The relative structure of valued fields

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- If K is a field and V an ordered abelian group, a valuation is a homomorphism $v : K^\times \to V$ satisfying

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

  This is generally extended to make $v(0) = \infty$.
- The valuation ring is $\mathcal{O} := \{x \in K \mid v(x) \geq 0\}$. $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m} := \{x \in \mathcal{O} \mid v(x) > 0\}$.
- The residue field of K is $R := \mathcal{O}/\mathfrak{m}$.
- We assume throughout that $\mathrm{char}(K) = 0$.

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Valued fields have a topology defined by balls.

Definition

A *ball* is a subset of K of the form

$$B_{>\gamma}(\alpha) = \{x \in K \mid v(x-\alpha) > \gamma\}$$

(an open ball) or $B_{\geq\gamma}(\alpha)$ (closed ball).
K and $\emptyset$ are also included.

Definition

A *swiss cheese* is a subset of K of the form $B \setminus (C_1 \cup \ldots \cup C_n)$, where $B, C_1, \ldots, C_n$ are all balls, with $C_i \subsetneq B$.

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Example

- *The p-adic valuation:* In $\mathbb{Q}$, if $\gcd(a,p) = \gcd(b,p) = 1$, let

$$v(p^n \frac{a}{b}) = n$$

Then $V = \mathbb{Z}$ and $R = \mathbb{F}_p$.

- In the field of rational functions $\mathbb{C}(t)$, define

$$v(\frac{P(t)}{Q(t)}) = \deg(Q(t)) - \deg(P(t))$$

Then $V = \mathbb{Z}$ and $R = \mathbb{C}$.

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Example

Let R be any field, and V any ordered abelian group. The *Hahn field* $R((t^V))$ consists of the formal power series over R:

$$\sum_{\delta \in V} c_\delta t^\delta$$

where the support $\{\delta \mid c_\delta \neq 0\}$ is well-ordered.

Taking

$$v(\sum c_\delta t^\delta) = \min\{\delta \mid c_\delta \neq 0\}$$

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K is *henselian* if it satisfies Hensel's Lemma:

For every $f(x) \in \mathcal{O}[x]$ and $a \in \mathcal{O}$, if $v(f(a)) > 0$ and $v(f'(a)) = 0$, then there exists $b \in \mathcal{O}$ such that $\bar{b} = \bar{a} \in R$ and $f(b) = 0$.

E.g.: The p-adic numbers $\mathbb{Q}_p$.

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Contrast this with the situation with *algebraically closed valued fields* (ACVF).
If (K, v) is algebraically closed, then:

- R is also algebraically closed, and therefore strongly minimal
- V is a divisible ordered abelian group, and therefore o-minimal

In particular R and V both admit quantifier elimination, and this is necessary to show that ACVF itself admits quantifier elimination (Robinson, 1956).

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Recall that a *swiss cheese* is a subset of K of the form $B \setminus (C_1 \cup \ldots \cup C_n)$, where $B, C_1, \ldots, C_n$ are all balls, with $C_i \subsetneq B$.

Robinson's quantifier elimination is a key element in:

Theorem (Holly, 1995)

Every definable subset of an algebraically closed valued field (K,v) can be expressed uniquely as a finite union of disjoint, non-trivially-nested swiss cheeses.

Definition

Let $S \subseteq M^n$ be a definable set in a (multi-sorted) model M. S is *coded* in M if there is a formula $\phi(\bar{x}, \bar{y})$ and $\bar{a} \in M$ such that

$$\forall \bar{b} \in M \ \left(S = \phi(M, \bar{b}) \Longleftrightarrow \bar{b} = \bar{a}\right)$$

Holly proved that in ACVF if we add to the 3-sorted structure (K, V, R) new sorts B and C for the open and closed balls, then one-dimensional definable subsets of K are coded:

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A main impetus for Holly's work was to give a first step towards elimination of imaginaries for ACVF.

Definition

A theory T admits *elimination of imaginaries* if for every $M \models T$ and definable $S \subseteq M^n$, S is coded in M.

