

# A decomposition theorem for characteristic 0 henselian fields

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- ▶ If  $K$  is a field and  $\Gamma$  an ordered abelian group, a valuation is a homomorphism  $v : K^\times \rightarrow \Gamma$  satisfying  $v(x+y) \geq \min\{v(x), v(y)\}$ . This is generally extended to make  $v(0) = \infty$ .
- ▶ The valuation ring is  $R := \{x \in K \mid v(x) \geq 0\}$ .  $R$  is a local ring with maximal ideal  $\mathfrak{m} := \{x \in R \mid v(x) > 0\}$ .
- ▶ The residue field of  $K$  is  $k := R/\mathfrak{m}$ .
- ▶ We assume throughout that  $\text{char}(K) = \text{char}(k) = 0$ .

Valued fields have a topology defined by balls.

### Definition

A *ball* is a subset of  $K$  of the form  $K$ ,  $\emptyset$ ,  $B_{\geq\gamma}(\alpha)$ , or  $B_{>\gamma}(\alpha) = \{x \in K \mid v(x - \alpha) > \gamma\}$ .

One unusual feature to keep in mind is that any element of a ball can serve as the center of the ball, and that for any two balls which intersect, one must be contained in the other.

## Definition

$K$  is *henselian* if it satisfies Hensel's Lemma:

For every monic  $f(x) \in R[x]$  and  $a \in R$ , if  $v(f(a)) > 0$  and  $v(f'(a)) = 0$ , then there exists  $b \in R$  such that  $\bar{b} = \bar{a}$  and  $f(b) = 0$ .

Every valued field has a *henselization*, an extension field which is henselian and which embeds into any other henselian extension.

The henselization of  $K$  carries the same value group  $\Gamma$  and residue field  $k$  as  $K$ .

In particular, no assumptions can be made about the value group or residue field of an arbitrary henselian field.

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In particular, no assumptions can be made about the value group or residue field of an arbitrary henselian field.

Model theoretically, there are various languages in which one may talk about valued fields. For example, a three-sorted language  $K, k, \Gamma$ . But instead we will use a language based on leading term structures.

## Definition

The *leading term structure*  $\text{RV}$  of  $K$  is  $K^\times / (1 + \mathfrak{m})$ .

The *leading term* of  $x \in K^\times$  is the image  $\text{rv}(x)$  of  $x$  under the quotient map.

(One may also define higher order structures  $\text{RV}_\delta$  for  $0 \leq \delta \in \Gamma$  as  $K^\times / (1 + \mathfrak{m}_\delta)$ , where  $\mathfrak{m}_\delta = \{x \in K \mid v(x) > \delta\}$  (so  $\text{RV} = \text{RV}_0$ ). These will not be used in this talk, but would be needed in the mixed characteristic case.)

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## Addition in RV

Besides the multiplication, RV also inherits addition from  $K$ , but this addition is only partially defined.

### Proposition

*Suppose  $v(x+y) = \min\{v(x), v(y)\}$ . Then for all  $z$  such that  $rv(z) = rv(x)$ ,  $rv(z+y) = rv(x+y)$ .*

*Conversely, if  $v(x+y) > \min\{v(x), v(y)\} = v(x)$ , then there exists  $z$  such that  $rv(z) = rv(x)$  but  $rv(z+y) \neq rv(x+y)$ .*

Consequently, extend  $rv$  to define  $rv(0) = \infty$  and

$$rv(x) + rv(y) = \begin{cases} rv(x+y) & v(x+y) = \min\{v(x), v(y)\} \\ \infty & \text{otherwise.} \end{cases}$$

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RV can be seen as a way of wrapping up residue field and value group into a single structure, as in

## Fact

*For all nonzero  $x, y \in K$ , the following are equivalent:*

1.  $\text{rv}(x) = \text{rv}(y)$
2.  $v(x - y) > v(x)$
3.  $v(x) = v(y)$  *and*  $\text{res}(y/x) = 1$

However, the two-sorted RV language is somewhat weaker than the usual three-sorted one...

## Proposition

*The value group  $\Gamma$  and residue field  $k$  are interpretable in  $\text{RV}$ .*

*Conversely,  $\text{RV}$  is interpretable in  $k \times \Gamma$ .*

*However, we only get a bi-interpretation by adding a cross-section.*

## Proof.

Recall from above that  $\text{rv}(x) = \text{rv}(y) \Rightarrow v(x) = v(y)$ , so we may speak unambiguously of  $v(\mathbf{x})$  ( $\mathbf{x} \in \text{RV}$ ).

Observe that  $v(\mathbf{x}) > 0 \iff \mathbf{x} = \infty \vee \mathbf{x} + \text{rv}(1) = \text{rv}(1)$ . This may be used to define also “ $v(\mathbf{x}) = 0$ ”.

