute the set S consisting of the four nonsingular solutions of $H(x,\lambda,1) = 0$.
- The four paths tracked in Step 3, which started at the points in S, all converged with the endpoints of the two paths ending at the real point coinciding. In particular, E and E_1 both consist of three points with $\pi(E) = E_1$ where

$$E_1 = \left\{ \begin{array}{c} (1/48, 0, -1/48), (-1/3+5i/9, 10/9+17i/24, -3/8+5i/9), \\ (-1/3-5i/9, 10/9-17i/24, -3/8-5i/9) \end{array} \right\}.$$

- Since $V = \mathcal{V}(f)$, we have $R = E_1 \cap \mathbb{R}^N = \{(1/48, 0, -1/48)\}$.

It is easy to verify that the point $(1/48, 0, -1/48)$ is the minimizer of the distance between the point y and $\mathcal{V}_{\mathbb{R}}(f)$, as shown in Figure 1.

Example 8 To illustrate the algorithm for an algebraic set, consider the polynomial system

$$f(x) = \begin{bmatrix} g_1(x) + r_1 g_3(x) \\ g_2(x) + r_2 g_3(x) \end{bmatrix} \quad \text{where} \quad g(x_1,x_2,x_3) = \begin{bmatrix} (x_1^2 - x_2)(x_1^2+x_2^2+x_3^2-1)(x_1-1) \\ (x_1x_2 - x_3)(x_1^2+x_2^2+x_3^2-1)(x_2-2) \\ (x_1x_3 - x_2^2)(x_1^2+x_2^2+x_3^2-1)(x_3-3) \end{bmatrix}$$

with $r_1 = 1/3$ and $r_2 = 1/7$. We want to investigate real points of the cubic curve $V = \{(x_1, x_1^2, x_1^3) \mid x_1 \in \mathbb{C}\} \subset \mathcal{V}(f)$ which is input into **RealPoints** via a witness set $\mathcal{W}$. Item 2 holds with

$$z = \begin{bmatrix} 1/5 \\ 1/9 \end{bmatrix}, \quad \gamma = 3/11 - i/13, \quad y = \begin{bmatrix} 1/4 \\ 1/6 \\ -3/2 \end{bmatrix}, \quad \text{and} \quad \alpha = \begin{bmatrix} 1/3 - i/7 \\ 6/11 + 3i/4 \\ 2/3 - 7i/8 \end{bmatrix} \quad \text{where } i = \sqrt{-1}.$$

Let H be the homotopy defined by (3).

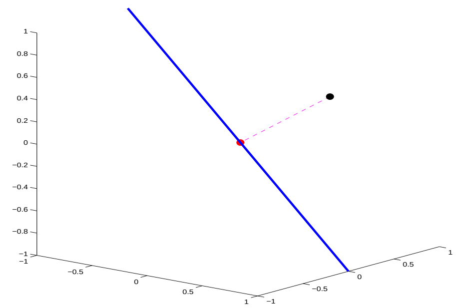

Figure 1: Plot of $\mathcal{V}_{\mathbb{R}}(f)$ and the point minimizing the distance between y and $\mathcal{V}_{\mathbb{R}}(f)$

- We used a standard 2-homogeneous homotopy, which required tracking 300 paths, to compute the set S in Step 2 consisting of the 95 nonsingular solutions of $H(x, \lambda, 1) = 0$.
- All 95 of the paths tracked in Step 3 starting from the points in S converged with the set $E_1 \cap \mathbb{R}^N$ consisting of 15 points.
- The homotopy membership test yields that 7 of these 15 points lie on V.

The point of minimum distance on $V \cap \mathbb{R}^N$ to y is approximately $(0.168, 0.028, 0.005)$, which is displayed in Figure 2 with the other 6 points on V.

