SOME INTERESTING BIRATIONAL MORPHISMS OF SMOOTH AFFINE QUADRIC 3-FOLDS

CINZIA BISI, JONATHAN D. HAUENSTEIN, AND TUYEN TRUNG TRUONG

To the late Nessim Sibony

ABSTRACT. We study a family of birational maps of smooth affine quadric 3-folds $x_1x_4 - x_2x_3 =$ constant, over $\mathbb{C}$, which seems to have some (among many others) interesting/unexpected characters: a) they are cohomologically hyperbolic, b) their second dynamical degree is an algebraic number but not an algebraic integer, and c) the logarithmic growth of their periodic points is strictly smaller than their algebraic entropy. These maps are restrictions of a polynomial map on $\mathbb{C}^4$ preserving each of the quadrics. The study in this paper is a mixture of rigorous and experimental ones, where for the experimental study we rely on the Bertini which is a reliable and fast software for expensive numerical calculations in complex algebraic geometry.

1. INTRODUCTION

A main theme in Complex Dynamics is that of equidistribution for periodic points. Roughly speaking, it is expected that if a rational map (more generally, meromorphic map) $f : X \dashrightarrow X$ of a complex projective manifold (more generally, compact Kähler manifold) satisfies a certain type of hyperbolicity, then it has an equilibrium measure with no mass on proper Zariski subsets, and to which the average of Dirac measures on hyperbolic periodic points converges.

To make precise the above statement, we first recall the definition of dynamical degrees of a map. Let X be a compact Kähler manifold of dimension m, and $f : X \dashrightarrow X$ a dominant meromorphic map (i.e.: there is a proper Zariski subset $I(f) \subset X$ - called the indeterminacy set of f - so that f is given as a holomorphic map $f : X\backslash I(f) \to X$ and the image of $X\backslash I(f)$ is dense in X). Let ω be the cohomological class of a Kähler form on X. For a number $0 \leq j \leq m$, the following limit exists and is independent of the choice of the Kähler form (see [45] when $X =$ a projective space, [27] and [26] for the general case)

$$\lim_{n\to\infty} ||(f^n)^*(\omega^j)||^{1/n}_{H^{2j}(X,\mathbb{C})}.$$

Here, $f^n = f\circ f\circ \ldots \circ f$ (n times) is the n-th iterate of f, and $||.||$ is any norm on the finite dimensional vector space $H^{2j}(X,\mathbb{C})$. We call the above limit the j-th dynamical degree of f, and denote by $\lambda_j(f)$. In the case X is a complex projective manifold, one can choose ω to be the class of an ample divisor and replace $H^{2j}(X,\mathbb{C})$ by $N^j_{\mathbb{R}}(X)$ the space of algebraic cycles of codimension j (with real

Date: August 30, 2022.

coefficients) modulo the numerical equivalence. $\lambda_0(f)$ is always 1, while $\lambda_m(f)$ is the topological degree of f (and hence is 1 when f is bimeromorphic). It is known that $\lambda_j(f)$'s are bimeromorphic invariants ([27] and [26]), that is if $\pi : Y \dashrightarrow X$ is a bimeromorphic map, and f_Y is the lifting of f to Y (i.e. $f_Y = \pi^{-1} \circ f \circ \pi$), then $\lambda_j(f_Y) = \lambda_j(f)$ for all j. There are also arithmetic analogs of these dynamical degrees, where the above mentioned results are largely unknown, and there are many challenging conjectures around (see [40][18]). Dynamical degrees can also be defined for maps on fields of positive characteristics [48][16], and the version for correspondence in [48] thus can be used to provide a generalization of Weil's Riemann hypothesis [49][38][37]. Dynamical degrees provide a useful way to compute topological entropy (a very fundamental dynamical invariant) of holomorphic selfmaps of compact Kähler manifolds [34][51]. Inspired by this fact, for a general meromorphic map, the logarithm of the maximum of its dynamical degrees is named its **algebraic entropy** in [9], and this is widely practiced in the literature. Moreover, they provide an upper bound for the topological entropy of dominant meromorphic maps of compact Kähler manifolds [27][26], and the same upper bound holds in the Berkovich space setting [30][31]. Besides being of interest in pure mathematics, birational maps appear naturally in some physical models (in lattice statistical mechanics), and their dynamical degrees are an indication of the complexity of these models, see e.g. [10][9][8][2][3][47].

A dominant meromorphic map $f : X \dashrightarrow X$ is cohomologically hyperbolic if there is one index j so that $\lambda_j(f) > \max_{i \neq j} \lambda_i(f)$. Cohomological hyperbolicity is a cohomological version of the well known notion of hyperbolic dynamics in differentiable dynamical systems, where many results around periodic points, equilibrium measures and topological entropy are known, see a survey in [36]. The rough idea is that since algebraic dynamical systems are more rigid than smooth dynamical systems, hyperbolicity of an algebraic dynamical system may be detected by the easier invariants on cohomology groups.

With this preparation, an explicit statement of a major folklore conjecture, which attracts a lot of attention and work, is the following:

Conjecture 1.1 (Folklore Conjecture). *Let X be a compact Kähler manifold of dimension m, and $f : X \dashrightarrow X$ a dominant meromorphic map. Assume that f is cohomologically hyperbolic. Then there is a birational map $\pi : Y \dashrightarrow X$ from a compact manifold Y so that for the lifting f_Y of f to Y, the following are true:*

1) There is a probability measure μ on Y with no mass on proper analytic subsets (hence, one can push it forward by f_Y) which is invariant by f, i.e. $(f_Y)_*\mu = \mu$.

2) Let $HP_n(f_Y)$ be the set of hyperbolic periodic points of period n of f_Y. Then:

a) The exponential growth of $\sharp HP_n(f_Y)$ is $\lambda(f)$, where $\lambda(f) = \max_j \lambda_j(f)$. This means that

$$\lim_{n\to\infty} \frac{\log \sharp HP_n(f_Y)}{n} = \log \lambda(f).$$

b) The hyperbolic periodic points of f equidistribute to μ. That is,

$$\lim_{n\to\infty}\frac{1}{\sharp HP_n(f_Y)}\sum_{x\in HP_n(f_Y)}\delta_x=\mu.$$

Here, δ_x is the Dirac measure at x.

c) The number of other isolated periodic points of f of order n is negligible compared to that of $\sharp HP_n(f_Y)$.

A stronger version (which also involves the topological entropy and Lyapunov exponents) of this conjecture was stated in [35]. A few representatives from the large known results in the literature resolving Conjecture 1.1 in the affirmative are: dimension 1 (see [14][42][32]); Hénon maps in dimension 2 (see [4][5]); automorphisms of K3 surfaces (see [15]); birational maps of surfaces satisfying a certain condition on Green currents (see [28], which relies on a precise estimate for the number of isolated periodic points: the lower bound is obtained therein by using laminar currents, and the upper bound is later provided by [39][21]) a large class of maps of surfaces satisfying an energy condition (see [19]); meromorphic maps of compact Kähler manifolds for which $\lambda_m > \lambda_{m-1}$ (see [22]); and analogs of Hénon-maps in higher dimensions (see [25]). For a comprehensive survey, see [24], where other topics of equidistribution - besides that of periodic points - are also discussed. Despite all of these partial results and many efforts, Conjecture 1.1 is still largely open.

There are some indications in the literature that Conjecture 1.1 may be too strong to be true. For example, one related conjecture is that $\lambda_1(f)$ is an algebraic number [8]. However, recent work [7][6] shows the existence of maps (can be chosen to be birational maps of $\mathbb{P}^d$ where $d\geq 3$) for which $\lambda_1(f)$ is a transcendental number, and hence the mentioned related conjecture does not hold. On the other hand, it is not known if the mentioned maps in [7][6] provide counter-examples to Conjecture 1.1, because these papers treat only the first dynamical degrees and concern neither periodic points nor equilibrium measures.

It is noteworthy that for the various maps in the literature where dynamical degrees can be actually computed (for a birational maps of surfaces a general procedure using point blowups has been given in [20], but besides that one must in general treat case by case), the dynamical degrees are either **algebraic integers** - i.e. roots of a polynomial $p(t)$ whose coefficients are integers and the leading coefficient is 1 - or **transcendental numbers** (i.e. not a root of any polynomial with integer coefficients). For some special maps, there are even particular speculations about that the dynamical degrees are algebraic integers of special types. Therefore, it is natural to ask what is the situation for the numbers in between these two types, that is algebraic numbers which are not algebraic integers. For example, recall that a real number λ is a Perron number if it is an algebraic integer and any of its Galois conjugate μ satisfies $|\mu|\leq\lambda$. There is the following conjecture [12][17], concerning the very actively studied polynomial maps of affine spaces:

Conjecture 1.2. *If $f : \mathbb{C}^d \to \mathbb{C}^d$ is a polynomial map, then the dynamical degrees (of the extension of f to $\mathbb{P}^d$) are all Perron numbers (in particular, algebraic integers).*

The above conjecture holds in dimension 2 (see [33][29]) and some automorphisms in $\mathbb{C}^3$ (see [43]), and some special automorphisms in higher dimensions ([13][12] and an unpublished result by Mattias Jonsson cited in these papers). For proper polynomials of $\mathbb{C}^m$ so that $\lambda_1^2 > \lambda_2$, [17] shows that $\lambda_1(f)$ is an algebraic number of degree at most m.