Or, if for every $\emptyset$-definable equivalence relation E on S, there is a $\emptyset$-definable function $f : S \to T$ such that

$$\forall a, b \in S \, (Eab \iff f(a) = f(b))$$

(The equivalence classes are the 'imaginaries'.)

Every model M can be expanded to one which eliminates imaginaries, M^{eq}, simply by adding new sorts for every $\emptyset$-definable equivalence relation. The point is to optimize the expansion in some way.

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It was conjectured that the same language from Holly's theorem might also suffice to eliminate imaginaries in ACVF. It turns out, however, that to code higher dimensional sets, a higher dimensional analog of the balls is needed.

Note that both open and closed balls centered at 0,

$$B_{\geq\gamma}(0) = \{x \in K \mid v(x) \geq \gamma\}$$

are modules over $\mathcal{O}$.

Balls centered away from 0 are cosets of these modules.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

ACVF eliminates imaginaries in a language expanded to include sorts for definable $\mathcal{O}$-submodules of K^n and cosets of those modules.

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Theorem (Ax-Kochen, 1965)

If K *and* L *are henselian and* $\mathrm{char}(R_K) = \mathrm{char}(R_L) = 0$*, then*

$$R_K \equiv R_L \ \& \ V_K \equiv V_L \iff K \equiv L.$$

Since Robinson proved completeness for ACVF, this is the first example of a relativized version of a theorem in ACVF holding for henselian fields (of characteristic 0).

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Theorem (Cohen, 1969)

The theory of the p-adics $\mathbb{Q}_p$ admits quantifier elimination, and is decidable.

This underscores how having a nice theory of henselian fields depends on nice theories for the residue field and value group: $\mathbb{Q}_p$ has residue field $\mathbb{F}_p$ and value group $(\mathbb{Z}, +)$.

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A full relative quantifier elimination for characteristic 0 henselian fields was first proved in 1994 by Kuhlmann (following a special case by Basarab, 1991).
One reason it may have taken longer is that the language with the residue field and value group is not really the right one...

Definition

The *leading term structure* RV of K is $K^\times/(1+\mathfrak{m})$.
The *leading term* of $x \in K^\times$ is the image $\mathrm{rv}(x)$ of x under the quotient map.

One may also define higher order structures RV_δ for $0 \leq \delta \in V$ as $K^\times/(1+\mathfrak{m}_\delta)$, where $\mathfrak{m}_\delta = \{x \in K \mid v(x) > \delta\}$ (so $\mathrm{RV} = \mathrm{RV}_0$).
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These will only be needed in the mixed characteristic case.

Example

In the valued field $\mathbb{C}((t))$, two elements will have the same leading term of order n if their value is the same and their first $n+1$ coefficients coincide:

Let

$$\begin{aligned} x &= t^{-2}+t^{-1}+1+t+2t^2+t^3+t^4+\dots \\ y &= t^{-2}+t^{-1}+1+t+t^2 \end{aligned}$$

Then $\mathrm{rv}_0(x)=\mathrm{rv}_0(y)$, $\mathrm{rv}_3(x)=\mathrm{rv}_3(y)$, but $\mathrm{rv}_4(x)\neq\mathrm{rv}_4(y)$.

$\mathrm{rv}(t^{-2}+t)$ contains the information only that the leading term is t^{-2}.

So $\mathrm{rv}(t^{-2}+t)=\mathrm{rv}(t^{-2})$ and this is enough to know that $\mathrm{rv}(t^{-2}+t)+\mathrm{rv}(t^{-2})$ should $=\mathrm{rv}(2t^{-2})$.

