ESTIMATING THE COMPLETE SOLUTION SET OF THE APPROXIMATE PATH SYNTHESIS PROBLEM FOR FOUR-BAR LINKAGES USING RANDOM MONODROMY LOOPS

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ABSTRACT

It is natural to employ an optimization algorithm for the approximate kinematic synthesis of linkages. The hope is to find some superior points in the design space that indicate dimensions that are practically useful. One way to achieve this is to find all minima of an objective, then to filter them so the best remain. However, the prospect of finding all minima is bleak unless the optimization problem at hand is particularly small. In this work, we show how to find nearly all minima for a large optimization problem using polynomial homotopy continuation in the approximate synthesis of a four-bar path generator. The system at hand has a Bézout bound of 543,848,665 and a Schnabel estimate to the maximum number of stationary points of $6 \cdot (303,249 \pm 713)$*, within a 95% confidence interval. At least with regards to mechanism synthesis, this work represents the largest scale deployment to date of homotopy continuation to solve an unconstrained optimization problem. The challenges of scaling and suggestions for design are given. Example usage for the design of a leg mechanism is given. On the mechanism design front, this is the first presentation of a nearly complete (within the limitations of numerical discernment) solution of the general four-bar optimal path synthesis problem.*

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1 INTRODUCTION

It is fitting to frame the approximate kinematic synthesis as an optimization problem. In doing so there is no shortage of extant tools [1], most of which cannot claim to generically find a complete set of local minima, including the global. However, applying homotopy continuation to find the zeros of first order conditions provides a means to find all local minima of unconstrained polynomial objectives. Such an approach has been applied to mechanism design problems in the past [2, 3]. In this work, we push this approach onto a much larger scale problem, advancing from a system of degree 25 to a system of estimated degree 303,249. In doing so, we point out the challenges encountered in terms of numerical precision, compute time, and design utility. As a result of this exercise, we also produce for the first time the nearly complete solution for the approximate path synthesis of a four-bar linkage with an arbitrary number of control point specifications and no dimensions pre-specified. This is a natural progression from the 5-9 precision point problems solved by researchers in the early 1990s [4–7]. The enabling computational technique of this work is the method of Random Monodromy Loops [8–11], which was preceded by Finite Root Generation [12, 13], and is executed on top of Bertini [14, 15].

The novelty of this work over other optimal path synthesis works is in the analytical formulation and in the use of numerical continuation to find nearly all possible design candidates, which

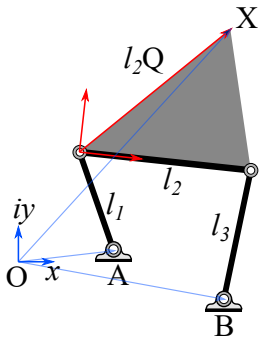

FIGURE 1: Schematic of a four-bar linkage in the complex plane

is not possible using other tools. The exact path synthesis problem of four-bars, known as Alt's problem [16], with a maximum of nine generic design specifications was first solved by Wampler et al. [7] which yielded a complete solution set of 1,442 Roberts' cognate triplets to a polynomial system. We report in this work that an optimization problem of path synthesis of four-bars admits approximately 303,249 ± 713 Roberts' cognate triplets as critical points with 95% confidence, computed using the statistical estimates provided by [11], for any number of design specifications greater than nine. This optimization model can accommodate as many design positions as required within the same framework, thus offering a key advantage over the exact synthesis approach albeit at the cost of significant computational effort. The optimization problem is about 200 times harder to solve compared to Alt's problem. The solution is made possible only by the use of a heuristic-based numerical continuation, namely, the recently developed technique of Random Monodromy Loops (RML) [8–11].

2 MATHEMATICAL MODEL

Consider a four-bar linkage as shown in Fig. 1. Let A and B be the fixed pivots of the linkage. The moving links are of length l_1, l_2, and l_3, respectively. Let ϕ_1, ϕ_2, and ϕ_3 be the respective angular displacements of these links measured counter-clockwise from the positive x-axis. The coupler trace point (normalized by the coupler base length l_2) is denoted in the coupler frame using the vector variable Q while X denotes the coupler trace point in the global frame. We use isotropic coordinates [17] which use a complex variable/parameter and its conjugate variable/parameter instead of two scalar components for any vector. This choice of coordinates offers a simpler mathematical description, reduced total degree of the resulting system, and other numerical advantages during computation.