In the case where the dynamical degrees are algebraic integers, there are three common approaches, which involves establishing a linear recurrence between the degrees of the iterates. The first one (for the first dynamical degree λ_1) is to observe directly a linear recurrence between the degrees of iterates of f. The second one is to construct a birational (or bimeromorphic) map $\pi : Y \to X$ for which the lifting map f_Y satisfies a so-called algebraic stability under which $\lambda_j(f)$ is the spectral radius of the pullback map $f_Y^* : H^{j,j}(Y) \to H^{j,j}(Y)$. (A generalization of this is the recent work [17] which shows that the pullback map on an associated Banach space $RZ(Y)$ is stable. However, unlike on the finite dimensional cohomology groups, there is no guarantee that the eigenvalues on $RZ(Y)$ are algebraic integers.) Finally, the third one is to use specialities of toric varieties and toric maps. In the case where the dynamical degree is transcendental, the method in [7][6] is to use Diophantine approximation.

The purpose of this paper is to present a family of birational maps of smooth quadratic 3-folds which are a candidate counter-example for Conjecture 1.1. At the same time, these birational maps come from polynomial maps on $\mathbb{C}^4$ whose second and third dynamical degrees seem to be an algebraic number but not an algebraic integer (and hence, can also be a counter-example for Conjecture 1.2). As such, the mentioned methods in the last part of the previous paragraph cannot be used to compute the conjectured second dynamical degree, and new ideas have to be developed for this task.

A disadvantage of the statement of Conjecture 1.1 is that some parts of it depend on an unspecified birational map $\pi : Y \dashrightarrow X$ and an unspecified equilibrium measure μ. As will be seen in Section 2, the following conjecture is a consequence of Conjecture 1.1, and concerns only the exponential growth of the set of isolated periodic points (which in theory can be explicitly found) and is independent of the birational model of a given map.

Conjecture 1.3. *Let X be a smooth complex projective variety of dimension d, and let $f : X \dashrightarrow X$ be a dominant rational map. Assume that f is cohomologically hyperbolic. Define $\lambda(f) = \max_j \lambda_j(f)$ and $IsoPer_n(f)$ the set of isolated periodic points of period n of f (multiplicities accounted). Let $Z \subset X$ be any Zariski open dense set. Then*

$$\limsup_{n\to\infty} \frac{\log \sharp(IsoPer_n(f) \cap Z)}{n} = \log \lambda(f).$$

Note that one part of the above conjecture is known [21] (valid in the Kähler setting, and the proof therein uses the theory of tangent currents developed in [23]): we always have

$$\limsup_{n\to\infty} \frac{\log \sharp(IsoPer_n(f)\cap Z)}{n} \leq \log \lambda(f).$$

(On the other hand, on differentiable dynamical systems, there are no such upper bounds on the number of periodic points, see [41].)

Hence, to disprove Conjecture 1.1 it suffices to display a counter-example to Conjecture 1.3. We next present our candidate counter-examples for Conjectures 1.3 (and hence 1.1) and 1.2, which are strikingly simple. We start with a polynomial $F:\mathbb{C}^4\rightarrow \mathbb{C}^4$ given by:

$$F(x_1,x_2,x_3,x_4)=(x_2,-x_4,x_1-x_1x_2^2,-x_3+x_1x_2x_4).$$

This is a birational map, with inverse

$$F^{-1}=(\frac{-x_3}{-1+x_1^2},x_1,\frac{x_1x_2x_3+x_4-x_1^2x_4}{-1+x_1^2},-x_2).$$

Remark 1.4. The above map F is an element of a more general family:

$$G(x_1,x_2,x_3,x_4)=(x_2,-x_4,x_1-x_2P(x_1,x_2,x_3,x_4),-x_3+x_4P(x_1,x_2,x_3,x_4)). \tag{1}$$

Here $P(x_1,x_2,x_3,x_4)$ is a polynomial (or more generally a rational function) of the form

$$P(x_1,x_2,x_3,x_4)=x_1Q_1(x_2,x_4)+x_3Q_3(x_2,x_4)+R(x_2,x_4).$$

Some special automorphisms of $\mathbb{C}^4$ in the family G have been studied in [11].

Let $\phi:\mathbb{C}^4\rightarrow\mathbb{C}$ be the map $\phi(x_1,x_2,x_3,x_4)=x_1x_4-x_2x_3$. It can be checked that the fibres of ϕ are invariant by f, that is $\phi=\phi\circ F$. In other words, via ϕ, F is semi-conjugate to the identity map on $\mathbb{C}$. For a generic $c\in\mathbb{C}$, let $Z_c=\phi^{-1}(c)=\{x_1x_4-x_2x_3=c\}\subset\mathbb{C}^4$ be the fibre of ϕ over c. Let $X_c=$ the closure in $\mathbb{P}^4$ of Z_c (hence, in homogeneous coordinates $[x_1:x_2:x_3:x_4:z]$ of $\mathbb{P}^4$, X_c is given by $x_1x_4-x_2x_3=cz^2$). Then both X_c and Z_c are smooth, invariant by F, and Z_c is a Zariski open dense subset of X_c. The remaining of this paper is to study the following conjecture for the map $f_c=F|_{Z_c}:Z_c\rightarrow Z_c$. Note that since f_c is a birational map in dimension 3, we have $\lambda_0(f_c)=\lambda_1(f_c)=1$. Hence, the only non-trivial dynamical degrees of f_c are $\lambda_2(f_c),\lambda_3(f_c)$.

Conjecture 1.5. *Let f_c be the above map, for a generic value $c\in\mathbb{C}$, and $\widehat{f_c}$ its extension to X_c. Then we have:*

1) The first dynamical degree $\lambda_1(\widehat{f_c})$ is the largest root ζ_1 of the polynomial t^3-t^2-t-1, and ζ_1 is approximately 1.8393.

2) The second dynamical degree $\lambda_2(\widehat{f_c})$ is the largest root ζ_2 of the polynomial $2t^3-3(t^2-1)-4t$. This polynomial is irreducible over $\mathbb{Q}$, and hence $\lambda_2(f_c)$ is an algebraic number but not an algebraic integer. Here, ζ_2 is approximately 2.1108.

3) Moreover,

Stronger estimate:

$$\limsup_{n\to\infty}\frac{\log(\sharp IsoFix_n(f_c))}{n}\leq \log 2.108.$$

Weaker estimate:

$$\limsup_{n\to\infty}\frac{\log(\sharp IsoFix_{2n+1}(f_c))}{2n+1}\leq \log 2.108.$$

Part 1 of the above conjecture is solved in Lemma 2.4. If f_c indeed satisfies Conjecture 1.5, then it is a primitive map, see Section 2 for detail. Conjecture 1.5, if holds, also implies that Conjecture 1.2 does not hold, see Section 2. Indeed, the experimental results in Section 2 seem to indicate that the limit

$$\lim_{n\to\infty}\frac{\log(\sharp IsoFix_n(f_c))}{n}$$

exists, and moreover is contained in the interval $[\log 2.0890, \log 2.1071]$. That the polynomial $2t^3-3(t^2-1)-4t$ in part 2 of Conjecture 1.5 is irreducible over $\mathbb{Q}$ can be easily checked by many means. (For example, if it were to be reducible over $\mathbb{Q}$, then it would have at least one root in $\mathbb{Q}$, which then must be half of an integer. Using computer softwares such as Mathematica, one finds that the given polynomial has 3 real roots lying between -2.5 and 2.5, and no half integer in this interval is a root of the polynomial.)

An intriguing phenomenon seen in the experiments and utilised as an approach towards part 3 of Conjecture 1.5 can be heuristically supported by the next result.

Proposition 1.6. *Let $X\subset\mathbb{P}^{2m}$ be a smooth quadric. Let $g: X \dashrightarrow X$ be a dominant rational map. Then the Lefschetz numbers $L(g^n)$ (i.e. the intersection number between the graph of g^n and the diagonal of X) are all positive, and satisfy log concavity, i.e.*

$$L(g^{n+n'})\leq L(g^n)L(g^{n'})$$

for all $n,n'\geq 0$. In particular, the sequence $\{L(g^n)^{1/n}\}_{n=1,2,\ldots}$ is decreasing.

Similarly, for every $0\leq j\leq 2m-1$, the degree sequence $\{||(g^n)^*|_{H^{2j}(X)}||\}_{n=1,2,\ldots}$ (where $||.||$ is a given norm on $H^{2j}(X)$) satisfies the log concavity.