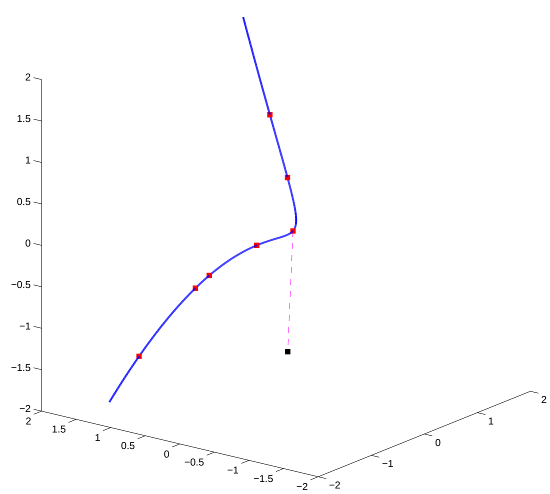

Figure 2: Plot of $V \cap \mathbb{R}^N$ and the point minimizing the distance between y and $V \cap \mathbb{R}^N$

3 Examples

The following examples were run using the software package Bertini v1.3.1 [10] on a server having four 2.3 GHz Opteron 6176 processors and 64 GB of memory that runs 64-bit Linux. The serial examples used one core while the parallel examples used one manager and 47 working cores. For the nonsingular solutions, we utilized alphaCertified [23, 24] to certify reality. For the singular solutions, we determined reality based upon the size of the imaginary parts using two different numerical approximations of the point.

3.1 Hypersurface example

Consider the polynomial provided in Example 5 of [40], namely

$$\begin{aligned} f(u_2,u_3,u_4,u_5) \quad = \quad & 110u_5^2u_4u_3 + 190u_5u_4^2u_3 + 80u_4^3u_3 + 80u_5^2u_3^2 + 270u_5u_4u_3^2 + 160u_4^2u_3^2 \\ & +80u_5u_3^3 + 80u_4u_3^3 - 32u_4u_3^2u_2 - 32u_3^3u_2 - 80u_5^2u_2^2 - 128u_5u_4u_2^2 \\ & -160u_5u_3u_2^2 - 112u_4u_3u_2^2 - 64u_3^2u_2^2 - 80u_5u_2^3 - 32u_3u_2^3 + 60u_5^2u_4 \\ & +220u_5u_4^2 + 160u_4^3 + 67u_5u_4u_3 + 136u_4^2u_3 - 24u_5u_3^2 - 88u_4u_3^2 - 64u_3^3 \\ & -100u_5^2u_2 + 32u_5u_4u_2 + 96u_4^2u_2 - 228u_5u_3u_2 - 108u_4u_3u_2 - 120u_3^2u_2 \\ & +20u_5u_2^2 + 96u_4u_2^2 - 56u_3u_2^2 + 110u_5u_4 + 80u_4^2 + 48u_4u_3 - 32u_3^2 \\ & +30u_5u_2 + 48u_4u_2 - 20u_3u_2. \end{aligned}$$

The approach of [40] computes 26 real points on the hypersurface which contains at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ using $y = 0$. Since $y = 0$ does not satisfy the hypotheses of Theorem 5, we used $y = \left[\begin{array}{cccc} 4/3 & -9/5 & -5/7 & 8/9 \end{array}\right]^T$ to compute at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$. In particular, we used serial processing with **RealPoints** taking $V = \mathcal{V}(f)$, $z = 1$, and $\gamma \in \mathbb{C}$ and $\alpha \in \mathbb{C}^2$ to be random of unit length.

Let H be the homotopy defined by (3).

- For Step 2, we used a 2-homogeneous regeneration [25] to compute the set S consisting of the 151 nonsingular solutions of $H(x, \lambda, 1) = 0$ in 10 seconds.
- For Step 3, each of the 151 paths converged with the set $E_1 \cap \mathbb{R}^N$ consisting of 28 distinct points, which was computed in 4 seconds.
- Since $V = \mathcal{V}(f)$, $R = E_1 \cap \mathbb{R}^N$ which consists of 28 points.

We note that since $|\mathrm{Sing}(f)| < \infty$, we could directly compute $\mathcal{V}(H(x, \lambda, 0))$ using a standard 2-homogeneous homotopy, which requires the tracking of 432 paths. Bertini performed this computation in serial in 27 seconds which yielded the same set R of 28 real critical points, as required by theory [35].