If $\Phi_k = e^{i\phi_k}$ for $k = 1,2,3$ are the rotation operators in 2D, then the vector loop equations associated with the path generation problem can be written via the left and right dyads as

$$A + l_1\Phi_1 + l_2 Q\Phi_2 = X, \tag{1}$$
$$B + l_3\Phi_3 + l_2(Q-1)\Phi_2 = X, \tag{2}$$

respectively. Their complex conjugate equations must also be included as part of the formulation in the isotropic coordinates framework, namely

$$A^* + l_1\frac{1}{\Phi_1} + l_2 Q^*\frac{1}{\Phi_2} = X^*, \tag{3}$$
$$B^* + l_3\frac{1}{\Phi_3} + l_2(Q^*-1)\frac{1}{\Phi_2} = X^*. \tag{4}$$

Note that the conjugate of a rotation operator is its reciprocal and l_1, l_2, and l_3 are real variables. The rotation operators are insignificant as they are not design specifications, so we eliminate them as follows. Eliminating Φ_1 from Eqs. (1) and (3), we obtain

$$\begin{aligned} &l_2 Q^*(A-X) + (XX^* - A^*X - AX^* - l_{1s})\Phi_2 \\ &\quad + l_2 Q(A^* - X^*)\Phi_2^2 = 0, \end{aligned} \tag{5}$$

where $l_{1s} = l_1^2 - l_2^2 QQ^* - AA^*$.

Similarly, eliminating Φ_3 from Eqs. (1) and (3), we obtain

$$\begin{aligned} &l_2(Q^*-1)(B-X) + (XX^* - B^*X - BX^* - l_{3s})\Phi_2 \\ &\quad + l_2(Q-1)(B^* - X^*)\Phi_2^2 = 0, \end{aligned} \tag{6}$$

where $l_{3s} = l_3^2 - l_2^2(Q-1)(Q^*-1) - BB^*$.

Finally, Φ_2 can be eliminated between Eqs. (5) and (6) via Sylvester's resultant to obtain a polynomial condition which represents the coupler trace of a given four-bar linkage:

$$\begin{vmatrix} Q^*(A-X) & g(X,X^*) & l_2Q(A^*-X^*) & 0 \\ 0 & l_2Q^*(A-X) & g(X,X^*) & Q(A^*-X^*) \\ (Q^*-1)(B-X) & h(X,X^*) & l_2(Q-1)(B^*-X^*) & 0 \\ 0 & l_2(Q^*-1)(B-X) & h(X,X^*) & (Q-1)(B^*-X^*) \end{vmatrix} = 0,$$

where

$$g(X,X^*) = XX^* - A^*X - AX^* - l_{1s} \quad \text{and}$$
$$h(X,X^*) = XX^* - B^*X - BX^* - l_{3s}.$$

When expanding this determinant condition, observe that l_2 only occurs in its quadratic form. Hence, substituting $l_{2s} = l_2^2$

simplifies the expression further. Let us denote this condition as $\eta(\boldsymbol{d}, X, X^*) = 0$ hereon, where

$$\boldsymbol{d} = \{A, A^*, B, B^*, l_{1s}, l_{2s}, l_{3s}, Q, Q^*\}$$

denotes the set of nine design variables. Note that l_{1s}, l_{2s}, and l_{3s} are still real variables based on their definitions. Some notable aspects of this condition are that it is real-valued, of algebraic degree six in X and X^*, and the circularity [17] is three. Further, this condition evaluates to the same value for any given four-bar linkage and its two corresponding Roberts' cognates [18]. Exploiting this in the numerical continuation implementation reduces computational efforts.

Suppose that the path generation problem is formulated for points (X_j, X_j^*) for $j = 1, \ldots, N$, where N is the number of design specifications, and let $\eta_j(\boldsymbol{d}) = \eta(\boldsymbol{d}, X_j, X_j^*)$. In particular, when $N = 9$ and the points are general, Alt's problem is equivalent to solving the nine-dimensional square polynomial system $\eta_j(\boldsymbol{d}) = 0$ for $j = 1, \ldots, 9$. A different formulation of this problem was used in the solution provided by [7]. Nonetheless, the solution set of Alt's problem includes 1,442 Roberts' cognate triplets. One way of accommodating more than nine design specifications is by formulating and minimizing an optimization objective function. To retain the polynomial nature of the system, we use a $\boldsymbol{L}^2$-norm based error measure as follows.

For $N > 9$, the objective function, which is based on the residual error of the coupler trace equation, is:

$$\frac{1}{2}\sum_{j=1}^{N} \eta_j^2. \tag{7}$$

This is a real-valued polynomial function. Hence, the first order necessary conditions for optimality are derived analytically as:

$$\sum_{j=1}^{N} \eta_j \frac{\partial \eta_j}{\partial \boldsymbol{d}} = \boldsymbol{0}. \tag{8}$$