The remaining of this paper is as follows. In Section 2, we will explain why Conjecture 1.3 is a consequence of Conjecture 1.1, and why an affirmative answer to Conjecture 1.5 provides a counter-example to Conjectures 1.1, 1.3 and 1.2. We then present various theoretical and experimental results concerning Conjecture 1.5. Based on these results, we present an approach towards resolving Conjecture 1.5 in the affirmative. We then stipulate some comments/suggestions on the direction of equidistribution for periodic points for rational and meromorphic maps.

Here is a brief summary of the idea for using experimental results reported later in this paper to give support to the validity of Conjecture 1.5. For to check part 2 of Conjecture 1.5, we compare the degree sequences of the iterates of F and some sequences related to the linear recurrence $w_n = 3/2\cdot(w_{n-1} - w_{n-3}) + 2w_{n-2}$, whose exponential growth is ζ_2. For to check the Weaker estimate in part 3 of Conjecture 1.5, we test if the sequence $(\sharp IsoFix_{2n+1}(f_c))^{1/(2n+1)}$ is **decreasing**, which is related to the log concavity phenomenon in Proposition 1.6, and to observe that for $n = 4, 5$ the number $(\sharp IsoFix_{2n+1}(f_c))^{1/(2n+1)}$ is ≤ 2.108. For to check the Stronger estimate in part 3 of Conjecture 1.5, we compare the two sequences $(\sharp IsoFix_{2n+1}(f_c))^{1/(2n+1)}$ and $(\sharp IsoFix_{2n}(f_c))^{1/(2n)}$ and use the Weaker estimate.

Acknowledgements. C. B. and T. T. T. thank Research in pair grants and the hospitality from EPFL (Centre Bernoulli) and the University of Trento and FBK Foundation, as well as travel grants from Trond Mohn Foundation, for their supports. They are also supported partly by Young Research talents grant 300814 from Research Council of Norway. J. D. H. was supported partly by National Science Foundation grant CCF 1812746.

2. Results

In this section we present our results around the map f and Conjecture 1.5. These results consist of both rigorously theoretically proven ones and experimental ones. Based on these results, we propose a road map toward resolving 1.5 in the affirmative.

Let $F: \mathbb{C}^4 \to \mathbb{C}^4$, X_c, Z_c and $f_c: Z_c \to Z_c$ be defined as in the introductive section. Throughout this section, we use the following notations:

$\widehat{F}$ = the extension of F to the projective space $\mathbb{P}^4$;

$\widehat{f_c}$ = the extension of f_c to X_c;

$d_n^{(1)}$ = the first degree of F^n (the n-th iterate of F): it is defined as the degree of the inverse image by $\widehat{F}^n$ of a generic linear 3-dimensional subspace H in $\mathbb{P}^4$;

$d_n^{(2)}$ = the second degree of F^n: it is defined as the degree of the inverse image by $\widehat{F}^n$ of a generic linear 2-dimensional subspace H^2 in $\mathbb{P}^4$;

$d_n^{(3)}$ = the third degree of F^n (the n-th iterate of F): it is defined as the degree of the inverse image by $\widehat{F}^n$ of a generic linear 1-dimensional subspace H^3 in $\mathbb{P}^4$ (it is indeed the same as the first degree of the inverse map F^{-n});

b_n is the sequence satsifying the linear recurrence $b_n = 3/2 \cdot (b_{n-1} - b_{n-3}) + 2b_{n-2}$, with the initial terms $b_1 = 3$, $b_2 = 7$ and $b_3 = 17$;

c_n is the sequence satsifying the linear recurrence $b_n = 3/2 \cdot (b_{n-1} - b_{n-3}) + 2b_{n-2}$, with the initial terms $b_1 = 5$, $b_2 = 9$ and $b_3 = 25$;

$IsoFix_n$ = the set of isolated fixed points of f_c^n;

$C \subset Z_c$: the curve defined by the ideal $< x_2 - x_1^2 x_2 - x_3, x_1 + x_4, x_1x_4 - x_2x_3 - c >$.

$D_1 \subset Z_c$: the curve with 2 components defined by the ideals $< x_2 - x_1^2 x_2 - x_3, x_1 + x_4, x_1x_4 - x_2x_3 - c >$ and $< -x_2 + x_1^2 x_2 - x_3, x_1 - x_4, x_1x_4 - x_2x_3 - c >$;

$D_2 \subset Z_c$: the curve with 2 components defined by the ideals $< x_2, x_3, x_1x_4 - x_2x_3 - c >$ and $< x_1, x_4, x_1x_4 - x_2x_3 - c >$.

Here are some useful comments concerning the above notations. First, by the definition of dynamical degrees, we have

$$\begin{aligned} \lim_{n\to\infty} [d_n^{(1)}]^{1/n} &= \lambda_1(\widehat{F}), \\ \lim_{n\to\infty} [d_n^{(2)}]^{1/n} &= \lambda_2(\widehat{F}), \\ \lim_{n\to\infty} [d_n^{(3)}]^{1/n} &= \lambda_3(\widehat{F}). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} \frac{d_{n+1}^{(1)}}{d_n^{(1)}} &= \lambda_1(\widehat{F}), \\ \lim_{n\to\infty} \frac{d_{n+1}^{(2)}}{d_n^{(2)}} &= \lambda_2(\widehat{F}), \\ \lim_{n\to\infty} \frac{d_{n+1}^{(3)}}{d_n^{(3)}} &= \lambda_3(\widehat{F}). \end{aligned}$$

Since $\widehat{F}$ is birational, we have $\lambda_0(\widehat{F}) = \lambda_4(\widehat{F}) = 1$. Similarly, we have $\lambda_0(f_c) = \lambda_3(f_c) = 1$. The relation between the dynamical degrees of $\widehat{F}$ and those of $\widehat{f}_c$ will be given in the next Subsection.

From our experiments, to be presented later in this section, the fixed point set of f_c^n seems to have the following structure:

The fixed point set of f_c^{4n+2} is the union of the curve C and $IsoFix_{4n+2}(f_c)$;

The fixed point set of f_c^{4n} is the union of the curve D (which is the union of D_1 and D_2 - of different multiplicities) and $IsoFix_{4n}(f_c)$;

The fixed point set of f_c^{2n+1} is $IsoFix_{2n+1}(f_c)$;

All isolated periodic points of f_c are hyperbolic, i.e. $IsoFix_n(f_c) = HP_n(f_c)$.

2.1. Theoretical results.

We first start with some relations between Conjectures 1.1, 1.3, 1.2 and 1.5.

Lemma 2.1. *1) If Conjecture 1.1 holds, then Conjecture 1.3 holds.*

2) Assume that parts 1 and 2 of Conjecture 1.5 hold.

a) If the Stronger estimate of part 3 of Conjecture 1.5 holds, then Conjecture 1.3 does not hold.

b) If the Weaker estimate of part 3 of Conjecture 1.5 holds, then Conjecture 1.1 does not hold.

3) If part 2 of Conjecture 1.5 holds, then Conjecture 1.2 does not hold.

Proof. 1) Assume that Conjecture 1.1 holds. Let $f : X \dashrightarrow X$ be a dominant rational map which is cohomologically hyperbolic. Let $\pi : Y \dashrightarrow X$ be the birational map given by Conjecture 1.1.

Let $Z \subset X$ be a Zariski open dense set. Since $\pi : Y \dashrightarrow X$ is birational, there is a Zariski open dense set $U \subset Z$ and a Zariski open dense set $V \subset Y$ so that π induces an isomorphism between V and U.

By Conjecture 1.1, $HP_n(Y)$ equidistributes to μ and where μ has no mass on proper analytic subsets. Since $Y \backslash V$ is a proper analytic subset of Y, it follows that $HP_n(f_Y) \cap V$ equidistributes to μ as well, and the exponential growth of $\sharp[HP_n(f_Y) \cap V]$ is also $\lambda(f_Y)$. Since $\pi : V \to U$ is an isomorphism, it follows that $HP_n(f_Y) \cap V = HP_n(f) \cap U$. Since $\lambda(f_Y) = \lambda(f)$ by the birational invariance of dynamical degrees, we obtain:

$$\lim_{n\to\infty} \frac{\log \sharp[HP_n(f) \cap U]}{n} = \log \lambda(f).$$

Since $HP_n(f) \cap U \subset IsoFix_n(f) \cap Z$, we get:

$$\liminf_{n\to\infty} \frac{\log \sharp[IsoFix_n(f) \cap Z]}{n} \geq \log \lambda(f).$$

On the other hand, by [21], we have

$$\limsup_{n\to\infty} \frac{\log \sharp IsoFix_n(f)}{n} \leq \log \lambda(f).$$

Combining the above two inequalities, we have finally

$$\lim_{n\to\infty} \frac{\log \sharp[IsoFix_n(f) \cap Z]}{n} = \log \lambda(f).$$

This means that Conjecture 1.3 holds.

