Positive-Dimensional Solution Sets (Lecture 2)

Ex 1: Compute a numerical irreducible decomposition of

$$f = \begin{bmatrix} x_1 \cdot x_5 - x_2 \cdot x_4 \\ x_2 \cdot x_6 - x_3 \cdot x_5 \end{bmatrix} \quad \text{in } \mathbb{C}^6$$

- Create input file:

```
CONFIG
  TrackType: 1;
END;
INPUT
    variable-group  x1,x2,x3,x4,x5,x6;
    function  f1,f2;
    f1 = x1*x5 - x2*x4;
    f2 = x2*x6 - x3*x5;
 END;
```

- Run Bertini (>> bertini input)

This uses the regenerative cascade algorithm to search dimension by dimension to locate irreducible components.

- Codimension 1: computes isolated solutions of

$$\begin{bmatrix} f_1 + \alpha \cdot f_2 \\ l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} = 0$$

where $\alpha \in \mathbb{C}$ random and $l_1, \ldots, l_5$ are random linear polynomials.

This has 2 solutions that don't satisfy $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0$.

(Note use of Bertini's Theorem)

Codimension 2: computes isolated solutions of

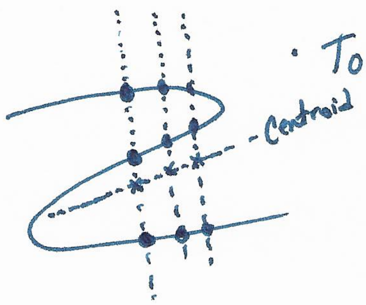
$$\begin{bmatrix} f_1 \\ f_2 \\ l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix} = 0$$

This has 4 solutions (all nonsingular).

Thus, $V(f_1, f_2)$ defines a codimension 2 variety of deg 4.

Next, we need to decompose into irreducible components.

- When there are many points, monodromy loops are used to try to identify which points must lie on the same irreducible component (smooth points are path connected)



- To verify a complete component: trace test.
  - Centroid of points moves linearly as the linear slice is moved in parallel

This results in 2 components: deg 1 & deg 3.

View main-data to see a generic point on each component:

deg 1: $x_2 = x_5 = 0$

deg 3: $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$ has rank ≤ 1

$$\begin{bmatrix} x_1 \cdot x_5 - x_2 \cdot x_4 \\ x_2 \cdot x_6 - x_3 \cdot x_5 \\ x_1 \cdot x_6 - x_3 \cdot x_4 \end{bmatrix} = 0.$$

- Restart regenerative cascade if problem using RegenStartLevel.
- If you do not want a decomposition, only the generic points: WitnessSupersetOnly.

Ex 2: Components of different dimension

- When the solution set consists of components of different dimension, the regenerative cascade will find points on higher dimensional components which must be identified and removed.

$$f = \begin{bmatrix} x(x^2 - y - z^2) \\ x(x + y + z^2 - 2)(y-5) \\ x(x^2 + x - 2)(z-2) \end{bmatrix}$$

Decomposition: 5 irreducible components

- codim 1: deg 1 $V(x)$
- codim 2: deg 2 $V(x-1, y+z^2-1)$
  deg 2 $V(x+2, y+z^2-4)$
- codim 3: deg 1 $V(x+3, y-5, z-2)$
  deg 1 $V(x-3, y-5, z-2)$

Nonisolated solutions can be identified with the local dimension test.
- If isolated, multiplicity is bounded by number of paths.
  - finitely generated local dual space
- If not isolated, compute more than the number of paths linearly independent elements in local dual space showing it is not isolated.

Most common example: endpoint of 1 path, but Jacobian is rank deficient ⇒ not isolated.

View "witness-superset" to see the points which were identified as "junk."

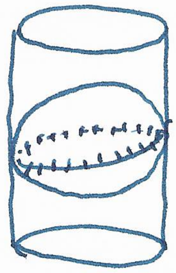
Ex 3: Deflating components

When a component has multiplicity > 1 with respect to the input system f, the component needs to be "deflated" in order to perform computations on it

↳ construct a polynomial system g such that the component has multiplicity 1 with respect to g.

Ex: $f = \begin{bmatrix} x^2+y^2-1 \\ x^2+y^2+z^2-1 \end{bmatrix}$.

intersection of cylinder and sphere



"Witness Points Deflated: 2"

↳ view main-data

Implemented: Replace f by $\begin{bmatrix} f \\ Jf \cdot \lambda \\ K(\lambda) \end{bmatrix}$ where $K(\lambda)$ is a system of random linear polynomials:
(Leykin-Verschelde-Zhao)

linears = dim null space of $Jf(x^*)$ at generic x^* on component.

