

# CREMONA TRANSFORMATIONS, SURFACE AUTOMORPHISMS AND THE GROUP LAW

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ABSTRACT. We give a method for constructing many examples of automorphisms with positive entropy on rational complex surfaces. The general idea is to begin with a quadratic cremona transformation that fixes some cubic curve and then use the group law to understand when the indeterminacy and exceptional behavior of the transformation may be eliminated by repeated blowing up.

## INTRODUCTION

Every automorphism of the complex projective plane  $\mathbf{P}^2$  is linear and therefore behaves quite simply when iterated. It is natural to seek other rational complex surfaces, for instance those obtained from  $\mathbf{P}^2$  by successive blowing up, that admit automorphisms with more interesting dynamics. Until recently, very few examples with positive entropy seem to have been known (see e.g. the introduction to [Can]).

Bedford and Kim [BK2] found some new examples by studying an explicit family of cremona transformations, i.e. birational self-maps of  $\mathbf{P}^2$ . McMullen [McM] gave a more synthetic construction of some similar examples. To this end he used the theory of infinite coxeter groups, some results of Nagata [Nag2, Nag1] about cremona transformations, and important properties of plane cubic curves. In this paper, we construct many more examples of positive entropy automorphisms on rational surfaces. Whereas [McM] seeks automorphisms with essentially arbitrary topological behavior, we limit our search to automorphisms that might conceivably be induced by cremona transformations of polynomial degree two (*quadratic transformations* for short). This restriction allows us to be more explicit about the automorphisms we find and to make do with less technology, using only the group law for cubic curves in place of coxeter theory and Nagata's theorems.

A quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  always acts geometrically by blowing up three points  $I(f) = \{p_1^+, p_2^+, p_3^+\}$  in  $\mathbf{P}^2$  and collapsing the lines joining them. We call the  $p_j^+$  *points of indeterminacy* and the lines  $\overline{p_i^+ p_j^+}$  *exceptional*. Typically, the points and the lines are distinct, but in general they can occur with multiplicity (see §1.2). Regardless,  $f^{-1}$  is also a quadratic transformation and  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$  consists of the images of the three exceptional lines.

Under certain fairly checkable circumstances a quadratic transformation  $f$  will lift to an automorphism of some rational surface  $X$  obtained from  $\mathbf{P}^2$  by a finite sequence of point

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blowups. Namely, suppose there are integers  $n_1, n_2, n_3 \in \mathbf{N}$  and a permutation  $\sigma \in \Sigma_3$  such that  $f^{n_j-1}(p_j^-) = p_{\sigma_j}^+$  for  $j = 1, 2, 3$ . Then<sup>1</sup> we can in effect cancel all indeterminate and exceptional behavior of  $f$  by blowing up the orbit segments  $p_j^-, f(p_j^-), \dots, f^{n_j-1}(p_j^-)$ . That is, if  $X$  is the rational surface that results from blowing up these segments, then  $f$  lifts to an automorphism  $\hat{f} : X \rightarrow X$ . General theorems of Gromov [Gro] and Yomdin [Yom] imply directly that the entropy of this automorphism is  $\log \lambda_1$ , where the *first dynamical degree*  $\lambda_1$  is the spectral radius of the induced pullback operator  $\hat{f}^*$  on  $H^2(X, \mathbf{R})$ .

Moreover, as Bedford and Kim observe (see the discussion surrounding Proposition 2.1), the action  $\hat{f}^*$  is entirely determined by the *orbit data*  $n_1, n_2, n_3, \sigma$ . Hence we say that  $\det(\hat{f}^* - \lambda \text{id})$  is the characteristic polynomial ‘for the orbit data  $n_1, n_2, n_3, \sigma$ .’ When the  $n_j$  are large enough (see [BK1, Theorem 5.1]), e.g.  $n_j \geq 3$  with at least one inequality strict, the characteristic polynomial has a root outside the unit disk, and hence  $\hat{f}$  has positive entropy.

Accordingly, one way to find positive entropy automorphisms induced by quadratic transformations would be to begin with some fixed quadratic transformation  $q$ , e.g.  $q(x, y) = (1/x, 1/y)$ , and look for  $T \in \text{Aut}(\mathbf{P}^2)$  such that  $f = T \circ q$  realizes the orbit data  $n_1, n_2, n_3, \sigma$ ; i.e. so that  $f^{n_j-1}(p_j^-) = p_{\sigma_j}^+$  for  $j = 1, 2, 3$ . This imposes essentially six conditions on  $f$ , so it seems plausible that some  $T$  in the eight parameter family  $\text{Aut}(\mathbf{P}^2)$  will serve. However, the degrees of the equations governing  $T$  increase exponentially with the  $n_j$ , and it therefore seems daunting to try to understand their solutions directly.