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Addition in RV

Besides the multiplication, RV_δ also inherits addition from K, but this addition is only partially defined. Define a relation:

$$\oplus_\delta(\mathbf{x},\mathbf{y},\mathbf{z}) \iff$$

$$\exists x,y,z \in K\,(\mathbf{x} = \mathrm{rv}_\delta(x) \wedge \mathbf{y} = \mathrm{rv}_\delta(y) \wedge \mathbf{z} = \mathrm{rv}_\delta(z) \wedge x+y=z)$$

Proposition

Given any $x,y \in K$, $v(x+y) = \min\{v(x),v(y)\}$ iff for all z such that $\mathrm{rv}(z) = \mathrm{rv}(x)$, $\mathrm{rv}(z+y) = \mathrm{rv}(x+y)$.

We generally write $\mathbf{x}+\mathbf{y}=\mathbf{z}$ rather than $\oplus_\delta(\mathbf{x},\mathbf{y},\mathbf{z})$, but the proposition shows that the $\mathbf{z}$ is only uniquely defined if $v(x+y) = \min\{v(x),v(y)\}$.

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RV can be seen as a way of wrapping up the algebra of the residue field and value group into a single structure, as well as the topology of balls:

Fact

For all nonzero $x, y \in K$, the following are equivalent:

1. $\mathrm{rv}(x) = \mathrm{rv}(y)$
2. $v(x - y) > v(x)$
3. $v(x) = v(y)$ *and* $\mathrm{res}(y/x) = 1$

Also, $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ *iff* $y \in B_{>v(x)+\delta}(x)$

It follows that R and V are both interpretable in RV.
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However, the two-sorted RV language is somewhat weaker than the three-sorted one.

The key complication arises where the addition in RV_δ is not well-defined.
As noted earlier, $\mathrm{rv}_\delta(P(x))$ is a well-defined function of $\mathrm{rv}_\delta(x)$ only when $v(P(x)) = \min\{v(a_i x^i)\}$ (for $P(x) = \sum_{i=0}^{d} a_i x^i$).
This motivates the definition

Definition

$P(x)$ has a *collision at* β *around* α if, for $P(x) = \sum_{i=0}^{d} a_i (x-\alpha)^i$, $v(P(\beta)) > \min_{i \leq d}\{v(a_i(\beta-\alpha)^i)\}$.

Notice that this idea of collision is not intrinsic to the polynomial alone, but depends on how P is expanded as a sum of monomials.

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Notice that this idea of collision is not intrinsic to the polynomial alone, but depends on how P is expanded as a sum of monomials.

In residue characteristic 0, collisions can only occur near a root of one of the (nonzero) derivatives of P.

Proposition

Let $\mathrm{char}(R) = 0$*, and suppose* $P(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i$ *has a collision at* β *around* α*. Then there are* $n < d$ *and* $\lambda \in K$ *with*

(i) $P^{(n)}(\lambda) = 0$*, and*

(ii) $\mathrm{rv}(\lambda - \alpha) = \mathrm{rv}(\beta - \alpha)$*, and in particular,* $v(\lambda - \beta) > v(\beta - \alpha)$*.*

Proof.

- Let m be maximal such that $\min_{i \leq d}\{v(a_i(\beta-\alpha)^i)\} = v(a_m(\beta-\alpha)^m)$
- Define $\sigma := a_m(\beta-\alpha)^m$ and $Q(x) := \frac{1}{\sigma}P((\beta-\alpha)x+\alpha)$
- So $Q \in \mathscr{O}[x]$, and $v(Q(1)) > 0$ (since $Q(1) = \frac{1}{\sigma}P(\beta)$ and $v(P(\beta)) > v(a_m(x-\alpha)^m) = v(\sigma)$ by definition of collision).

We will attempt to find a root of a derivative of Q using Hensel's Lemma.

