A counter example to an ideal membership test

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Abstract

An ideal membership test is presented by A. Leykin in *Numerical primary decomposition* using dual bases. A counter example is presented to demonstrate that the proposed test is invalid.
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1 Introduction

The proposed ideal membership test of [4] is based on the classical dual space ideal membership test, described in Section 2, at a generic point on each primary component of the ideal. This proposed test, described in Section 3, states that one may truncate the ideal membership test based on the degree of the polynomial being tested. A counter example to this test is presented in Section 4.

2 Dual space membership test

Following standard notation, e.g., [1, 3], for $\alpha \in (\mathbb{Z}_{\geq 0})^N$ and $z \in \mathbb{C}^N$, define $\partial_\alpha[z] : \mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}$ as

$$\partial_\alpha[z](f) = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(z)$$

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where $\alpha! = \alpha_1! \cdots \alpha_N!$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$. When the context is clear, ∂_α and ∂_{x^α} mean $\partial_\alpha[z]$. Let

$$D_z = \operatorname{span}_{\mathbb{C}} \left\{ \partial_\alpha[z] \ : \ \alpha \in (\mathbb{Z}_{\geq 0})^N \right\}$$

be the space of the differential functionals at z. For each $k \geq 0$, let

$$D_z^k = \operatorname{span}_{\mathbb{C}} \left\{ \partial_\alpha[z] \ : \ \alpha \in (\mathbb{C}_{\geq 0})^N, |\alpha| \leq k \right\}$$

be the space of differential functionals at z of order at most k. For an ideal I on $\mathbb{C}[x_1, \ldots, x_N]$, let

$$D_z(I) = \{ \partial \in D_z \mid \partial(f) = 0 \text{ for every } f \in I \}$$

be the *dual space of I at z*. For each $k \geq 0$, let $D_z^k(I) = D_z(I) \cap D_z^k$ be the *k^{th} order dual space of I at z*. The set $D_z(I)$ is a $\mathbb{C}$-vector space and a basis is called a *dual basis of I at z*.

Let I be an ideal and let $V(I) = \{x \mid f(x) = 0, \forall f \in I\}$. The ideal I has a primary decomposition of the form $I = I_1 \cap \cdots \cap I_M$ where I_j is a primary ideal (see [2] for more details). The following theorem, stated in the language of [4], is the classical dual space membership test using a primary decomposition.

Theorem 1. *Define $R = \mathbb{C}[x_1, \ldots, x_N]$ and let $I \subset R$ be an ideal such that $I = I_1 \cap \cdots \cap I_M$ is a primary decomposition and $g \in R$. Then, $g \in I$ if and only if for each $j = 1, \ldots, M$, $\partial(g) = 0$ for every $\partial \in D_{z_j}(I_j)$ where z_j is a generic point in $V(I_j)$.*

3 Proposed membership test

The following is a restatement of the proposed ideal membership test of [4].

Proposed Theorem 2 (Theorem 4.6 of [4])**.** *Define $R = \mathbb{C}[x_1, \ldots, x_N]$ and let $I \subset R$ be an ideal such that $I = I_1 \cap \cdots \cap I_M$ is a primary decomposition and $g \in R$. Then, $g \in I$ if and only if for each $j = 1, \ldots, M$, $\partial(g) = 0$ for every $\partial \in D_{z_j}^{\deg g}(I_j)$ where z_j is a generic point in $V(I_j)$.*

It is important to note that the only difference between Theorem 1 and Proposed Theorem 2 (Theorem 4.6 of [4]) is that Theorem 1 uses the whole dual space while Proposed Theorem 2 uses only the dual space of order $\deg g$. That is, since the whole dual space can be infinite dimensional, this difference turns the proposed ideal membership test of [4] into a finite dimensional computation, namely Algorithm 4.7 of [4]. Unfortunately, as the example presented in Section 4 demonstrates, Theorem 4.6 and Algorithm 4.7 of [4] are invalid since truncating the dual space at order $\deg g$ is not sufficient for determining ideal membership in general.

4 Counter example

Let $I = \langle y - x^2, y^2 \rangle \subset \mathbb{C}[x, y]$ and $g(x, y) = y$. Clearly, $\deg g = 1$ and $g \notin I$. Since $V(I) = \{(0, 0)\}$, one needs to compute the dual space at $z = (0, 0)$. It is easy to verify that

$$D_z(I) = \operatorname{span}_{\mathbb{C}} \{\partial_1, \partial_x, \partial_y + \partial_{x^2}, \partial_{xy} + \partial_{x^3}\}.$$

Since $(\partial_y + \partial_{x^2})(g) = 1 \neq 0$, Theorem 1 concludes that $g \notin I$. However, since $D_z^1(I) = \operatorname{span}_{\mathbb{C}} \{\partial_1, \partial_x\}$ and $\partial_1(g) = \partial_x(g) = 0$, Proposed Theorem 2 (Theorem 4.6 of [4]) incorrectly concludes that $g \in I$.

References

[1] D.J. Bates, J.D. Hauenstein, C. Peterson, and A.J. Sommese. A numerical local dimension test for points on the solution set of a system of polynomial equations. *SIAM J. Numer. Anal.*, 47(5), 3608–3623, 2009.

[2] D. Cox, J. Little and D. O'Shea. *Ideals, varieties, and algorithms*, Second Edition. Undergraduate Texts in Mathematics, Springer, New York, 1996.

[3] B. Dayton and Z. Zeng. Computing the multiplicity structure in solving polynomial systems. *ISSAC 2005*, 116–123, *ACM*, New York, 2005.

[4] A. Leykin. Numerical primary decomposition. *ISSAC 2008*, 165–172, *ACM*, New York, 2008.