This resulting system of polynomial equations is also a square nine-dimensional system. The monomial structure of the system is invariant with respect to N for any $N > 9$, thus providing a unified framework. A measure of complexity of this polynomial system is the 5-homogeneous Bézout bound [17] of 543,848,665 for the variable grouping $\{\{A, A^*, l_{1s}\}, \{B, B^*, l_{3s}\}, \{Q\}, \{Q^*\}, \{l_{2s}\}\}$ which is an upper bound on the number of solutions to Eq. (8) for $N > 9$. Note that the corresponding bound for solving Alt's problem with $N = 9$ and the same variable grouping is 2,580,480[1]. By comparison, the optimization problem is about 200 times harder to solve than Alt's problem if a multi-homogeneous homotopy technique [7] is used. This involves numerically tracking 543,848,665 homotopy paths, which is a phenomenal computational effort. From other corroborating evidences in the literature, e.g., [8,20], we know that only a small percentage (typically <1%) of these paths will lead to finite solutions while the rest diverge off to infinity. In such a scenario, it is advantageous to resort to heuristic numerical continuation techniques such as Random Monodromy Loops (RML) [8–11]. Here the computational effort is associated only with the number of finite solutions, which is usually much lower than the Bézout bound. In the following section, we describe the use of RML to solve our derived polynomial system.

[1]Better upper bounds have been reported in literature, e.g., [7, 19] for Alt's problem using a different formulation.

3 NUMERICAL CONTINUATION

Random Monodromy Loops (RML) is a numerical continuation technique that collects finite roots to a system of polynomial equations starting from a seed solution. A monodromy loop is a group action that permutes the starting solution set onto the same system via randomly constructed paths [8–11]. When applied iteratively, the aggregation of new roots follows a probabilistic model that estimates the system's total root count [11]. We use the implementation of RML put forth in [8], wherein monodromy loops are carried over the total coefficient space of our system. A generic polynomial system is posed *ab initio* with a randomly chosen set of $N = 10$ design specifications (X_j, X_j^*) for $j = 1, 2, .., 10$, which is one greater than that of Alt's problem. These specifications can be generic complex choices without adhering to the conjugate relationship. Then, an initial seed solution is found for the polynomial system by using a local method such as Newton's method. Starting from the seed solution, a monodromy loop is tracked numerically. This procedure results in potentially new roots for the original system. When this process is repeated with the cumulative set of solutions after each iteration, the solution set grows in a capture-mark-recapture model as elucidated in [11]. Since the total number of isolated solutions to the polynomial system is finite, the cumulative set of solutions saturate over the iterations and approach the total number of roots.

As previously mentioned, for every critical point found during an iteration, its two Roberts' cognates are also critical points in view of the objective function being invariant with respect to the cognates. Furthermore, there is a redundant representation in the design space where the left and right dyads of the linkage can be swapped to obtain an apparently distinct solution; therefore, critical points to this optimization problem occur in groups of six. The computational effort can be reduced by retaining only one member of each group and treating subsequent captures as repeats during the implementation. Since the homotopy is invariant with respect to this six-way symmetry, monodromy loops will preserve the group and thus map a whole group of six at

the start to possibly a different group of six at the end. Refer to Appendix A for the transformations that identify the members of a group. Consequently, our reported root-count estimate is one-sixth of the actual root-count estimate. Since the Roberts' cognates can always be back-calculated using well-known algebraic transformations, provided in Appendix A, this strategy allows us to mitigate computational effort. We used the software `Bertini` [14, 15] in parallel mode on a four node dual 24 core Intel® Xeon® 2.30 GHz system (totaling 192 cores) in the Center for Research Computing at the University of Notre Dame to track the numerical continuations.

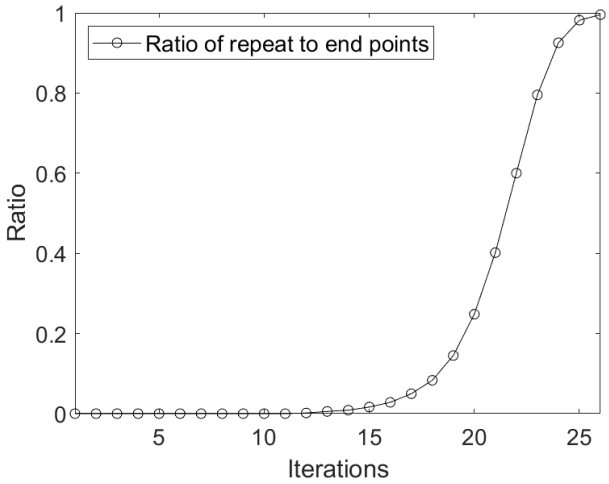

FIGURE 2: Progression of the RML runs. The beginning runs result in new end points; the later runs with ratios close to one indicate most end points are repeats and convergence has been achieved.

For the first few RML iterations, generally new points are found and few, if any, are repeated. In our data, the first time that a repeated point was found was on iteration 12. As the solution set grows, more points are discovered and the ratio of repeats to end points grows, which is used in calculating statistical estimates on the total number of solutions [11]. Consider iteration 21, where approximately 40.2% of the end points were repeat solutions; by iteration 26, 99.6% of end points were repeats. Our final set of computed roots is 303,226, which took a total computation time of approximately 21 hours.