Direct computation of valuations shows that

$$v(Q^{(m)}(1)) = v\left(\frac{1}{\sigma}\sum_{i=m}^{d}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i 1^{i-m}\right) = v(m!) = 0$$

So, let $n < m$ be least with $v(Q^{(n+1)}(1)) = 0$.
Apply Hensel's Lemma to $Q^{(n)}(x)$ to find a root $u \in K$ of $Q^{(n)}$ with $\bar{u} = \bar{1}$.
The desired root of $P^{(n)}(x)$ is $\lambda := u(\beta-\alpha)+\alpha$. □

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In residue characteristic p, this proof breaks down because $v(m!) \neq 0$ if $m \geq p$.
However, it can be shown that if Hensel's Lemma cannot give us a root of Q, then the 'severity' of the collision is bounded:

Proposition

Assume $\mathrm{char}(R) = p > 0$. *If* $P(x)$ *has a collision at* β *around* α, *then either there exists a root* λ *of a derivative of* P *as in the previous Proposition, or there is an integer* $n \geq 0$ *such that* $\mathrm{rv}(x - \alpha) = \mathrm{rv}(\beta - \alpha)$ *implies*

$$\min_{i \leq d} \left\{ v(a_i (x - \alpha)^i) \right\} < v(P(x)) \leq \min_{i \leq d} \left\{ v(a_i (x - \alpha)^i) \right\} + v(n)$$

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However, it can be shown that if Hensel's Lemma cannot give us a root of Q, then the 'severity' of the collision is bounded:

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Assume $\mathrm{char}(R) = p > 0$. *If* $P(x)$ *has a collision at* β *around* α, *then either there exists a root* λ *of a derivative of* P *as in the previous Proposition, or there is an integer* $n \geq 0$ *such that* $\mathrm{rv}(x - \alpha) = \mathrm{rv}(\beta - \alpha)$ *implies*

$$\min_{i \leq d} \left\{ v(a_i(x-\alpha)^i) \right\} < v(P(x)) \leq \min_{i \leq d} \left\{ v(a_i(x-\alpha)^i) \right\} + v(n)$$

The next result uses this to decompose K into swiss cheeses on each of which $v(P(x))$ is simple, being a function of $v(x-\alpha)$ for some α. We again assume $\mathrm{char}(R) = 0$.

Proposition

Given $P(x) \in K[x]$, there exist (disjoint) swiss cheeses $V_1, \ldots, V_n$ partitioning K, and $\alpha_1, \ldots, \alpha_n \in K$ such that for all $x \in V_i$,

$$v(P(x)) = \min_{j \leq d} \left\{ v(a_{ij}(x-\alpha_i)^j) \right\},$$

with $P(x) = \sum_{j=0}^{d} a_{ij}(x-\alpha_i)^j$.

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Vaguely:

The proof proceeds by seeking out a collision, finding a root λ of a derivative $P^{(n)}$ near them, and recentering P as $\sum b_i(x-\lambda)^i$ there.

Since there are only finitely many such λ, this process must stop, at which point there can be no further collisions nearby.

In particular, the α_i in the proposition can be chosen from among the roots of derivatives $P^{(n)}$.

Finally, translating this into the language of leading terms gives

Proposition

Assume $\mathrm{char}(R) = 0$*. Let* $P(x) = \sum_{i=0}^{d} a_i (x-\alpha)^i \in K[x]$*. Then there are disjoint swiss cheeses* $V_1, \ldots, V_n$ *partitioning* K *and* $\alpha_1, \ldots, \alpha_n \in K$ *such that for all* $x \in V_i$*,*

$$\mathrm{rv}(P(x)) = \sum_{j=0}^{d} \mathrm{rv}(a_{ij})\,\mathrm{rv}(x-\alpha_i)^j$$

is well-defined. (As before $P(x) = \sum_{j=0}^{d} a_{ij}(x-\alpha_i)^j$*.)*

The proposition provides a method of pushing questions about the field into the RV structures.
Two applications illustrate how definability in characteristic 0 henselian fields is controlled by the leading terms:

- decidability relative to the leading terms
- a characterization of definable subsets of K in terms of definable subsets in the leading terms

Both require *relative quantifier elimination*, namely in our theory every formula is equivalent to one with no *field-sorted* quantifiers.

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Both require *relative quantifier elimination*, namely in our theory every formula is equivalent to one with no *field-sorted* quantifiers.