Now  $v(\mathbf{x}) = v(\mathbf{y}) \iff \exists \mathbf{c} \in \text{RV} (v(\mathbf{c}) = 0 \wedge \mathbf{x} = \mathbf{c}\mathbf{y})$ .

The proofs of the remaining statements are similarly elementary. □

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A *swiss cheese* is a subset of  $K$  of the form  $B \setminus (C_1 \cup \dots \cup C_n)$ , where  $B, C_1, \dots, C_n$  are all balls (or  $K$  itself), with  $C_i \subsetneq B$ .

We turn briefly to algebraically closed valued fields to cite:

## Theorem (Holly)

*Every definable subset of an algebraically closed valued field  $K$  can be expressed uniquely as a finite union of disjoint, non-trivially-nested swiss cheeses.*

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Holly's theorem is to a large extent a consequence of the quantifier elimination in algebraically closed valued fields. Since any field and ordered abelian group can be the residue field and value group of a henselian field, and RV interprets both, there is no hope for a general quantifier elimination theorem as in ACVF. Instead, we have relative results:

### Theorem (Kuhlmann)

*Let  $M$  be a henselian valued field of characteristic and residue characteristic 0.  $\text{Th}(M)$  eliminates field-sorted quantifiers in the language with sorts  $K$  and  $\text{RV}$ . (In the finitely ramified mixed characteristic case, sorts  $\text{RV}_\delta$  for  $\delta \in \{v(p^n) \mid n \in \mathbb{N}, p = \text{char}(k)\}$  are used.)*

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The goal is to prove some kind of relativized analogue of Holly's theorem.

In the relative setting, in place of balls and swiss cheeses we must consider pullbacks of arbitrary definable sets in  $RV^n$ . After recentering, we obtain every definable subset of  $K$ :

### Theorem

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$$\{x \in K \mid \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D\}$$

*with  $\alpha_1, \dots, \alpha_n \in K$ ,  $D$  a definable subset of  $RV^n$ .*

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## Valuations of polynomials

Ideally, the valuation of a polynomial will simply be given by the minimum of the valuations of its terms.

However, where multiple terms have the same valuation, a 'collision' can occur, making the valuation larger than would be expected.

Notice that this idea of collision is not intrinsic to the polynomial, but depends on how  $P$  is expanded as a sum of monomials. We make this more precise in

### Definition

$P(x)$  has a *collision at  $\beta$  around  $\alpha$*  if, for  $P(x) = \sum_{i=0}^d a_i(x - \alpha)^i$ ,  
 $v(P(\beta)) > \min_{i \leq d} \{v(a_i(\beta - \alpha)^i)\}$ .

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Collisions can only occur near a root of one of the (nonzero) derivatives of  $P$ .

### Proposition

Suppose  $P(x) = \sum_{i=0}^d a_i(x - \alpha)^i$  has a collision at  $\beta$  around  $\alpha$ .

Then there are  $n < d$  and  $\lambda \in K$  with

- (i)  $P^{(n)}(\lambda) = 0$ , and
- (ii)  $\text{rv}(\lambda - \alpha) = \text{rv}(\beta - \alpha)$ , and in particular,  
 $v(\lambda - \beta) > v(\beta - \alpha)$ .

## Proof.

- ▶ Let  $m$  be maximal such that  
$$\min_{i \leq d} \{v(a_i(\beta - \alpha)^i)\} = v(a_m(\beta - \alpha)^m)$$
- ▶ Define  $\sigma := a_m(\beta - \alpha)^m$  and  $Q(x) := \frac{1}{\sigma}P((\beta - \alpha)x + \alpha)$
- ▶ So  $Q \in R[x]$ , and  $v(Q(1)) > 0$  (since  $Q(1) = \frac{1}{\sigma}P(\beta)$  and  $v(P(\beta)) > v(a_m(x - \alpha)^m) = v(\sigma)$  by definition of collision).

We will attempt to find a root of a derivative of  $Q$  using Hensel's Lemma.

Direct computation of valuations shows that

$$v(Q^{(m)}(1)) = v\left(\frac{1}{\sigma} \sum_{i=m}^d \frac{i!}{(i-m)!} a_i (\beta - \alpha)^i 1^{i-m}\right) = 0$$

So, let  $n < m$  be least with  $v(Q^{(n+1)}(1)) = 0$ .

Apply Hensel's Lemma to  $Q^{(n)}(x)$  to find a root  $u \in K$  of  $Q^{(n)}$  with  $\bar{u} = \bar{1}$ .

The desired root of  $P^{(n)}(x)$  is  $\lambda := u(\beta - \alpha) + \alpha!$  □

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The next result uses this to decompose  $K$  into swiss cheeses on each of which  $v(P(x))$  is simple. Define

$$m(P, \alpha, S) := \max\{i \leq d \mid \exists x \in S \forall j \leq d v(a_i(x - \alpha)^i) \leq v(a_j(x - \alpha)^j)\}$$

where the  $a_i$  are the coefficients of the expansion of  $P$  around  $\alpha$ .