Following [11], we employed three statistical methods based on the aforementioned capture-mark-recapture model to approximate the root count: Lincoln-Petersen, Chapman, and Schnabel estimates. The Lincoln-Petersen and Chapman estimates use data collected from a single monodromy loop. While the Chapman estimate is unbiased, the Lincoln-Petersen estimate is biased for

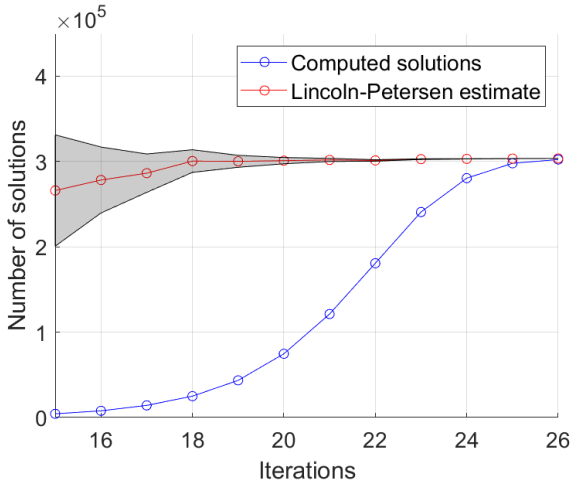

FIGURE 3: Lincoln-Petersen estimation of the root-count of the polynomial system.

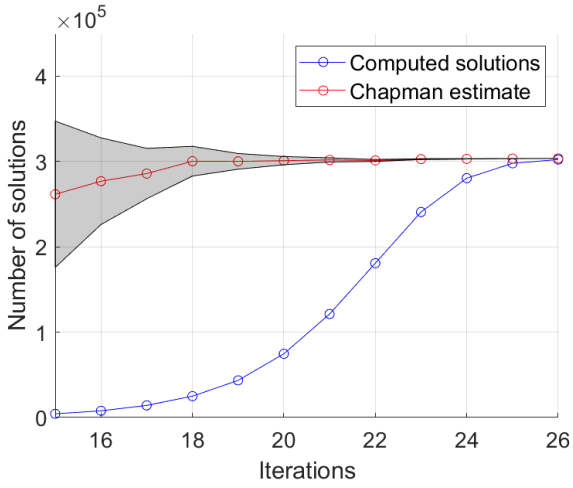

FIGURE 4: Chapman estimation of the root-count of the polynomial system.

small sample sizes. The Schnabel estimate incorporates data from multiple monodromy loops. We used a moving window of size three for the Schnabel estimate, that is, our estimate incorporates data from the current and previous two monodromy loops. Each of these estimates have corresponding variances that are used to compute a 95% confidence interval. For an overview to monodromy loops and expressions for the three mentioned estimates and their respective variances, refer to [11].

Figures 3-5 show the progression of the computed number of solutions, the corresponding statistical estimate, and a shaded area representing the 95% confidence interval for the solution estimate. Since the Schanbel estimate relies on data from previous

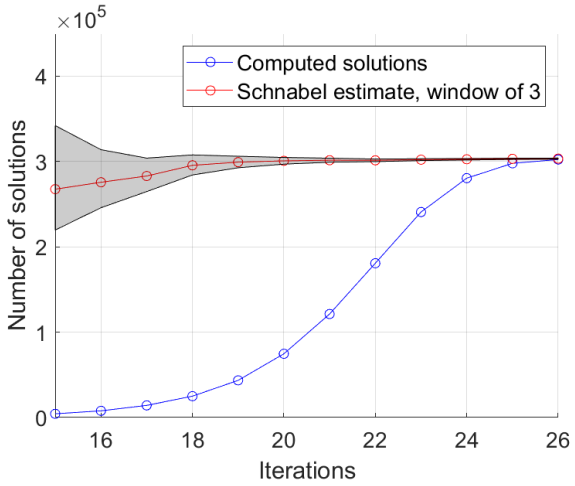

FIGURE 5: Schnabel estimation of the root-count of the polynomial system, window of 3.

monodromy loops and to maintain consistency amongst the three estimates, these plots all start from iteration 15. Clearly, the confidence interval quickly narrows in on the estimate as the trials progress. At iteration 26, our number of computed roots is 303,226. The final Schnabel estimate, using data from iterations 24, 25, and 26, is 303,249 ± 713. Considering this estimate, we likely have computed approximately 99.7% of the complete solution set.