2) We prove for part a) only, the proof of part b) is similar. Assume that parts 1 and 2 of Conjecture 1.5 hold, and also that the Stronger estimate of part 3 of Conjecture 1.5 holds. Since X_c is of dimension 3, and the dynamical degrees of $\widehat{f_c}$ are: $\lambda_0(\widehat{f_c}) = 1$, $\lambda_1(\widehat{f_c}) = \zeta_1 \sim 1.8393$, $\lambda_2(\widehat{f_c}) = \zeta_2 \sim 2.1108$, and $\lambda_3(\widehat{f_c}) = 1$, the map $\widehat{f_c}$ is cohomologically hyperbolic. We also have that Z_c is a Zariski open dense set of X_c, and $\widehat{f_c}|_{Z_c} = f_c$. Hence, $IsoFix_n(\widehat{f_c}) \cap Z_c = IsoFix_n(f_c)$. By Conjecture 1.5 we have

$$\limsup_{n\to\infty} \frac{\log \sharp IsoFix_n(f_c)}{n} \leq \log 2.108 < \log \zeta_2 = \log \lambda(f_c).$$

This contradicts Conjecture 1.3.

3) Assume that part 2 of Conjecture 1.5 holds. Then we have $\lambda_2(\widehat{f_c}) = \zeta_2$ is an algebraic number, but not an algebraic integer. By Lemma 2.3, we have $\lambda_3(\widehat{F}) = \lambda_2(\widehat{f_c})$, hence $\lambda_3(\widehat{F})$ is not an algebraic integer. In this case, also $\lambda_2(\widehat{F}) = \zeta_2$ is not an algebraic integer. Thus the polynomial map $F : \mathbb{C}^4 \to \mathbb{C}^4$ is a counter-example to Conjecture 1.2. $\square$

We recall that [52] a dominant rational map $f : X \dashrightarrow X$ is primitive, if there **do not exist** a variety W of dimension $1 \leq \dim(W) \leq \dim(X) - 1$, a dominant rational map $\pi : X \dashrightarrow W$, and a dominant rational map $g : W \dashrightarrow W$ so that $\pi \circ f = g \circ \pi$. Primitive maps can be viewed as building blocks from which all maps can be constructed. It is clear from the definition that being primitive is a birational invariant. We have the following result.

Lemma 2.2. *Assume that parts 1 and 2 of Conjecture 1.5 holds. Then the map f_c is primitive.*

Proof. Let $\widehat{f}_c^{-1}$ be the inverse of $\widehat{f}_c$. If Conjecture 1.5 holds, then

$$\lambda_1(\widehat{f}_c^{-1}) = \lambda_2(\widehat{f}_c) > \lambda_1(\widehat{f}_c) = \lambda_2(\widehat{f}_c^{-1}).$$

This inequality implies that $\widehat{f}_c^{-1}$ is primitive, [44]. Hence f_c^{-1} (the inverse of f_c) is also primitive.

From this, we will show that f_c is primitive. Assume by contradiction that f_c is not primitive. Then there are $\pi : X \dashrightarrow W$ and $g : W \dashrightarrow W$ dominant rational maps (with $1 \leq \dim(W) \leq 2$) so that $\pi \circ f_c = g \circ \pi$. Since f_c is a birational map, it is easy to see that g is also a birational map. Then, it follows that $\pi \circ f_c^{-1} = g^{-1} \circ \pi$, which contradicts the fact that f_c^{-1} is primitive. □

We next relate dynamical degrees of $\widehat{F}$ and those of $\widehat{f}_c$.

Lemma 2.3. *We have*

1) $\lambda_1(\widehat{f}_c) = \lambda_1(\widehat{F})$, *and* $\lambda_2(\widehat{f}_c) = \lambda_3(\widehat{F})$.

2) $\lambda_2(\widehat{F}) = \max\{\lambda_1(\widehat{F}), \lambda_3(\widehat{F})\}$.

Proof. We know from the introductive section that F is semi-conjugated to the identity map $id_{\mathbb{C}}$ on the curve $\mathbb{C}$, via the map $\phi : \mathbb{C}^4 \to \mathbb{C}$ with $\phi(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$. The dynamical degrees of (the extension to $\mathbb{P}^1$ of) $id_{\mathbb{C}}$ are $\lambda_0(id_{\mathbb{C}}) = \lambda_1(id_{\mathbb{C}}) = 1$.

Let $\lambda_j(\widehat{F}|\phi)$ $(j = 0, 1, 2, 3)$ be the relative dynamical degrees w.r.t. ϕ of $\widehat{F}$ (CITE). Since F preserves the fibres of ϕ, it follows that $\lambda_j(\widehat{F}|\phi) = \lambda_j(\widehat{f}_c)$.

By CITE, we have

$$\begin{aligned}
\lambda_1(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_1(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_0(\widehat{f}_c)\} = \lambda_1(\widehat{f}_c), \\
\lambda_3(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_3(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_2(\widehat{f}_c)\} = \lambda_2(\widehat{f}_c), \\
\lambda_2(\widehat{F}) &= \max\{\lambda_0(id_{\mathbb{C}})\lambda_2(\widehat{f}_c), \lambda_1(id_{\mathbb{C}})\lambda_1(\widehat{f}_c)\} \\
&= \max\{\lambda_1(\widehat{f}_c), \lambda_2(\widehat{f}_c)\} = \max\{\lambda_1(\widehat{F}), \lambda_2(\widehat{F})\}.
\end{aligned}$$

□

Next, we compute $\lambda_1(f_c)$.

Lemma 2.4. $\lambda_1(\widehat{f}_c) = \zeta_1$, *the largest root of the polynomial* $t^3 - t^2 - t - 1$.

Proof. By Lemma 2.3, we have $\lambda_1(\widehat{f_c}) = \lambda_1(\widehat{F})$. Thus we only need to show that $\lambda_1(\widehat{F}) = \zeta_1$. Recall that $\lambda_1(\widehat{F}) = \lim_{n\to\infty}[d_n^{(1)}]^{1/n}$. Hence, the proof is finished if we can show that the degree sequence $d_n^{(1)}$ of F satisfies the linear recurrence:

$$d_n^{(1)} = d_{n-1}^{(1)} + d_{n-2}^{(1)} + d_{n-3}^{(1)},$$

for all n.

The leading term of $F(x_1,x_2,x_3,x_4) = ((F)_1,(F)_2,(F)_3,(F)_4)$ in terms of degree is $x_1x_2x_4$ in $(F)_4$. Moreover, $(F)_3$ has a unique leading monomial, and $x_1x_2x_4$ is the unique leading monomial in $(F)_4$. By direct calculation, we find that $F^2 = ((F^2)_1,(F^2)_2,(F^2)_3,(F^2)_4)$ is given by:

$$(-x_4, x_3 - x_1x_2x_4, x_2 - x_2x_4^2, -x_1 + x_1x_2^2 + x_2x_3x_4 - x_1x_2^2x_4^2).$$

Hence, again, the leading term of F^2 is contained in $(F^2)_4$, and $(F^2)_4$ has a unique leading monomial. One can directly check the same phenomenon for F^3 and F^4.

We will prove by induction that the leading term of $F^n = ((F^n)_1,(F^n)_2,(F^n)_3,(F^n)_4)$ is contained in $(F^n)_4$, and moreover $(F^n)_4$ has a unique leading monomial. In addition the concerned linear recurrence for the degree sequence holds. Assume by induction that this claim hold until $n-1$. We will show that it also holds for n. Since $F^n(x_1,x_2,x_3,x_4)$ is

$$((F^{n-1})_2, -(F^{n-1})_4, (F^{n-1})_1 - (F^{n-1})_1 \cdot {(F^{n-1})_2}^2, -(F^{n-1})_3 + (F^{n-1})_1 \cdot (F^{n-1})_2 \cdot (F^{n-1})_4)$$

and since

$$(F^{n-1})_2 = -(F^{n-2})_4$$

and

$$(F^{n-1})_1 = (F^{n-2})_2 = -(F^{n-3})_4$$

it follows that the leading degree of F^n is that of $(F^{n-1})_4 \cdot (F^{n-2})_4 \cdot (F^{n-3})_4$ which is contained in $(F^n)_4$, and that $(F^n)_4$ has a unique leading monomial, which is the product of the unique leading monomials in $(F^{n-1})_4$, $(F^{n-2})_4$ and $(F^{n-3})_4$. From this, we have immediately the linear recurrence:

$$d_n^{(1)} = d_{n-1}^{(1)} + d_{n-2}^{(1)} + d_{n-3}^{(1)}.$$

□

Finally, we prove Proposition 1.6.

Proof of Proposition 1.6. First, we have the known fact that the cohomology groups of X come from the pullback on cohomology of the embedding $\iota : X \hookrightarrow \mathbb{P}^{2m}$. [For the convenience of the readers, we briefly recall the arguments. By the Lefschetz hyperplane theorem, for $j \leq 2m-2$, the pullback $\iota^* : H^j(\mathbb{P}^{2m}) \to H^j(X)$ is an isomorphism. For $j \geq 2m+2$ one can use Poincare duality to determine the cohomology group. It remains to show that $H^{2m-1}(X) = 0$. This can be done by computing the Euler characteristic of X, and the latter can be done by using the normal bundle sequence and Whitney sum formula.]