A key idea in [McM], which we follow here, is to look only at cremona transformations  $f$  that preserve some fixed cubic curve  $C$ . Various aspects of such transformations have been studied in several recent papers (e.g. [DJS, Pan1, Pan2, BPV]). We say that  $f$  *properly fixes*  $C$  if  $f(C) = C$  and no singular point of  $C$  is indeterminate for  $f$  or  $f^{-1}$ . Then  $f$  preserves both regular and singular points  $C_{reg}, C_{sing} \subset C$  separately, and degree considerations imply that  $I(f), I(f^{-1}) \subset C$ . As a Riemann surface, each component of  $C_{reg}$  is equivalent to  $\mathbf{C}/\Gamma$  for some (possibly rank 0 or 1) lattice  $\Gamma \subset \mathbf{C}$ . Hence  $f|_{C_{reg}}$  is covered by an affine transformation  $z \mapsto az + b$  of  $\mathbf{C}$ , where the *multiplier*  $a \in \mathbf{C}^*$  is admissible in the sense that  $a\Gamma = \Gamma$ . Our first result determines the prevalence and nature of the quadratic transformations that preserve a given cubic curve. For simplicity, we limit the statement here to irreducible cubics. See Propositions 1.2, 1.3 and Theorem 1.4 below for a more complete story.

**Theorem 1.** *Let  $C \subset \mathbf{P}^2$  be an irreducible cubic curve. Suppose we are given points  $p_1^+, p_2^+, p_3^+ \in C_{reg}$ , a multiplier  $a \in \mathbf{C}^*$ , and a translation  $b \in C_{reg}$ . Then there exists at most one quadratic transformation  $f$  properly fixing  $C$  with  $I(f) = \{p_1^+, p_2^+, p_3^+\}$  and  $f|_{C_{reg}} : z \mapsto az + b$ . This  $f$  exists if and only if the following hold.*

- $p_1^+ + p_2^+ + p_3^+ \neq 0$ ;
- $a$  is admissible for  $C_{reg}$ ;
- $a(p_1^+ + p_2^+ + p_3^+) = 3b$ ;

*Finally, the points of indeterminacy for  $f^{-1}$  are given by  $p_j^- = ap_j^+ - 2b$ ,  $j = 1, 2, 3$ .*

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<sup>1</sup>We assume here that the  $n_j$  are taken to be minimal. To keep the present discussion simple we also assume that  $f^k(p_j^-) \neq f^\ell(p_i^-)$  for any  $k, \ell \geq 0$  and  $i \neq j$ . We do not make the latter assumption outside this paragraph. See §2.1 for a more complete discussion.

Addition in the hypotheses and conclusions of this theorem is with respect to the natural group structure on  $C_{reg}$ . The condition  $\sum p_j^+ \neq 0$  is, by the group law, equivalent to saying that  $I(f)$  is not equal to the intersection of  $C$  with a line. The third item constrains the translation  $b$  for  $f|_{C_{reg}}$  up to addition of an inflection point on  $C_{reg}$ . We point out that several of the observations needed to prove Theorem 1 appear already in the paper [PS], which is concerned with a restricted version of the family treated in [BK2].

We apply Theorem 1 to study quadratic transformations that fix each of the three basic types of irreducible cubic, and in particular to identify those transformations that lift to automorphisms on some blowup of  $\mathbf{P}^2$ . For irreducible curves, the results of our analysis are as follows.

**Theorem 2.** *Let  $n_1, n_2, n_3 \in \mathbf{N}$ ,  $\sigma \in \Sigma_3$  be orbit data whose characteristic polynomial has a root outside the unit circle. Suppose that  $C$  is an irreducible cubic curve and  $f$  is a quadratic transformation that properly fixes  $C$  and realizes the orbit data. Then  $C$  is one of the following.*

- *The cuspidal cubic  $y = x^3$ .*
- *A torus  $\mathbf{C}/\Gamma$  with  $\Gamma = \mathbf{Z} + i\mathbf{Z}$  or  $\Gamma = \mathbf{Z} + e^{2\pi i/6}\mathbf{Z}$ .*

*Both cases occur, but only finitely many sets of orbit data can be realized in the second one.*