Due to the high degree of the polynomial system and the use of double precision path tracking, the failure rate of the homotopy paths via RML was around 20% per iteration. Although the three statistical estimates account for path failures, the underlying statistical model assumes that such failures are uniformly distributed amongst the paths. This assumption is not upheld in our work since ill-conditioned end points have a higher chance of failure. This results in the Lincoln-Petersen and Chapman methods underestimating the root count and reporting tighter bounds. Therefore, we consider the Schnabel estimate for its calculation over a window of data and wider reported confidence interval.

Adaptive precision path tracking [21] addresses the ill-conditioning and reduces the failure rate, but it comes at the computational expense of high precision computations. We did not employ this since our goal was to find a reasonably complete solution set and use it to compute nearly all minima for the ensuing example problem.

4 ILLUSTRATION

We used four-bar linkage approximating a hopping motion to illustrate computing nearly all local minima via homotopy continuation. The solutions computed via RML in the previous section serve as start points for a coefficient parameter homotopy [22] that

TABLE 1: DESIGN SPECIFICATION FOR A HOPPING MOTION.

#	x	y
1	0	1.000000
2	0.031944	0.994898
3	0.063888	0.979592
4	0.095831	0.954082
5	0.127775	0.918367
6	0.159719	0.872449
7	0.191663	0.816327
8	0.223607	0.750000
9	0.255551	0.673469
10	0.287494	0.586735
11	0.319438	0.489796
12	0.351382	0.382653
13	0.383326	0.265306
14	0.415270	0.137755
15	0.447214	0

yields designs of interest. Recall that RML can be used to solve any system with $N > 9$ design specifications. For the hopping motion, we specify $N = 15$ specifications (x_j, y_j), listed in Table 1. The isotropic specifications are $X_j = x_j + iy_j$ and $X_j^* = x_j - iy_j$ for $j = 1, \ldots, 15$ and $i = \sqrt{-1}$ which are used to construct the target system Eq. (8). Within the `Bertini` environment, we employed the user-defined homotopy module to track 303,226 paths in double precision yielding 195,100 successful end points. The computation time was about 400 minutes using the previously mentioned computational resource from Notre Dame.

Since the end points satisfy Eq. (8), they are stationary points of the objective function in Eq. (7). Out of these, we retain the physical solutions which respect the conjugate relationship between A and A^*, and so on, and have real values for l_{1s}, l_{2s}, and l_{3s}. This set includes both the minima and saddle points of the objective function. By analyzing the Hessian matrix of Eq. (7), we can identify the minima of the objective function and obtain a set of 39 solutions, which, considering the cognates, yields $39 \cdot 3 = 117$ four-bar linkages. Four of these 39 Roberts' cognate triplets are exhibited in Figs. 6-9 with the dimensions provided in

TABLE 2: LIST OF FOUR-BAR DESIGNS FOR A HOPPING MOTION.

#	Cog.	A	B	l_1	l_2	l_3	Q
I	1	2.110+1.561i	-0.930 + 0.017i	2.018	0.477	1.142	0.527+0.215i
	2	0.839+0.093i	2.110+1.561i	0.650	1.149	0.272	-0.626+0.663i
	3	-0.930 + 0.017i	0.839+0.093i	0.248	0.593	1.048	1.753+0.797i
II	1	1.077+0.886i	-0.255 + 0.124i	0.136	0.733	0.894	1.288-0.143i
	2	-0.747+0.096i	1.077+0.886i	1.158	0.177	0.950	0.233-0.085i
	3	-0.255 + 0.124i	-0.747+0.096i	0.236	0.288	0.044	-2.783-1.386i
III	1	-0.702-0.016i	0.480+0.587i	1.190	0.347	0.339	0.096+0.161i
	2	-0.686 + 0.232i	-0.702-0.016i	0.0635	0.223	0.065	-1.736+4.581i
	3	0.480+0.587i	-0.686+0.232i	0.319	0.311	1.092	1.072+0.191i
IV	1	0.330 + 0.563i	-0.925 + 0.439i	0.302	0.290	1.268	0.742 + 0.589i
	2	-0.529 - 0.268i	0.330 + 0.563i	1.201	0.286	0.275	0.173 + 0.656i
	3	-0.925 + 0.439i	-0.529 - 0.268i	0.186	0.814	0.194	0.624 + 1.426i

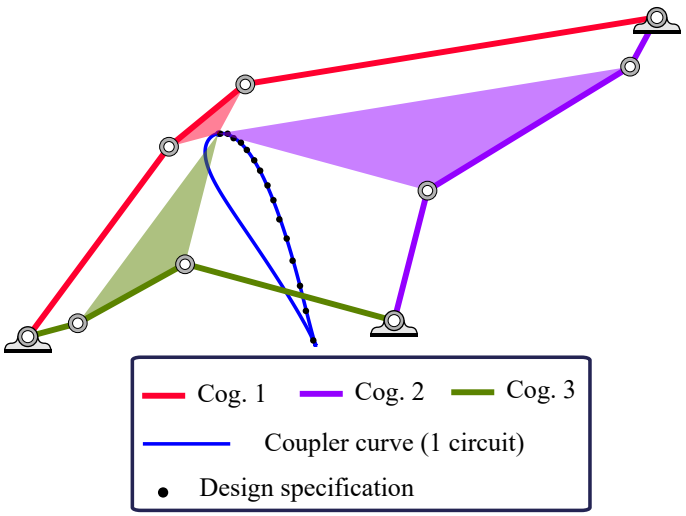

FIGURE 6: A feasible Roberts' cognate triplet, Design I, shown in Table 2.

