Lines in $\mathbf{R}^n$

Definition 0.1. *Let $A \in \mathbf{R}^n$ be a point and $\mathbf{v} \in \mathbf{R}^n$ be a non-zero vector. The line through A in direction $\mathbf{v}$ is the set*

$$L = \{A + t\mathbf{v} \in \mathbf{R}^n : t \in \mathbf{R}\}.$$

Let us further say that two lines are *parallel* if the associated direction vectors are parallel (i.e. one is a scalar multiple of the other). In class I stated a criterion for deciding when two lines are the same. I restate this criterion in a slightly slicker form here and give a detailed proof. Note that the proof illustrates three common features of many proofs:

- proofs of 'if and only if' statements have two parts;
- the importance of relying on definitions (in this case of 'line' and 'parallel');
- how to show two sets (in this case L and $\tilde{L}$) are equal.

Proposition 0.2. *Two lines are the same if and only if they are parallel and have at least one point in common.*

Proof. To fix notation, I call the lines L and $\tilde{L}$; I let $A, \mathbf{v} \in \mathbf{R}^n$ be the point and vector that determine L; and I let $B, \mathbf{w} \in \mathbf{R}^n$ be the point and vector that determine $\tilde{L}$.

Suppose first that $L = \tilde{L}$. Then certainly L and $\tilde{L}$ have a point in common: for instance $A = A + 0\mathbf{v} \in L$, and since $L = \tilde{L}$ we also have $A \in \tilde{L}$. To see that the lines are parallel, note that (by the same argument) $A + \mathbf{v}$ is also a point in $\tilde{L}$. Therefore there are scalars $c_1, c_2 \in \mathbf{R}$ such that

$$A = B + c_1\mathbf{w} \text{ and } A + \mathbf{v} = B + c_2\mathbf{w}.$$

Subtracting the first equation from the second gives $\mathbf{v} = (c_2 - c_1)\mathbf{w}$. That is, $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$ and the two lines are parallel. This finishes the 'only if' part of the proof.

For the 'if' part we begin again, supposing instead that the lines are parallel and that there is at least one point $p \in L \cap \tilde{L}$. In other words, there are scalars $c_1, c_2, c_3 \in \mathbf{R}$ such that $\mathbf{v} = c_1\mathbf{w}$, $p = A + c_2\mathbf{v}$ and $p = B + c_3\mathbf{w}$.

Seeking to show $L \subset \tilde{L}$, I let $q \in L$ be any given point. That is, there is a scalar $c \in \mathbf{R}$ such that $q = A + c\mathbf{v}$. From the assumptions above, I can rewrite this as

$$q = (p - c_2\mathbf{v}) + c\mathbf{v} = (B + c_3\mathbf{w}) + (c - c_2)\mathbf{v} = B + (c_3 + c_1(c - c_2))\mathbf{w}.$$

In other words, $q \in \tilde{L}$, too. So $L \subset \tilde{L}$. In the same fashion, one shows that $\tilde{L} \subset L$. This proves that the lines are equal. $\square$