3.2 An example from filter banks

Consider the polynomial system named F633 [19] that was considered in [1], which is available at [20]. This polynomial system consists of 9 polynomials in 10 variables. Since two of the polynomials are linear and linearly independent, we utilized intrinsic coordinates to reduce the number of variables to 8 and the number of polynomials to 7, all of which are bilinear. Since these 7 polynomials are not independent, we further reduced down to a system of 6 bilinear polynomials in 8 variables, namely

$$f(u_3,\ldots,u_6,U_3,\ldots,U_6) = \begin{bmatrix} g(u_3,\ldots,u_6,U_3,\ldots,U_6) \\ g(U_3,\ldots,U_6,u_3,\ldots,u_6) \\ u_3U_3-1 \\ u_4U_4-1 \\ u_5U_5-1 \\ u_6U_6-1 \end{bmatrix}$$

where

$$g(x_1,\ldots,x_4,y_1,\ldots,y_4) = 8(x_1y_2+x_1y_3+x_2y_3+x_1y_4+x_2y_4+x_3y_4)+4(x_1+x_2+x_3+x_4)+13.$$

The algebraic set $\mathcal{V}(f)$ is an irreducible surface of degree 32. We used **RealPoints** to compute a set of points containing a point from each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ by taking

$$y = \begin{bmatrix} 1/5 & -3/4 & -2/3 & 7/9 & -4/7 & 12/13 & 1/2 & -10/11 \end{bmatrix}^T,$$

$z \in \mathbb{R}^6$, $\gamma \in \mathbb{C}$, and $\alpha \in \mathbb{C}^7$ to be random of unit length, and $\mathcal{W}$ a witness set for $V = \mathcal{V}(f)$. Let H be the homotopy defined by (3).

- For Step 2, we used a standard 2-homogeneous homotopy, which required tracking 1792 paths, to compute the set S consisting of the 274 nonsingular solutions of $H(x,\lambda,1) = 0$. This computation took 120 seconds in serial (4 seconds in parallel).
- For Step 3, each of the 274 paths converged with the set $E_1 \cap \mathbb{R}^N$ consisting of 36 distinct points. This computation took one second in serial.
- Since $V = \mathcal{V}(f)$, $R = E_1 \cap \mathbb{R}^N$ which consists of 36 points.

In Step 2, we could have used a 3-homogeneous homotopy since the system itself is naturally 2-homogeneous. However, this would increase the number of paths from 1792 to 1960. Also, since $\mathrm{Sing}(f) = \emptyset$, we could directly compute $\mathcal{V}(H(x,\lambda,0))$ using a standard 2-homogeneous homotopy, which requires the tracking of 1792 paths. Bertini performed this computation in serial in 120 seconds yielding the same set R of 36 real critical points, as required by theory [35].

3.3 A cubic-centered 12-bar linkage

Consider the 12-bar spherical linkage obtained by locking the scissors of the collapsible cube with 12 scissors linkages presented in [53], which is displayed in Figure 3 of [52]. Following the setup in [52], we will consider the cube with side length 2 where we fix the center at the origin and two adjacent vertices, say $P_7 = (-1,1,-1)$ and $P_8 = (-1,-1,-1)$. Let $P_1,\ldots,P_6$ denote the position of the other 6 vertices yielding 18 variables. The constraints on these vertices is that they must maintain their initial relative distances yielding a polynomial system f consisting of the following 17 polynomials:

$$\begin{aligned} g_{ij} &= |P_i - P_j|^2 - 4, \\ &\qquad \{i,j\} \in \{(1,2),(3,4),(5,6),(1,5),(2,6),(3,7),(4,8),(1,3),(2,4),(5,7),(6,8)\}; \\ h_i &= |P_i|^2 - 3, \qquad i \in \{1,2,3,4,5,6\}. \end{aligned}$$

The algebraic set $\mathcal{V}(f)$ consists of 8, 34, and 2 irreducible components of dimension 1, 2, and 3, respectively. Table 1 presents the degrees of these components. Let V be the union of the

dimension	degree	# components
3	8	2
2	4	2
	8	14
	12	12
	16	1
	20	4
	24	1
1	4	6
	6	2