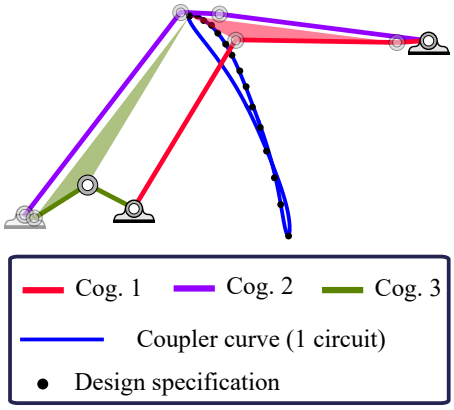

FIGURE 7: A feasible Roberts' cognate triplet, Design II, shown in Table 2.

Table 2. Of these, Figs. 6-8 are approximately tracing the path as desired; the design in Fig. 9 is circuit-defective where the design positions are split between the two circuits of the linkage [23]. Additionally, there exists designs wherein the coupler trace show undesirable loops in their path as shown in Fig. 10 even though the design is not circuit-defective.

None of the designs found showed desired packaging of the mechanism with sufficient ground clearance. This issue can be circumvented by affixing a carefully chosen dyad and converting the four-bar into a Stephenson-III type six-bar linkage as shown in Fig. 11. The transferred coupler curve retains the essential features of the original specification. Additionally, the end-effector point is the bottom most point of the linkage in all its configurations as desired.

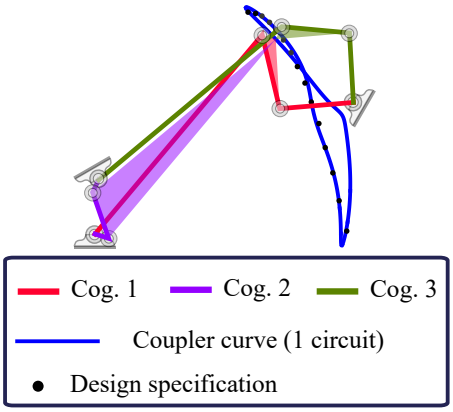

FIGURE 8: A feasible Roberts' cognate triplet, Design III, shown in Table 2.

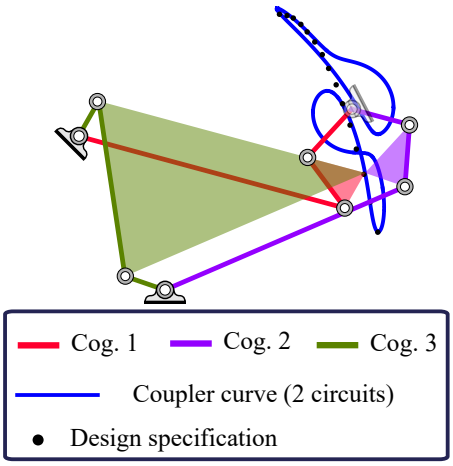

FIGURE 9: A Roberts' cognate triplet, Design IV, shown in Table 2. which is circuit-defective.

5 SUMMARY

In this paper, we present the solutions to an unconstrained optimization problem of path synthesis of four-bar linkages using numerical continuation technique. The problem formulation is generic and invariant with respect to the number of design specifications. We pose a polynomial objective function based on residual error in the coupler trace equation and derive the first-order necessary conditions of optimality. This results in a nine-dimensional square system of highly non-linear equations with at most 543,848,665 solutions via the Bézout bound. This is one of the largest optimization problems attempted using a numerical continuation approach. We employ Random Monodromy Loops to compute our solution set; the Schnabel estimate of the system's total root count is $6 \cdot (303,249 \pm 713)$, with 95% confidence. The multiplication of the estimate by 6 represents the six-way symmetry of the solution set due to Roberts' cognates