This method can easily be overwhelmed:

Ex: $\begin{bmatrix} x-y \\ (x+z)^6 \end{bmatrix}$ ~ requires 5 deflations.

Simplified method of H.-Mourrain-Szanto is first to have polynomial many new functions without increasing number of variables (not yet implemented in Bertini)

Ex 4: Sampling

One feature of numerical algebraic geometry is the ability to sample arbitrarily many general points from a component.

This has many applications, especially when it is very difficult to compute a polynomial generating set (e.g., Cifuentes - Parrilo (2017)).

Once a numerical irreducible decomposition is computed, Bertini can compute arbitrarily many points arbitrarily close to the variety: uses "witness_data" file.

```
CONFIG
   TrackType: 2;
   SharpenDigits: 30;
END;
INPUT
   variable_group x,y,z;
   function f1,f2;
   f1 = x^2+y^2-1;
   f2 = x^2+y^2+z^2-1;

END;
```

~ Follow the onscreen menu to sample points.

(One application: interpolation of low degree polynomials that vanish on variety)

Ex 5: Membership Testing

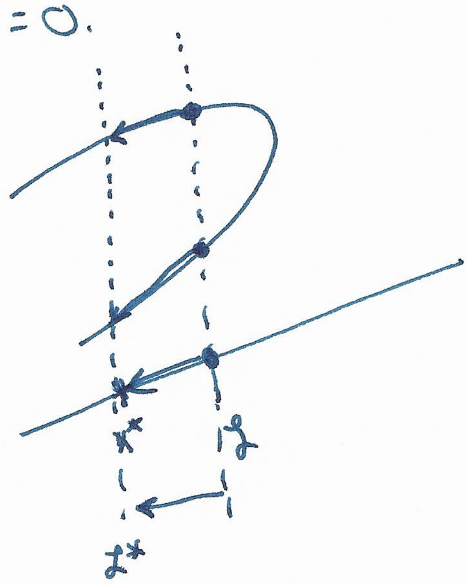
A witness set for an irreducible component provides a method to test if a given point is contained in the irreducible component or not.

(e.g. $M_2 \notin \sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4)$ in H.-Ikenmeyer-Landsberg)

For example, $(1,0,2,3,0,4)$, $(2,3,4,4,6,8)$, $(2,0,3,4,0,6)$

all satisfy $\begin{bmatrix} x_1 \cdot x_5 - x_2 \cdot x_4 \\ x_2 \cdot x_6 - x_3 \cdot x_5 \end{bmatrix} = 0.$

Which componets do they live on?



Use "witness-data" that contains the num. irred. decomposition and create "member-points".

```
CONFIG
  TrackType: 3;
END;
```

Bertini reports that 1^{st} point is on one component and 2^{nd} point is on the other component. The 3^{rd} point is on both componets.

Ex 6: Printing a witness set

One can view the witness sets computed by Bertini using the "printing" option starting with "witness-data":

```
CONFIG
  TrackType:4;
END;
```

Follow the on screen menu to print points and linear slices.

Ex 7: Projection

Bertini can compute dimensions and degrees of images of irreducible components under coordinate projections.

For example, compute the projection onto (x_1, x_3, x_4, x_6) of each irreducible component of

$$\begin{bmatrix} x_1 \cdot x_5 - x_2 \cdot x_4 \\ x_2 \cdot x_6 - x_3 \cdot x_5 \end{bmatrix}.$$

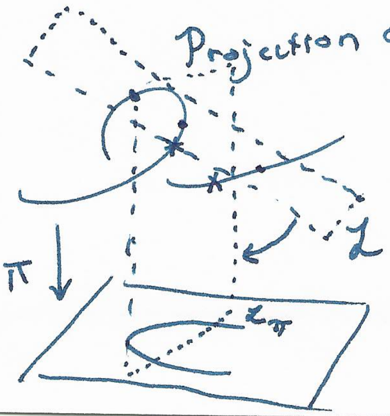
- Use "witness_data" from previous computation.
- Create "projection" which defines the projection map coordinatewise using binary $\begin{cases} 0: \text{do not project onto this coordinate} \\ 1: \text{project onto this coordinate.} \end{cases}$

So, $\pi(x) = (x_1, x_3, x_4, x_6)$ is defined by:

1 0 1 1 0 1.