### Proposition

*Let  $P(x) \in K[x]$  and  $S$  be a swiss cheese in  $K$ . Then there exist (disjoint) sub-swiss cheeses  $V_1, \dots, V_k$  partitioning  $S$ , and  $\alpha_1, \dots, \alpha_k \in K$ , all algebraic over the coefficients of  $P$  and parameters defining  $S$ , such that for all  $x \in V_i$ ,*

$$v(P(x)) = v(a_{im_i}(x - \alpha_i)^{m_i}),$$

*with  $P(x) = \sum_{n=0}^d a_{in}(x - \alpha_i)^n$  and  $m_i = m(P, \alpha_i, V_i)$ .*

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For simplicity, assume  $S$  is a ball  $B_{\geq \gamma}(\alpha)$ . Let  $P(x) = \sum_{n=0}^d a_n(x - \alpha)^n$ , and set  $m = m(P, \alpha, S)$ . The proof proceeds by induction on  $m$ .

- ▶ If  $m = 0$ , then  $v(P(x)) = v(a_0)$  for all  $x \in S$ .
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The picture shows  $S$  broken up into three pieces:

$$S = B_{>\delta}(\alpha) \cup B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) \cup B_{\geq\gamma}(\alpha) \setminus B_{\geq\delta}(\alpha)$$

On the last of these,  $v(P(x)) = v(a_m(x - \alpha)^m)$ . On the first,  $m(P, \alpha, B_{>\delta}(\alpha)) < m$ , so that the induction hypothesis applies.

It therefore remains only to consider  $B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) =: C$ , i.e. when  $v(x - \alpha) = \delta$ .

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It therefore remains only to consider  $B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) =: C$ , i.e. when  $v(x - \alpha) = \delta$ .

- ▶ Let  $T := \{x \in C \mid v(P(x)) \neq v(a_m(x - \alpha)^m)\}$ , i.e.  $T$  is the set of elements of  $C$  at which  $P$  has a collision around  $\alpha$ .
- ▶  $T$  is the disjoint union of equivalence classes under the equivalence  $x \sim y \Leftrightarrow v(x - y) > \delta \Leftrightarrow \text{rv}(x - \alpha) = \text{rv}(y - \alpha)$ .
- ▶ By the preceding Proposition, each equivalence class  $[\beta]_{\sim}$  contains a root  $\lambda$  of a nonconstant derivative of  $P$ . (In particular, there are finitely many, and  $C \setminus T$  is a swiss cheese.)

Choose one such  $\lambda$  and ball  $D := B_{>\delta}(\lambda)$  of  $T$ .

Inside  $D$ , we shift  $P$  to look at  $P(x) = \sum_{i=0}^d b_i(x - \lambda)^i$ .

Interestingly,  $m(P, \lambda, D) \leq m(P, \alpha, S)$ , though equality may occur:

- ▶ If the inequality is strict, the induction hypothesis applies.
- ▶ If  $m(P, \lambda, D) = m(P, \alpha, S)$ , repeat the whole argument to shift to  $D' := B_{>\delta}(\lambda')$ . But inside  $D'$ , there are strictly fewer roots of  $P^{(n)}$ , so a second induction on the number of roots of derivatives of  $P$  saves the day, and we're done.



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Call  $P$  *linearly analyzable* in  $S$  if there is a term  $t[x_1, \dots, x_n]$  in RV (each  $x_i$  is a variable for the sort RV) and elements  $\alpha_1, \dots, \alpha_n \in K$  such that for every  $x \in S$ ,

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Let  $S$  be a swiss cheese and  $P(x) = \sum_{i=0}^d a_i(x - \alpha)^i \in K[x]$ . Then there are sub-swiss cheeses  $W_1, \dots, W_k$  partitioning  $S$  such that  $P(x)$  is linearly analyzable on each  $W_i$ .

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Now we're ready to make the reduction from formulas defining subsets of  $K$  down to equivalent formulas of a much simplified form.

Starting with a formula

$$\phi(x, \bar{a})$$

Using quantifier elimination, assume  $\phi$  is of the form

$$\bigvee \bigwedge (f(x) = 0 \wedge g(x) \neq 0 \wedge \langle \text{rv}(h_1(x)), \dots, \text{rv}(h_k(x)) \rangle \in D)$$

The  $f$ 's,  $g$ 's, and  $h$ 's are polynomials in  $K[x]$ ,  $D$ 's definable sets in  $\text{RV}$ .