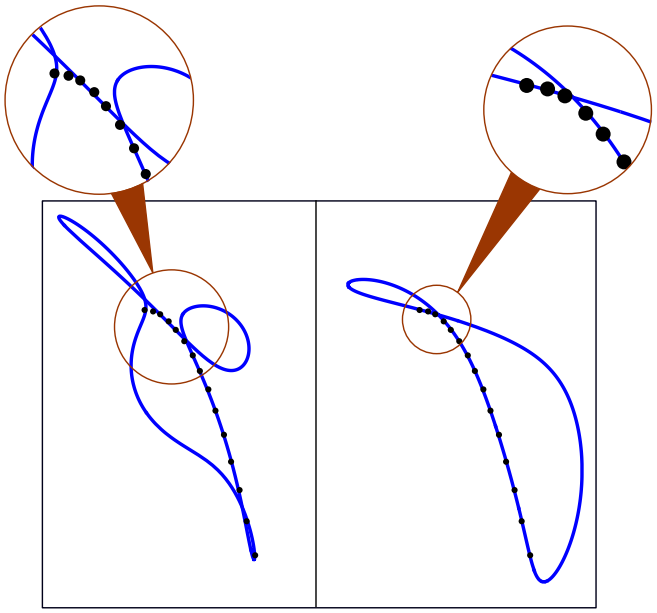

FIGURE 10: Two examples of coupler curves traced by four-bar linkages which contain other undesirable characteristics despite not being circuit-defective.

and swapping the left and right dyads. We then apply this result to design a mechanism for generating a hopping motion.

This work is a significant step-up from other contemporary works in terms of the computational effort expended. A key advantage is that the designer has the option to specify as many as 50 design positions or higher albeit at the cost of additional functional evaluations. Here, we showcase an example with 15 specified design positions. Further, we show that numerical continuation-based optimization is capable of finding nearly all local minima, thus leading to multiple alternative designs other conventional optimization toolboxes may not afford. While this methodology reliably finds effective designs, the computational effort may be more effective solving problems of a smaller scale, such as linkages with specified ground pivots. Scaling down in this way would be a better design strategy in terms of achieving real time computations and shorter lead time.

Although direct six-bar designs may be out of scope, we show an indirect approach in which these linkages are developed by adding on to four-bar linkages designed via numerical continuation-based optimization. The occurrence of significant failure rates in homotopy path tracking for real-world applications hampers the appeal of this method, which otherwise could reach into other engineering domains. A potential solution to this open problem would be developing effective numerical routines that handle ill-conditioned polynomial systems without requiring high numerical precision.

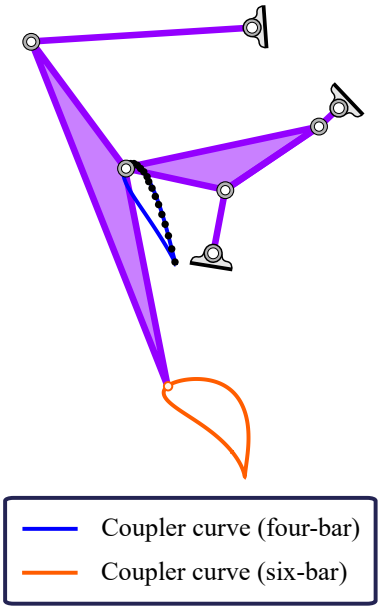

FIGURE 11: Design of a six-bar mechanism for generating a hopping motion, derived from the four-bar Design I (second cognate).

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REFERENCES

[1] Amodeo, L., Talbi, E.-G., and Yalaoui, F., 2018. *Recent Developments in Metaheuristics*, Vol. 62 of *Operations Research/Computer Science Interfaces Series*. Springer, Cham, Switzerland.

[2] Liu, A.-X., and Yang, T.-L., 1999. "Finding All Solutions to Unconstrained Nonlinear Optimization for Approximate Synthesis of Planar Linkages Using Continuation Method". *Journal of Mechanical Design,* ***121***(3), Sept., pp. 368–374.

[3] Baskar, A., and Plecnik, M., 2021. "Computing all solutions to a discretization-invariant formulation for optimal mechanism design". In 2021 IEEE International Conference on Robotics and Automation (ICRA), pp. 9745–9751.

[4] Morgan, A. P., and Wampler, II, C. W., 1990. "Solving a Planar Four-Bar Design Problem Using Continuation". *Journal of Mechanical Design,* ***112***(4), Dec., pp. 544–550.

[5] Tsai, L.-W., and Lu, J.-J., 1990. "Coupler-Point-Curve Synthesis Using Homotopy Methods". *Journal of Mechanical Design,* ***112***(3), Sept., pp. 384–389.

[6] Subbian, T., and Flugrad, Jr., D. R., 1991. "Four-Bar Path Generation Synthesis by a Continuation Method". *Journal of Mechanical Design,* ***113***(1), Mar., pp. 63–69.

[7] Wampler, C., Morgan, A., and Sommese, A., 1992. "Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages". *Journal of Mechanical Design,* ***114***(1), pp. 153–159.

[8] Baskar, A., and Plecnik, M., 2020. "Synthesis of Six-Bar Timed Curve Generators of Stephenson-Type Using Random Monodromy Loops". *Journal of Mechanisms and Robotics,* ***13***(1), 08. 011005.