In particular, this means the following: $H^{2j+1}(X)=0$ for all j; if $2j$ is even and $\leq 2m-2$ then $H^{2j}(X)$ is generated by h^j where $h=\iota^*(H)$ is the pullback of a hyperplane H in $\mathbb{P}^{2m}$, while the remaining groups are computed using Poincare duality. Hence $H^*(X)$ is generated by h, and we have $h^{2m-1}=H^{2m-1}.X=2$.

Thus, if $\pi_1,\pi_2:X\times X\rightarrow X$ are the two canonical projections, then the cohomology class of the diagonal of X is

$$\{\Delta_X\}=\frac{1}{2}\sum_{j=0}^{2m-1}\pi_1^*(h^j).\pi_2^*(h^{2m-1-j}).$$

From this, we get the following formula for the Lefschetz number of a map $g:X\rightarrow X$:

$$L(g)=\sum_{j=0}^{2m-1}g^*(\frac{1}{2}h^j).h^{2m-1-j}.$$

Each of the summand in the right hand side is >0, and hence $L(g)>0$. We denote by $d_j(g):=g^*(\frac{1}{2}h^j).h^{2m-1-j}$ the j-th degree of g.

Now, we will show that for any two dominant rational maps $g_1,g_2:X\dashrightarrow X$, and any $j\leq 2m-1$ then $d_j(g_1\circ g_2)\leq d_j(g_1)d_j(g_2)$. To this end, we follow the ideas in [45][27] of using automorphisms of X to regularise positive closed currents. We recall that the quadric X is a homogeneous space, whose automorphism group is the subgroup of the linear automorphisms of $\mathbb{P}^{2m}$ preserving the quadratic form. Hence, by using a convolution process with the aid of the Haar measure of the automorphism group of X, for any positive closed (j,j) current T on X, there is a sequence of positive closed smooth (j,j) forms T_ϵ weakly converging to T so that (recall that the cohomology of X is generated by h) in cohomology $\{T_\epsilon\}=\{T\}$.

Now we recall how $d_j(g_1\circ g_2)$ can be computed. There is a Zariski open set $Z\subset X$ so that if we choose generic algebraic varieties in X: V_j of codimension j representing the cohomology class h^j and W_{2m-j} of codimension $2m-j$ representing the cohomology class h^{2m-j}, then

$$d(g_1\circ g_2)=(g_2|_Z)^{-1}(g_1^{-1}(V_j/2)).W_{2m-j}.$$

Here $g_1|_Z$ is a proper map of finite fibres, and hence the preimage of $g_1|_Z$ of any variety is again a variety of the same dimension. Moreover, we have the following property: If T_ϵ is a sequence of positive closed smooth forms weakly converging to the current of integration over $g_1^{-1}(V_j)/2$ and being of the same cohomology class as that of $g_1^{-1}(V_j)/2$ (which is proven in the above), then

$$(g_2|_Z)^{-1}(g_1^{-1}(V_j/2)).W_{2m-j}\leq\lim_{\epsilon\rightarrow 0}g_2^*(\{T_\epsilon\}).h^{2m-j}.$$

Since

$$\begin{aligned} \{T_\epsilon\} &= \{g_1^*(V_j/2)\} = g_1^*(h^j/2) \\ &= [g_1^*(h^j/2).h^{2m-j}]/2.h^j = d_j(g_1)\frac{h^j}{2}, \end{aligned}$$

we obtain

$$(g_2|_Z)^{-1}(g_1^{-1}(V_j/2)).W_{2m-j} \leq d_j(g_1)g_2^*(h^j/2).h^{2m-j} = d_j(g_1)d_j(g_2).$$

Therefore, the degree sequences are log concave. From this, we obtain

$$\begin{aligned} L(g_1 \circ g_2) &= \sum_j d_j(g_1 \circ g_2) \leq \sum_j d_j(g_1)d_j(g_2) \\ &\leq (\sum_j d_j(g_1))(\sum_j d_j(g_2)) = L(g_1)L(g_2). \end{aligned}$$

Applying for $g_1 = g^n$ and $g_2 = g^{n'}$ we obtain the log concavity $L(g^{n+n'}) \leq L(g^n)L(g^{n'})$ needed. From this log concavity property, it is well known that the sequence $n \mapsto L(g^n)^{1/n}$ is decreasing. □

Remark 2.5. One can also prove Proposition 1.6 by a purely algebraic proof, which replaces regularisation of currents by Chow's moving lemma and which is valid on any algebraically closed field, see [48].

2.2. **Experimental results.** In this subsection we calculate the fixed point sets of the iterates f_c^n, as well as the degree sequences $d_n^{(1)}$, $d_n^{(2)}$ and $d_n^{(3)}$, for as large as possible n's, to help study Conjecture 1.5. We also study how close the degree sequence $d_n^{(2)}$ and $d_n^{(3)}$ are to the linear recurrence $v_n = 3/2 \cdot (v_{n-1} - v_{n-3}) + 2v_{n-2}$ related to the polynomial $t^3 - 3/2 \cdot (t^2 - 1) - 2t$ (for which ζ_2 is the largest root). These calculations will be used in the next subsection, where we propose an approach towards solving Conjecture 1.5. Here we recall the relevant relations from Lemmas 2.3 and 2.4:

$$\begin{aligned} \lim_{n\to\infty}[d_n^{(1)}]^{1/n} &= \zeta_1 = \lambda_1(\widehat{F}) = \lambda_1(\widehat{f}_c), \\ \lim_{n\to\infty}[d_n^{(3)}]^{1/n} &= \lambda_3(\widehat{F}) = \lambda_2(\widehat{f}_c), \\ \lim_{n\to\infty}[d_n^{(2)}]^{1/n} &= \lambda_2(\widehat{F}) = \max\{\lambda_1(\widehat{F}), \lambda_3(\widehat{F})\} = \max\{\lambda_1(\widehat{f}_c), \lambda_2(\widehat{f}_c)\}. \end{aligned}$$

Formal computer algebra techniques (such as Gröbner basis routine on the softwares Mathematica and Maple) can only compute up to about $N = 5$. Hence, we need to utilize Bertini [1], which is a numerical routine - with a high level guarantee - having reliable performances, and which allows for parallel computation thus can speed up the performance. We note that the computations for finding the periodic points are more expensive and difficult than those for calculating the degree sequences.

The first table reports on the calculations for periodic points of f_c (up to period $n = 12$), as well as the exponential growth of isolated periodic points. All the isolated periodic points are hyperbolic.

N	fixed points on general fiber	$[\sharp IsoFix_N(f_c)]^{1/N}$
1	4	**4**
2	C (occurring with multiplicity 1)	0
3	10	**2.15443469003**
4	D_1 (multiplicity 1) & D_2 (multiplicity 2)	0
5	44	**2.13152551327**
6	C (multiplicity 1) AND 12 points	1.51308574942
7	186	**2.10967780991**
8	D_1 (multiplicity 1) & D_2 (multiplicity 2) AND 128 points	1.83400808641
9	820	**2.10744910267**
10	C (multiplicity 1) AND 1440 points	2.06936094886
11	3634	**2.10703309279**
12	D_1 (multiplicity 1) & D_2 (multiplicity 2) AND 6908 points	2.08903649661

The next table computes the degree sequence for the iterates F^n (up to $n = 14$), as well as their exponential growth:

N	$d_N^{(1)}$	$d_N^{(2)}$	$d_N^{(3)}$	$[d_N^{(1)}]^{1/N}$	$[d_N^{(2)}]^{1/N}$	$[d_N^{(3)}]^{1/N}$
1	3	5	3	3	5	3
2	5	9	7	2.2360679775	3	2.64575131106
3	9	25	17	2.08008382305	2.92401773821	2.57128159066
4	17	49	37	2.03054318487	2.64575131106	2.46632571456
5	31	109	79	1.98734075466	2.55555539674	2.39621299048
6	57	225	167	1.96175970274	2.46621207433	2.34667391139
7	105	477	353	1.94420174432	2.41348988334	2.3118934527
8	193	1005	745	1.93061049898	2.37285258221	2.28570160944
9	355	2117	1571	1.92025412137	2.34166378698	2.26532588341
10	653	4465	3311	1.91201510161	2.31729938473	2.24903346712
11	1201	9401	6977	1.90527844956	2.29719383004	2.23575612581
12	2209	19817	14701	1.8996910486	2.28079626154	2.22473817189
13	4063	41741	30975	1.89497551023	2.2668767672	2.2154523255
14	7473	87961	65263	1.89094202127	2.25508846088	2.2075207175

We can see from the above two tables that, as predicted by Conjecture 1.5, the exponential growth of the degree sequence is much larger than that of the isolated periodic points. From the table for the degree sequence above, we can readily check that the sequence $d_n^{(1)}$'s indeed satisfies the linear recurrence $w_n = w_{n-1}+w_{n-2}+w_{n-3}$, as proven in Lemma 2.4. If Conjecture 1.5 holds, then we must have $\lambda_2(\widehat{F}) = \lambda_3(\widehat{F})$, and a first idea towards actually showing that $\lambda_2(\widehat{F}) = \lambda_2(\widehat{F}) = \zeta_2$ is to show that both sequences $d_n^{(2)}$'s and $d_n^{(3)}$'s satisfy the linear recurrence $w_n = 3/2\cdot(w_{n-1}-w_{n-3})+2w_{n-2}$. It turns out that neither of these sequences satisfies this linear recurrence, but the next two tables show that the differences to this linear recurrence are relatively small (in comparison to the size of the concerned degree sequences).