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Since  $f(x) = 0 \wedge g(x) \neq 0$  defines a swiss cheese  $S$  in  $K$ , we rewrite this as

$$\bigvee \bigwedge (x \in S \wedge \langle \text{rv}(h_1(x)), \dots, \text{rv}(h_k(x)) \rangle \in D)$$

Now apply the partition given by the last theorem, which gave a swiss cheese decomposition such that  $\text{rv}(h_i(x))$  becomes  $t[x - \alpha_{i1}, \dots, x - \alpha_{in_i}]$  on one piece of the partition:

$$\bigvee \bigwedge (x \in S \wedge \langle t_1[\text{rv}(x - \alpha_{11}), \dots, \text{rv}(x - \alpha_{1n_1})], \dots, t_k[\text{rv}(x - \alpha_{k1}), \dots, \text{rv}(x - \alpha_{kn_k})] \rangle \in D)$$

(These are different  $S$ 's.)

Given

$$\bigvee \bigwedge (x \in S \wedge \langle t_1[\text{rv}(x - \alpha_{11}), \dots, \text{rv}(x - \alpha_{1n_1})], \dots, t_k[\text{rv}(x - \alpha_{k1}), \dots, \text{rv}(x - \alpha_{kn_k})] \rangle \in D)$$

The terms  $t_i$  can be considered part of the formula defining  $D$ :

$$\bigvee \bigwedge (x \in S \wedge \langle \text{rv}(x - \alpha_{11}), \dots, \text{rv}(x - \alpha_{1n_1}), \dots, \text{rv}(x - \alpha_{k1}), \dots, \text{rv}(x - \alpha_{kn_k}) \rangle \in D)$$

$$\bigvee \bigwedge (x \in S \wedge \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D)$$

We proved that  $\Gamma$  is interpretable in RV, so  $x \in S$  is actually another formula of the form  $\langle \text{rv}(x - \beta_1), \dots, \text{rv}(x - \beta_m) \rangle \in E$ . Therefore we can reduce to a big disjunction of conjunctions of the form

$$\bigvee \bigwedge \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D$$

Lastly notice that this  $\vee \wedge$  can also be thought of as part of the formula defining  $D$ . For example:

$$\langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D \wedge \langle \text{rv}(x - \beta_1), \dots, \text{rv}(x - \beta_m) \rangle \in E$$

is equivalent to

$$\langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n), \text{rv}(x - \beta_1), \dots, \text{rv}(x - \beta_m) \rangle \in (D \wedge E)$$

where if  $D$  is defined by  $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$  and  $E$  by  $\psi(\mathbf{x}_1, \dots, \mathbf{x}_m)$ , then  $D \wedge E$  will be defined by  $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n) \wedge \psi(\mathbf{x}_{n+1}, \dots, \mathbf{x}_{n+m})$ .

By this method we can assume our

$$\bigvee \bigwedge \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D$$

to actually be of the form

$$\langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D.$$

This finished the proof of the main theorem.

Holly's theorems, built around the quantifier elimination for ACVF, suggested a starting point for the elimination of imaginaries theorem of Haskell, Hrushovski, and Macpherson. In particular, she proved as a corollary of her swiss cheese decomposition theorem

### Theorem

*ACVF admits coding of 1-variable sets (that is, definable subsets of  $K$ ) in a language that contains additional sorts for the disks, sets of the form  $B_{\geq \gamma}(\alpha) = \{x \in K \mid v(x - \alpha) \geq \gamma\}$ .*

Similarly we may prove analogously

## Theorem

$\text{Th}(K, \text{RV})$  admits coding of 1-variable sets (that is, definable subsets of  $K$ ) in a language that contains additional sorts for sets of the form

$$\{x \in K \mid \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D\}$$

$(\alpha_i \in K, D \text{ definable in } \text{RV}^n)$ .

The hope would be that this could also serve as a starting pointing for an elimination of imaginaries in  $K$  relative to  $\text{RV}^{\text{eq}}$  and some additional structure.

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( $\alpha_i \in K$ ,  $D$  definable in  $\text{RV}^n$ ).

The hope would be that this could also serve as a starting point for an elimination of imaginaries in  $K$  relative to  $\text{RV}^{\text{eq}}$  and some additional structure.

## A relative elimination of imaginaries?

Using quantifier elimination and syntactical arguments as above it is easy to see that every definable subset of  $K^n$  has the form

$$\{(x_1, \dots, x_n) \in K \mid \langle \text{rv}(f_1(\bar{x})), \dots, \text{rv}(f_k(\bar{x})) \rangle \in D\}$$

with  $D$  definable in  $\text{RV}^k$ , and  $f_i \in K[x_1, \dots, x_n]$ .

As we are unable to say anything about the possible structure of  $D$ , it seems the natural way to proceed would be to give some restricted class of polynomials  $f_i(\bar{x})$  which suffices. This seems difficult.

Thank you to everyone at McMaster for inviting me to speak and for your hospitality!