Projection of deg 1 component: $\begin{cases} \overline{\pi(V)} : \text{dim } 4, \text{deg } 1 \quad (= \mathbb{C}^4) \\ \text{general fiber: dim } 0, \text{deg } 1 \end{cases}$

Projection of deg 3 component: $\begin{cases} \overline{\pi(V)} : \text{dim } 3, \text{deg } 2 \quad (= V(x_1 \cdot x_6 - x_3 \cdot x_4)) \\ \text{general fiber: dim } 1, \text{deg } 1 \end{cases}$



Ex 8: Isosingular local dimension test & reconstructing witness set from 1 point.

In some problems, a solution is known and you want to determine if the dimension of the tangent space at the point is equal to the dimension of the component passing through the point.

Ex: For $f = \begin{bmatrix} x_1 \cdot x_5 - x_2 \cdot x_4 \\ x_2 \cdot x_6 - x_3 \cdot x_5 \end{bmatrix}$,

$x^* = (2,0,3,4,0,6)$ satisfies $f(x^*) = 0$

but dim null $Jf(x^*) = 5$.

Does x^* lie on 5-dim'l irreducible component?

Create a start file with the point ("start_x")

```
CONFIG
  TrackType: 6;
END;
```

>> bertini input start_x

~ "Unable to verify"

What about $y^* = (2,3,4,4,6,8)$: dim ~~null~~ $JF(y^*) = 4$.

Create a "start" file with the point ("start-y")

>> bertini input start-y

"verified"

From a smooth point, one can use monodromy loops to compute other points in witness set and trace test to verify that all points in witness set have been found.

Construct Witness Set : 1;

>> bertini input-witness start-y

~ dim 4, deg 3 component.

(For larger problems, use more efficient monodromy solvers that keep information about each loop and reuse this information when additional points are found: Leykin et al.)

Ex 9: Intersection

One way to solve large systems is to break it into smaller systems and use solutions to subsystems to reconstruct solutions to original system: $V(f_1,\dots,f_n) = V(f_1,\dots,f_k) \cap V(f_{k+1},\dots,f_n)$.

For example, compute the irreducible components of the so-called first-order deflated variety for the cyclic-4 system:

$$F(x,\lambda) = \begin{bmatrix} f(x) \\ Jf(x)\cdot\lambda \end{bmatrix} \quad \text{where } f(x) = \begin{cases} x_1 + x_2 + x_3 + x_4 \\ x_1\cdot x_2 + x_2\cdot x_3 + x_3\cdot x_4 + x_4\cdot x_1 \\ x_1\cdot x_2\cdot x_3 + x_2\cdot x_3\cdot x_4 + x_3\cdot x_4\cdot x_1 + x_4\cdot x_1\cdot x_2 \\ x_1\cdot x_2\cdot x_3\cdot x_4 - 1 \end{cases}$$

$x, \lambda \in \mathbb{C}^4$.

Step 1: Compute a numerical irreducible decomposition of $f(x)=0$

>> bertini input-f

~ 2 irreducible curves of deg 2.

~ keep witness-data file ~ call it witness_data-f.

Step 2: Extend from $V(f)$ to $V(F)$.

```
CONFIG
   TrackType: 7;
END;
```

>> bertini input-extend

Onscreen menu:

1	nontrivial component to start with
input-f	name of input file for f
witness-data-f	name of witness_data file for f
1	dimension of components
-2	regenerate all components of dim 1

Irreducible decomp of $V(F)$: 10 components of dim 2

- 8 components of deg 1 : singular points of $V(f)$ with 2 dim'l null space
- 2 components of deg 4 : lifts of irreducible components of $V(f)$ with 1 dim'l null space.

For example, we can now project these components onto the x coordinates to verify:

projection: $\underbrace{1\ \ 1\ \ 1\ \ 1}_{x\text{ coordinates}}\ \ \underbrace{0\ \ 0\ \ 0\ \ 0}_{\lambda\text{ coordinates}}$

>> bertini input-project

- One of the 8 components of deg 1: $\begin{cases} \overline{\pi(V)}: \text{dim } 0, \text{deg } 1 \\ \text{gen. fiber: dim } 2, \text{deg } 1 \end{cases}$

- One of the 2 components of deg 4: $\begin{cases} \overline{\pi(V)}: \text{dim } 1, \text{deg } 2 \\ \text{gen. fiber: dim } 1, \text{deg } 1 \end{cases}$