N	$d_N^{(2)}$	$3/2\cdot(d_{N-1}^{(2)}-d_{N-3}^{(2)})+2d_{N-2}^{(2)}$	$d_N^{(2)}-[3/2\cdot(d_{N-1}^{(2)}-d_{N-3}^{(2)})+2d_{N-2}^{(2)}]$
1	5		
2	9		
3	25		
4	49	48	1
5	109	110	−1
6	225	224	1
7	477	482	−5
8	1005	1002	3
9	2117	2124	−7
10	4465	4470	−5
11	9401	9424	−23
12	19817	19856	−39
13	41741	41830	−89
14	87961	88144	−183

N	$d_N^{(3)}$	$3/2 \cdot (d_{N-1}^{(3)} - d_{N-3}^{(3)}) + 2d_{N-2}^{(3)}$	$d_N^{(3)} - [3/2 \cdot (d_{N-1}^{(3)} - d_{N-3}^{(3)}) + 2d_{N-2}^{(3)}]$
1	3		
2	7		
3	17		
4	37	35	2
5	79	79	0
6	167	167	0
7	353	353	0
8	745	745	0
9	1571	1573	-2
10	3311	3317	-6
11	6977	6991	-14
12	14701	14731	-30
13	30975	31039	-64
14	65263	65399	-136

2.3. **Further analysis & A road map towards an affirmative answer to Conjecture 1.5.** In this Subsection we further analyze the experimental findings in the previous Subsection, in connection to Conjecture 1.5. Since part 1 of Conjecture 1.5 is solved by Lemma 2.4, it remains to treat parts 2 and 3 of Conjecture 1.5.

2.3.1. *An approach towards establishing the upper bound $\lambda_2(\widehat{f_c}) \leq \zeta_2$.* As seen from before, this is equivalent to establishing that

$$\lim_{n\to\infty}[d_n^{(3)}]^{1/n} \leq \zeta_2.$$

Here is an approach to showing this. From the last 2 tables in the previous Subsection, it seems very evident that for $n \geq 5$, we should have $d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)}$. This is indeed enough to proving the desired upper bound, as seen in the next lemma.

Lemma 2.6. *Assume that for all $n \geq 5$, we have*

$$d_n^{(3)} \leq 3/2 \cdot (d_{n-1}^{(3)} - d_{n-3}^{(3)}) + 2d_{n-2}^{(3)}.$$

Then $\lambda_3(\widehat{F}) \leq \zeta_2$.

Proof. Since $\lim_{n\to\infty}[d_n^{(3)}]^{1/n} = \lambda_3(\widehat{F})$, it follows that for all $j \geq 0$ we have

$$\lim_{n\to\infty}\frac{d_{n+j}^{(3)}}{d_n^{(3)}} = \lambda_3(\widehat{F})^j.$$

Assume that $d_n^{(3)} \leq 3/2\cdot(d_{n-1}^{(3)}-d_{n-3}^{(3)})+2d_{n-2}^{(3)}$ for all $n\geq 5$. Dividing $d_{n-3}^{(3)}$ on both side of the inequality, and taking limit when $n\to\infty$, we obtain

$$\lambda_3(\widehat{F})^3 \leq 3/2\cdot(\lambda_3(\widehat{F})^2-1)+2\lambda_3(\widehat{F}).$$

Looking at the graph of the function $t\mapsto t^3-3/2\cdot(t^2-1)+2t$ (for $t\in[1,\infty)$, recalling that dynamical degrees of a map are ≥ 1), we see that the above inequality holds only if $\lambda_3(\widehat{F})\leq\zeta_2$. □

In the last 2 tables of the previous Subsection, it seems also very evident that we should have $d_n^{(2)} \leq 3/2\cdot(d_{n-1}^{(2)}-d_{n-3}^{(2)})+2d_{n-2}^{(2)}$ for all $n\geq 9$. If this is indeed true, then similarly to Lemma 2.6 we will have that $\lambda_2(\widehat{F})\leq\zeta_2$, which by Lemma 2.3 also implies that $\lambda_3(\widehat{F})\leq\zeta_2$.

Hence, one promising approach towards establishing $\lambda_2(\widehat{f_c})\leq\zeta_2$ is to answer in the affirmative the following question:

Question 1:

a) Is it true that $d_n^{(3)} \leq 3/2\cdot(d_{n-1}^{(3)}-d_{n-3}^{(3)})+2d_{n-2}^{(3)}$ for n large enough?

b) Is it true that $d_n^{(2)} \leq 3/2\cdot(d_{n-1}^{(2)}-d_{n-3}^{(2)})+2d_{n-2}^{(2)}$ for n large enough?

2.3.2. *Two approaches towards establishing the lower bound* $\lambda_2(\widehat{f_c})\geq\zeta_2$. Again, to prove that $\lambda_2(\widehat{f_c})\geq\zeta_2$ is the same as proving that $\lambda_3(\widehat{F})\geq\zeta_2$, and also is the same as proving that $\lambda_2(\widehat{F})\geq\zeta_2$.

To this end, we have two approaches. One is again to base on the last two tables in the previous Subsection, while the other is based on a new viewpoint. This new viewpoint gives even more support to that we should have $\lambda_2(\widehat{f_c})\geq\zeta_2$.

Approach 1: As we mentioned, the last two tables seem to show that while $d_n^{(3)}$ (as well as $d_n^{(2)}$) does not satisfy the linear recurrence $w_n=3/2\cdot(w_{n-1}-w_{n-3})+2w_{n-2}$, it is very close to satisfying the linear recurrence. More precisely, the difference is relatively small, in the sense that:

$$\frac{d_n^{(3)}-3/2\cdot(d_{n-1}^{(3)}-d_{n-3}^{(3)})-2d_{n-2}^{(3)}}{d_n^{(3)}}$$

is small. This prompts us to ask the following question:

Question 2:

a) Is it true that

$$\lim_{n\to\infty}\frac{d_n^{(3)}-3/2\cdot(d_{n-1}^{(3)}-d_{n-3}^{(3)})-2d_{n-2}^{(3)}}{d_n^{(3)}}=0?$$

b) Is it true that

$$\lim_{n\to\infty}\frac{d_n^{(2)}-3/2\cdot(d_{n-1}^{(2)}-d_{n-3}^{(2)})-2d_{n-2}^{(2)}}{d_n^{(2)}}=0?$$

We have the following result.

Lemma 2.7. *Assume that either part a) or part b) of Question 2 has an affirmative answer. Then* $\lambda_3(\widehat{F}) \geq \zeta_2$.

Proof. As in the proof of Lemma 2.6, from the assumption in the statement of Lemma 2.7, we obtain $\lambda_3(\widehat{F})^3 \geq 3/2 \cdot (\lambda_3(\widehat{F})^2 - 1) + 2\lambda_3(\widehat{F})$. Since the polynomial $t^3 - 3/2 \cdot (t^2 - 1) - 2t$ has 3 real roots with approximate values -1.202, 0.591 and $\zeta_2 \sim 2.1108$, while $\lambda_3(\widehat{F}) \geq 1$ by definition, we conclude that we must have $\lambda_3(\widehat{F}) \geq \zeta_2$.

□

Hence, a promising approach to establishing the lower bound $\lambda_3(\widehat{F}) \geq \zeta_2$ is to solve in the affirmative Question 2. However, Approach 2 below seems to have more evidence to support than this Approach 1.

Approach 2: In this approach, we compare the degree sequence $d_n^{(3)}$'s (respectively $d_n^{(2)}$'s) with the sequence b_n's (correspondingly c_n's). The sequence b_n's satisfies the linear recurrence $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$ and has the first 3 initial values the same as that for the sequence $d_n^{(3)}$'s. Similarly, the sequence c_n's satisfies the linear recurrence $w_n = 3/2 \cdot (w_{n-1} - w_{n-3}) + 2w_{n-2}$ and has the first 3 initial values the same as that for the sequence $d_n^{(2)}$'s. From the experimental results in the previous Subsection, we get the following two tables.