Start with a formula of the form (suppressing parameters and free variables, where φ is a definable predicate in the RV language and the f_i are polynomials):

$$\exists x \in K \left(\varphi\left(\mathrm{rv}(f_1(x)), \ldots, \mathrm{rv}(f_n(x))\right)\right)$$

$$\Updownarrow$$

$$\exists y_1, \ldots, y_m \in K \left(\bigwedge_i g_i(y_i) = 0 \ \wedge \ \exists x \left(\psi\left(\mathrm{rv}(x-y_1), \ldots, \mathrm{rv}(x-y_m)\right)\right) \right)$$

This can be translated into a formula of the above form using the decomposition. Now ψ is another predicate definable in RV and the g_i are polynomials with degrees $\leq \max\{\deg(f_i(x))\}$.

QE proof sketch

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Now, in the new formula, the $\exists x$ can be eliminated by replacing each $\mathrm{rv}(x - y_i)$ with a new bound RV-variable $\mathbf{z}_i$ (this is the base case for an induction, which we've omitted).

If the degrees of the g_i are actually *strictly less* than the degrees of the f_i, we can proceed inductively.

Otherwise, we finally have to turn to a more involved argument of making successive approximations and as before relying on the fact that we know only finitely many things can go wrong.

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The quantifier elimination proof yields an explicit process of producing a field-quantifier-free formula, so that

- Given a decision procedure for the RV structure, we have also a decision procedure for the valued field
- Alternatively, there is a decision procedure for characteristic 0 henselian fields given an oracle for the RV structure

Note that the same proofs will work under any expansion of the RV sorts.

This is especially useful because we may wish to

- add a cross section (to make RV bi-interpretable with $R \times V$)
- expand RV to $\mathrm{RV}^{\mathrm{eq}}$ (to work towards a relative elimination of imaginaries)

Holly proved that if K is algebraically closed, then every definable subset of K has a canonical presentation as a finite union of swiss cheeses.

Using the relative quantifier elimination and the swiss cheese decomposition, we can now show:

Theorem

Every definable subset of a characteristic 0 henselian field K can be written as

$$\{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x-\alpha_1), \ldots, \mathrm{rv}_{\delta_n}(x-\alpha_n) \rangle \in D\}$$

where D is definable in $\mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_n}$.

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Holly's swiss cheeses in ACVF can be seen as the combination of a pullback of a finite set (from the residue field) and an interval (the value group).

The pullback of an interval $[\gamma, \infty) \subseteq V$ to K produces a ball $B_{\geq\gamma}(0)$ around 0. Shifting to balls centered elsewhere can be taken as analogous to our shifting by $\langle \alpha_1, \ldots, \alpha_k \rangle$ in the theorem.

To obtain a one-dimensional elimination of imaginaries, Holly introduced new sorts for the balls. It follows by the same reasoning that:

Theorem

Definable subsets (in one variable) of henselian valued fields of characteristic 0 *are coded in the leading term language after expanding* RV *to* RV^{eq} *and adding new sorts for definable sets of the form*

$$\left\{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x-\alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x-\alpha_k)\rangle \in D\right\}.$$

Just as a key insight in proving elimination of imaginaries for ACVF came in finding the proper generalization of the balls to higher dimensions, it seems likely that the same thing may be true in our case...

It is an immediate consequence of the relative quantifier elimination that definable subsets of K^n all take the form

$$\{\langle x_1,\ldots,x_n\rangle \in K \mid \langle \mathrm{rv}_{\delta_1}(f_1(\bar{x})),\ldots,\mathrm{rv}_{\delta_k}(f_k(\bar{x}))\rangle \in E\}$$

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Therefore a trivial result:

Theorem

The theory of a henselian valued field (K, v) of characteristic 0 eliminates imaginaries in the leading term language expanded to include new sorts for sets of the form

$$\left\{\langle x_1, \ldots, x_n\rangle \in K \mid \langle \mathrm{rv}_{\delta_1}(f_1(\bar{x})), \ldots, \mathrm{rv}_{\delta_k}(f_k(\bar{x}))\rangle \in E\right\}.$$

Question

Is there a natural subclass $\mathscr{P}$ of polynomials over K such that in

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