[9] Duff, T., Hill, C., Jensen, A., Lee, K., Leykin, A., and Sommars, J., 2018. "Solving polynomial systems via homotopy continuation and monodromy". *IMA Journal of Numerical Analysis,* ***39***(3), 04, pp. 1421–1446.

[10] Hauenstein, J. D., Oeding, L., Ottaviani, G., and Sommese, A. J., 2019. "Homotopy techniques for tensor decomposition and perfect identifiability". *Journal für die reine und angewandte Mathematik (Crelles Journal),* ***2019***(753), pp. 1–22.

[11] Hauenstein, J. D., and Sherman, S. N., 2020. "Using monodromy to statistically estimate the number of solutions". In IMA Conference on Mathematics of Robotics, Springer, pp. 37–46.

[12] Plecnik, M. M., and Fearing, R. S., 2017. "Finding Only Finite Roots to Large Kinematic Synthesis Systems". *Journal of Mechanisms and Robotics,* ***9***(2), Mar., pp. 021005–021005–8.

[13] Plecnik, M. M., and Fearing, R. S., 2017. "A Study on Finding Finite Roots for Kinematic Synthesis". In ASME 2017 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, American Society of Mechanical Engineers, pp. V05BT08A083–V05BT08A083.

[14] Bates, D. J., Hauenstein, J. D., Sommese, A. J., and Wampler, C. W., 2006. Bertini: Software for numerical algebraic geometry. Permanent doi: dx.doi.org/10.7274/R0H41PB5.

[15] Bates, D. J., Hauenstein, J. D., Sommese, A. J., and Wampler, C. W., 2013. *Numerically solving polynomial systems with Bertini*. SIAM, Philadelphia.

[16] Alt, H., 1923. "Über die erzeugung gegebener ebener kurven mit hilfe des gelenkviereckes". *ZAMM,* ***3***(1), pp. 13–19.

[17] Wampler, C. W., 1996. "Isotropic coordinates, circularity, and Bézout numbers: planar kinematics from a new perspective". In Proceedings of ASME Design Engineering Technical Conference and Computers in Engineering Conference, Irvine, California, August, 18-22.

[18] Roberts, S., 1875. "On three-bar motion in plane space". *Proceedings of the London Mathematical Society,* ***s1-7***(1), pp. 14–23.

[19] Hauenstein, J. D., and Rodriguez, J. I., 2020. "Multiprojective witness sets and a trace test". *Advances in Geometry,* ***20***(3), pp. 297–318.

[20] Plecnik, M. M., and McCarthy, J. M., 2015. "Computational

Design of Stephenson II Six-Bar Function Generators for 11 Accuracy Points". *Journal of Mechanisms and Robotics,* ***8***(1), 08. 011017.

[21] Bates, D. J., Hauenstein, J. D., Sommese, A. J., and Wampler, II, C. W., 2008. "Adaptive multiprecision path tracking". *SIAM J. Numer. Anal.,* ***46***(2), pp. 722–746.

[22] Morgan, A. P., and Sommese, A. J., 1989. "Coefficient-parameter polynomial continuation". *Appl. Math. Comput.,* ***29***(2, part II), pp. 123–160.

[23] Balli, S. S., and Chand, S., 2002. "Defects in link mechanisms and solution rectification". *Mechanism and Machine Theory,* ***37***(9), pp. 851–876.

A Cognate transformations

For a given four-bar linkage with design parameters

$$\boldsymbol{d}_1 = \{A, A^*, B, B^*, l_1, l_2, l_3, Q, Q^*\}$$

following the convention in Fig. 1, its two other Roberts' cognates can be computed as follows:

$$\boldsymbol{d}_2 = \Big\{A + Q(B-A), A^* + Q^*(B^*-A^*), A, A^*, l_3\sqrt{QQ^*}, \\ l_1\sqrt{QQ^*}, l_2\sqrt{QQ^*}, \frac{Q-1}{Q}, \frac{Q^*-1}{Q^*}\Big\}.$$

$$\boldsymbol{d}_3 = \Big\{B, B^*, A + Q(B-A), A^* + Q^*(B^*-A^*), l_2\sqrt{(1-Q)(1-Q^*)}, \\ l_3\sqrt{(1-Q)(1-Q^*)}, l_1\sqrt{(1-Q)(1-Q^*)}, \frac{1}{1-Q}, \frac{1}{1-Q^*}\Big\},$$

Further, for any design $\boldsymbol{d}_1$, its symmetric representation is

$$\boldsymbol{d}_1' = \{B, B^*, A, A^*, l_3, l_2, l_1, 1-Q, 1-Q^*\}.$$

When taken together, all $3 \cdot 2 = 6$ members of a group can be identified/computed using these rules.