N	$d_N^{(3)} = 3^{\text{rd}}$ degree of F	$b_N = 3/2 \cdot (b_{N-1} - b_{N-3}) + 2b_{N-2}$	difference=$d_N^{(3)} - b_N$
1	3	3	0
2	7	7	0
3	17	17	0
4	37	35	2
5	79	76	3
6	167	158.5	8.5
7	353	337.25	15.75
8	745	708.875	36.125
9	1571	1500.0625	70.9375
10	3311	3161.96875	149.03125
11	6977	6679.765625	297.234375
12	14701	14093.4921875	607.5078125
13	30975	29756.81640625	1218.18359375
14	65263	62802.0560546875	2460.9439453125015

N	$d_N^{(2)} = 2^{\text{nd}}$ degree of F	$c_N = 3/2 \cdot (c_{N-1} - c_{N-3}) + 2c_{N-2}$	difference$=d_N^{(2)} - c_N$
1	5	5	0
2	9	9	0
3	25	25	0
3	49	48	1
5	109	108.5	0.5
6	225	221.25	3.75
7	477	476.875	0.125
8	1005	995.0625	9.9375
9	2117	2114.46875	2.53125
10	4465	4446.515625	18.484375
11	9401	9406.1171875	−5.1171875
12	19817	19830.50390625	−13.50390625
13	41741	41888.216796875	−147.216796875
14	87961	88384.1572265625	−423.1572265625

It seems very evident that we should have $d_n^{(3)} \geq b_n$ for all $n \geq 1$. This prompts us the following question.

Question 3: Is it true that we have $d_n^{(3)} \geq b_n$ for all $n \geq 1$? (In the proof, we only need $d_n^{(3)} \geq \epsilon b_n$ for all $n \geq 1$ and a constant $\epsilon > 0$.)

We have the following result.

Lemma 2.8. *Assume that Question 3 has an affirmative answer. Then $\lambda_3(\widehat{f_c}) \geq \zeta_2$.*

Proof. This follows easily from the fact that $\lim_{n\to\infty} b_n^{1/n} = \zeta_2$. □

(Note that, on the other hand, it seems that for $n \geq 11$ then $d_n^{(2)} \leq c_n$. Using the latter inequality, we obtain only the upper bound $\lambda_2(F) \leq \zeta_2$, which is already discussed in the previous Subsubsection and not the lower bound wanted. However, one can ask whether $d_n^{(2)} \geq \epsilon c_n$ for all $n \geq 1$ and a constant $\epsilon > 0$. This inequality seems to be supported by the data, and is also enough to deduce that $\lambda_3(\widehat{f_c}) \geq \zeta_2$.)

Hence, a promising approach towards showing $\lambda_2(\widehat{f_c}) \geq \zeta_2$ is to solve Question 3 in the affirmative.

2.3.3. *An approach to part 3 of Conjecture 1.5.* We divide this into two tasks: one concerning the Weaker estimate (which provides a counter-example to Conjecture 1.1) and one concerning the Stronger estimate (which provides a counter-example to Conjecture 1.3).

An approach towards the Weaker estimate in part 3 of Conjecture 1.5:

From the table for the number of isolated periodic points, we find that the sequence $[\sharp IsoFix_{2n+1}(f_c)]^{(1/2n+1)}$ (for $0 \leq n \leq 5$) is a **decreasing sequence**: 4, 2.15443469003, 2.13152551327, 2.10967780991, 2.10744910267, and 2.10703309279. Also, we find that the fixed point set of f_c^{2n+1} (for $0 \leq n \leq 5$) consists of isolated points only. These facts do not look like a random coincidence, and hence naturally lead to the following question:

Question 4: Is it true that the sequence $[\sharp IsoFix_{2n+1}(f_c)]^{1/(2n+1)}$ (for $n = 0, 1, 2, \ldots$) is a decreasing sequence?

If Question 4 has an affirmative answer, then since $[\sharp IsoFix_9]^{1/9} = 2.1074... < 2.108$, we obtain right away a proof of the Weaker estimate in part 3 of Conjecture 1.5.

Here is a heuristic explanation for why Question 4 can have an affirmative answer: We know from Proposition 1.6 that the Lefschetz numbers $\{L(\widehat{f}_c^n)^{1/n}\}$ for the map $\widehat{f}_c : X_c \dashrightarrow X_c$ is decreasing. It seems that in this special case, we can localise this property to the map $f_c = \widehat{f}_c|_{Z_c}$. If this is so, and if we can show that the fixed point set of f_c consists of only isolated points (as seen in the experiments), then Question 4 is solved in the affirmative (since then the Lefschetz number is the same as the number of fixed points).

An approach towards the Stronger estimate in part 3 of Conjecture 1.5:

This part is probably more difficult to establish. Our clue is that from the table we observe the following phenomenon: for all $0 \leq n \leq 5$, we have $[\sharp IsoFix_{2n+1}(f_c)]^{1/2n+1} \geq [\sharp IsoFix_{2n+2}(f_c)]^{1/2n+2}$. Thus comes another question:

Question 5: Is it true that $[\sharp IsoFix_{2n+1}(f_c)]^{1/2n+1} \geq [\sharp IsoFix_{2n+2}(f_c)]^{1/2n+2}$ for all $n = 0, 1, 2, \ldots$?

If Question 5 has an affirmative answer, and if moreover the Weaker estimate in part 3 of Conjecture 1.5 holds, then the Stronger estimate in part 3 of Conjecture 1.5 follows readily.

Why Question 5 should have an affirmative answer could be again contributed to a localisation of Proposition 1.6. While the log concavity of the sequence $IsoFix_n(f_c)$ is violated (see the next paragraph), still some parts could be preserved, allowing us to have affirmative answers to both Questions 4 and 5.

In stead of Question 5, the following variant is also enough for our purpose

Question 6: Is it true that $\sharp IsoFix_{2n+1}(f_c) \geq \sharp IsoFix_{2n}(f_c)$ for all $n = 1, 2, 3, \ldots$?

2.3.4. *A speculation on the exponential growth of the isolated periodic points.* Another curious phenomenon, which we do not need in resolving Conjecture 1.5, is that the sequence $[\sharp IsoFix_{2n+2}(f_c)]^{1/2n+2})$ (for $n = 0, 1, 2, \ldots$) seems to be **increasing**. We do not know of a possible explanation for this interesting phenomenon.

However, if this increasing phenomenon is true, and the phenomena mentioned in Questions 4 and 5 are also true, then we will obtain

$$\log 2.0890 \leq \lim_{n\to\infty} \frac{\log \sharp IsoFix_{2n}(f_c)}{2n} \leq \lim_{n\to\infty} \frac{\log \sharp IsoFix_{2n+1}(f_c)}{2n+1} \leq \log 2.1071,$$

and it is then reasonable to speculate that indeed

$$\lim_{n\to\infty} \frac{\log \sharp IsoFix_n(f_c)}{2}$$

also exists. (In this case, the limit of course must be contained in the interval $[\log 2.0890, \log 2.1071]$.)

3. Conclusions

In this paper, we presented a simple family of birational maps on smooth affine quadric 3-folds, coming from those on $\mathbb{C}^4$, which seem to be cohomologically hyperbolic while having less periodic points than expected. Moreover, the second dynamical degree of these maps seem to be an algebraic number, but not an algebraic integer.

This kind of maps requires the development of new tools/ideas stronger than those in the current literature. Among the theoretical tools, it needs the development of more effective non-proper intersections of varieties (not only for projetcive varieties, but also for affine varieties), which help to establish various log concavity phenomena. It also requires taking into account periodic points located in proper analytic subspaces (only the case of bimeromorphic self-maps of surfaces has been dealt with in the literature [46][39][21]). Besides Conjecture 1.3, another consequence of Conjecture 1.1 is the following:

Conjecture 3.1. *Let X be a smooth complex projective variety, and $f : X \dashrightarrow X$ a dominant rational map. If f is cohomologically hyperbolic, then the set of isolated periodic points of f is Zariski dense.*

Conjecture 3.1 is independent of the birational model of f. Besides the cases where Conjecture 1.1 is known to be valid, one nontrivial case where Conjecture 3.1 is solved in the affirmative is when X is of dimension 2 and f is a birational map [50]. (In this case, f being cohomologically hyperbolic is equivalent to $\lambda_1(f) > 1$, and the latter fact can be checked by a finite algorithm following the paper [20] mentioned above. Indeed, there is a sufficient criterion for $\lambda_1(f) > 1$ in terms of some inequalities involving only several first terms in the degree sequence for f, see [50].) In light of the map considered in Conjecture 1.5, it is interesting to check whether Conjecture 3.1 holds for every cohomologically hyperbolic birational maps in dimension 3.

Among the experimental tools, one needs faster and less expensive (while still being highly reliable) methods to deal with numerical calculations, in particular those specially designed for compositions of simple maps. Indeed, the calculations presented in Section 2 are more or less at the upper limit of what current calculation methods can afford us (even for the very simple map F).

Since dynamical degrees can be defined over any algebraically closed field [48][16], it is also meaningful to extend the study mentioned above to arbitrary algebraically closed fields.

References

[1] D. J. Bates, J. D. Hauenstein, A. J. Sommese and C. W. Wample, *Bertini: Software for Numerical Algebraic Geometry,* Available at bertini.nd.edu with permanent doi: dx.doi.org/10.7274/R0H41PB5.