Table 1: Irreducible decomposition of $\mathcal{V}(f)$

one-dimensional irreducible components of $\mathcal{V}(f)$, which has degree 36, and $\mathcal{W}$ be a witness set for V. Let $C_1, \ldots, C_6$ denote the six irreducible curves of degree 4 contained in V, and C_7 and C_8 denote the two irreducible curves of degree 6 contained in V. The components $C_1, \ldots, C_6$ are self-conjugate while C_7 and C_8 are conjugates of each other. That is, $C_7 \cup C_8$ contains only finitely many real points which must be contained in $C_7 \cap C_8$.

We used **RealPoints** to compute a finite set of points containing a point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ contained in V by taking

$$\begin{aligned} y \quad = \quad & [0.142, 0.319, -0.286, -0.167, 0.276, 0.238, 0.217, -0.268, -0.089, \\ & -0.198, 0.287, -0.042, -0.243, 0.119, 0.309, -0.312, 0.305, 0.162]^T, \end{aligned}$$

$z \in \mathbb{R}^{17}$, $\gamma \in \mathbb{C}$, and $\alpha \in \mathbb{C}^{18}$ to be random of unit length. Let H be the homotopy defined by (3).

- For Step 2, we computed S using a diagonal homotopy [44] by computing $A \cap B$ where $A = \mathcal{V}(f - \gamma z) \times \mathbb{C}^{18}$ and

  $$B = \mathcal{V}(\lambda_0(x - y) + \lambda_1 \nabla f_1(x)^T + \cdots + \lambda_{17} \nabla f_{17}(x)^T, \alpha_0\lambda_0 + \cdots + \alpha_{17}\lambda_{17} - 1).$$

  Since $\mathcal{V}(f - \gamma z)$ is a curve of degree 480 and $B_{\mathcal{L}} = B \cap \left(\mathcal{L} \times \mathbb{C}^{18}\right)$, where $\mathcal{L}$ is a random line in $\mathbb{C}^{18}$, consists of 13 points, the diagonal homotopy required tracking $480 \cdot 13 = 6240$ paths, which yielded the 1536 points in S. A witness set for $\mathcal{V}(f - \gamma z)$ was computed using regeneration [25] and $B_{\mathcal{L}}$ was computed using a standard 2-homogeneous homotopy. Overall, this computation took 5.5 minutes in parallel.

- For Step 3, only 1440 of the 1536 paths converged and $\pi(E) = E_1$. This computation took 18.5 minutes in serial (26 seconds in parallel) and found that the set $E_1 \cap \mathbb{R}^N$ consists of 283 distinct points.

- For Step 4, the homotopy membership test found that $R = V \cap E_1 \cap \mathbb{R}^N$ consists of 24 points, which took 80 seconds in serial.

The set $R \setminus \mathrm{Sing}(f)$ consists of 16 points and meets C_i for $i = 1, \ldots, 6$. This yields that $C_i \cap \mathbb{R}^{18}$ is also one dimensional for $i = 1, \ldots, 6$. Additionally, two points of R lie in $C_7 \cap C_8$, one of which is presented in Figure 3 of [52]. Each of the other six points of R, which arose from 30 homotopy paths in Step 4, lies in the intersection of V with some higher-dimensional components of $\mathcal{V}(f)$.

4 Conclusion

Infinitesimal deformations are widely used in real algebraic geometric algorithms. By utilizing homotopy continuation to model the deformation, we have demonstrated that one can obtain an algorithm for computing a finite set of real roots of a polynomial system containing a point on each connected component. In particular, this algorithm computes a finite superset of the isolated roots over the real numbers. This is similar to basic homotopy continuation in that one computes a finite superset of the isolated roots over the complex numbers. The isolated complex roots can be identified by, for example, using the local dimension test of [6], but a similar test currently does not exist over the real numbers. Nonetheless, since many of the algorithms in numerical algebraic geometry depend only on the ability to compute a superset of the isolated roots, we will investigate what other computations can be performed in numerical *real* algebraic geometry building from the algorithm presented here.

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