[2] E. Bedford and K.-H. Kim, *Degree growth of matrix inversion: birational maps of sym- metric, cyclic matrices,* Disc. Contin. Dyn. Syst. 21 (4), 977-1013, 2008.

[3] E. Bedford and T. T. Truong, *Degree complexity of birational maps related to matrix inversion,* Comm. Math. Phys. 298 (2010), no 2, 357–368.

[4] E. Bedford, M. Lyubich and J. Smillie, *Distribution of periodic points of polynomial diffeomorphisms of* $\mathbb{C}^2$*,* Invent. Math. 114(2) (1993) 277–288.

[5] E. Bedford, M. Lyubich and J. Smillie, *Polynomial diffeomorphisms of* $\mathbb{C}^2$*. IV. The measure of maximal entropy and laminar currents,* Invent. Math. 112(1) (1993) 77–125.

[6] J. P. Bell, J. Diller, M. Jonsson and H. Krieger, *Birational maps with transcendental dynamical degree,* arXiv:2107.04113.

[7] J. P. Bell, J. Diller and M. Jonsson, *A transcendental dynamical degree,* Acta Math. 225 (2020), no 2, 193–225.

[8] M. P. Bellon and C.-M. Viallet, *Algebraic entropy,* Commun. Math. Phys. 204, 425–437 (1999).

[9] M. P. Bellon, J.-M. Maillard and C.-M. Viallet, *Integrable Coxeter groups,* Phys. Lett. A 159 (1991), no 4–5, 221–232.

[10] M. P. Bellon, J.-M. Maillard and C.-M. Viallet, *Rational mappings, arborescent iterations, and the symmetries of integrability,* Phys. Rev. Lett. 67 (1991), no 11, 1373–1376.

[11] C. Bisi, J.-P. Furter and S. Lamy, *The tame automorphism group of an affine quadric threefold acting on a square complex,* J. Éc. polytech. Math. 1 (2014), 161–223.

[12] J. Blanc and I. van Santen, *Automorphisms of the affine 3-space of degree 3,* Indiana Univ. Math. J. 71 (2022), no 2, 857–912.

[13] J. Blanc and I. van Santen, *Dynamical degrees of affine-triangular automorphisms of affine spaces,* arXiv:1912.01324.

[14] H. Brolin, *Invariant sets under iteration of rational functions,* Ark. Mat. 6 (1965), 103–144.

[15] S. Cantat, *Dynamique des automorphismes des surfaces K3,* Acta Math. 187 (2001) 1–57.

[16] N.-B. Dang, *Degrees of iterates of rational maps on normal projective varieties,* Proc. Lond. Math. Soc. (3), 121 (2020), no 5, 1268–1310.

[17] N.-B. Dang and C. Favre, *Spectral interpretations of dynamical degrees and applications,* Ann. of Math. (2) 194 (2021), no 1, 299–359.

[18] N.-B. Dang, D. Ghioca, F. Hu, J. Lesieutre and M. Satriano, *Higher arithmetic degrees of dominant rational self-maps,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 23 (2022), no 1, 463–481.

[19] J. Diller, R. Dujardin and V. Guedj, *Dynamics of meromorphic maps with small topological degree III: Geometric currents and ergodic theory,* Ann. Sci. Ecole Norm. Sup. (4), 43(2) (2010) 235–278.

[20] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces,* Amer. J. Math. 123 (2001), no 6, 1135–1169.

[21] T-C Dinh, V.-A. Nguyen and T. T. Truong, *Growth of the number of periodic points for meromorphic maps,* Bull. Lond. Math. Soc. 49 (2017), no 6, 947–964.

[22] T-C Dinh, V.-A. Nguyen and T. T. Truong, *Equidistribution for meromorphic maps with dominant topological degree,* Indiana J. Math. 64(6) (2015) 1805–1828.

[23] T-C Dinh and N. Sibony, *Density of positive closed currents, a theory of non-generic intersections,* J. Algebraic Geom. 27 (2018), no 3, 497–551.

[24] T-C Dinh and N. Sibony, *Equidistribution problems in complex dynamics of higher dimension,* Internat. J. Math. 28 (2017), no 7, 1750057, 31 pp.

[25] T-C Dinh and N. Sibony, *Equidistribution of saddle periodic points for Hénon-type automorphisms of* $\mathbb{C}^k$*,* Math. Ann. 366(3–4) (2016) 1207–1251.

[26] T-C Dinh and N. Sibony, *Regularization of currents and entropy,* Ann. Sci. Ecole Norm. Sup. (4), 37 (2004), no 6, 959–971.

[27] T-C Dinh and N. Sibony, *Une borne supérieure de l'entropie topologique d'une application rationnelle,* Annals of Math., 161 (2005), 1637–1644.

[28] R. Dujardin, *Laminar currents and birational dynamics,* Duke Math. J. 131(2) (2006) 219–247.

[29] C. Favre and M. Jonsson, *Dynamical compactifications of* $\mathbb{C}^2$*,* Ann. of Math. (2) 173(1):211–248, 2011.

[30] C. Favre and J. Rivera-Letelier, *Théorie ergodique des fractions rationnelles sur un corps ultramétrique,* Proc. London Math Soc. (3) 100 (2010), no 1, 116–154.

[31] C. Favre, T. T. Truong and J. Xie, *Topological entropy of a rational map over a complete metrized field,* arXiv:2208.00668.

[32] A. Freie, A. Lopes and R. Mane, *An invariant measure for rational maps,* Bol.Soc.Brasil.Mat.14(1983), no. 1, 45–62.

[33] S. Friedland and J. Milnor, *Dynamical properties of plane polynomial automorphisms,* Ergodic Theory Dynam. Systems, 9(1):67–99, 1989.

[34] M. Gromov, *On the entropy of holomorphic maps,* Enseign. Math. (2), 49 (2003) no 3–4, 217–235. Manuscript (1977).

[35] V. Guedj, *Propriétés ergodiques des applications rationnelles,* (French), Quelques aspects des systèmes dynamiques polynomiaux, 97–202, Panor. Synthèses 30, Sociéteé Mathématique de France, Paris 2010, x+341 pp.

[36] B. Hasselblatt and Y. Pesin, *Hyperbolic dynamics,* doi:10.4249/scholarpedia.2208.

[37] F. Hu and T. T. Truong, *An inequality on polarized endomorphisms,* arXiv:2104.12660.

[38] F. Hu and T. T. Truong, *A dynamical approach to generalized Weil's Riemann hypothesis and semisimplicity,* arXiv:2102.04405.

[39] K. Iwasaki and T. Uehara, *Periodic points for area-preserving birational maps of surfaces,* Math. Z. 266 (2010) 289–318.

[40] S.Kawaguchi and J. H. Silverman, *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties,* J. Reine Angew. Math. 713 (2016), 21–48.

[41] V. Y. Kaloshin, *Generic diffeomorphisms with superexponential growth of number of periodic orbits,* Comm. Math. Phys. 211 (2000) 253–271.

[42] M. J. Lyubich, *Entropy properties of rational endomorphisms of the Riemann sphere,* Ergodic Theory Dyn. Syst. 3(3) (1983) 351–385.

[43] K. Maegawa, *Classification of quadratic polynomial automorphisms of* $\mathbb{C}^3$ *from a dynamical view point,* Indiana Univ. Math. J., 50(2):935–951, 2001.

[44] K. Oguiso and T. T. Truong, *Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy,* J. Math. Sci. Univ. Tokyo 22 (2015), no 1, 361–385.

[45] A. Russakovskii and B. Shiffman, *Value distributions for sequences of rational mappings and complex dynamics,* Indiana Univ. Math. J. 46 (1997), 897–932.

[46] S. Saito, *General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings,* Amer. J. Math. 109 (1987) 1009–1042.

[47] T. T. Truong, *Degree complexity of birational maps related to matrix inversion: symmetric case,* Math. Z. 270 (2012), no 3–4, pp 725–738.

[48] T. T. Truong, *Relative dynamical degrees of correspondences over a field of arbitrary characteristic,* J. Reine Angew. Math. 758 (2020), 139–182.

[49] T. T. Truong, *Relations between dynamical degrees, Weil's Riemann hypothesis and the standard conjectures,* arXiv:1611.01124.

[50] J. Xie, *Periodic points of birational transformations on projective surfaces,* Duke Math. J. 164 (2015), no 5, 903–932.

[51] Y. Yomdin, *Volume growth and entropy,* Israel J. Math. 57 (3), (1987) 285–300.

[52] D.-Q. Zhang, *Dynamics of automorphisms on projective complex manifolds,* J. Differential Geom. 82 (2009), no 3, 691–722.

Department of Mathematics and Computer Science, University of Ferrara, Italy

E-mail address: bsicnz@unife.it

Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, USA

E-mail address: hauenstein@nd.edu

Department of Mathematics, University of Oslo, Norway

E-mail address: tuyentt@math.uio.no