When  $C$  is a torus, the multiplier of the restriction  $f|_C$  is necessarily a root of unity. The problem with the nodal cubic and tori without additional symmetries is that the multiplier of a realization must be  $\pm 1$ , which implies (see Corollary 2.3 and Theorem 2.4) that all roots of the characteristic polynomial lie on the unit circle. In the case of tori with square or hexagonal symmetries, where multipliers can be  $i$  or  $e^{\pi i/3}$ , one actually can get realizations that lift to automorphisms with positive entropy. An interesting feature of these examples is that by passing to a fourth or sixth iterate, one obtains a positive entropy automorphism of a rational surface  $X$  that nevertheless fixes the original cubic curve  $C$  *pointwise*. We note that the group of cremona transformations fixing a cubic was considered in [Bla].

In general, realizations of orbit data by transformations whose multipliers are roots of unity seem to be somewhat sporadic, and we do not know how to characterize them in simple terms. We have a better understanding when the multiplier is not a root of unity.

**Theorem 3.** *Suppose in the previous theorem that the multiplier  $a$  of  $f|_{C_{reg}}$  is not a root of unity. Then*

- (1)  *$C$  is cuspidal;*
- (2)  *$a$  is a root of the characteristic polynomial for the given orbit data;*
- (3) *if  $n_1 = n_2 = n_3$ , then  $\sigma$  is the identity.*
- (4) *if  $n_i = n_j$  for  $i \neq j$ , then  $\sigma$  does not interchange  $i$  and  $j$ .*

*Conversely, when these conditions are met by  $C$  and  $a$ , there is a quadratic transformation  $f$ , unique up to conjugacy by a linear transformation fixing  $C$ , such that  $f$  realizes the given orbit data, properly fixes  $C$  and has multiplier  $a$  on  $C_{reg}$ . Consequently,  $f$  lifts to an automorphism on some rational surface  $\pi : X \rightarrow \mathbf{P}^2$  whose entropy is  $\log \lambda_1$ , where  $\lambda_1 > 1$  is Galois conjugate to  $a$ .*

This result is reminiscent of those proved in §7 of [McM]. In particular, the special cases discussed in §11 of that paper are included here. These fix a cusp cubic and realize orbit

data of the form  $n_1 = n_2 = 1$ ,  $n_3 \geq 8$ , with  $\sigma$  cyclic. On the other hand, some of the maps in Theorem 3 do not appear [McM]. For instance, when  $n_1 = n_2 = n_3 \geq 4$ ,  $\sigma = \text{id}$ ,  $I(f)$  degenerates to a single point, which is not permitted in McMullen's analysis. To use the terminology from [McM], coincidence of two points in  $I(f)$  implies the existence of a 'geometric nodal root' for the action  $\hat{f}^*$  of the induced automorphism.

We also consider quadratic transformations fixing reducible cubics  $C$ . If  $C$  is reducible with one singularity, then things turn out much as they did for the cuspidal cubic. The arguments used to prove Theorem 3 remain valid once one accounts for the facts that  $f$  permutes the components of  $C_{\text{reg}}$  and that this permutation must be compatible with the one prescribed in the given orbit data. The end result (Theorem 4.1) is that one can realize somewhat fewer, though still infinitely many, different sets of orbit data.

If  $C$  has two or three singular points, things turn out differently. Any quadratic transformation  $f$  that properly fixes  $C$  must have multiplier  $f|_{C_{\text{reg}}}$  equal to  $\pm 1$ . Nevertheless, by judiciously choosing the translations for  $f|_{C_{\text{reg}}}$  we are still able to realize infinitely many sets of orbit data. We treat the case  $\#C_{\text{sing}} = 3$  more thoroughly (see Theorem 4.4).

**Theorem 4.** *Let  $n_1, n_2, n_3 \geq 1$  and  $\sigma \in \Sigma_3$  be orbit data whose characteristic polynomial has a root outside the unit circle. If the orbit data is realized by some quadratic transformation  $f$  that properly fixes  $C = \{xyz = 0\}$ , then  $\sigma = \text{id}$ , and  $f$  maps each component of  $C_{\text{reg}}$  to itself with multiplier 1. Conversely, when  $\sigma = \text{id}$  and  $n_1, n_2, n_3 \geq 6$ , there exists at least one such realization.*

The proof amounts to an extended exercise in arithmetic mod 1. Unlike Theorem 3, the conclusion gives little idea of how many different realizations are possible. We simply show that for any given orbit data, there are finitely many quadratic transformations that might serve as realizations, and then we find one candidate from among these that works.

We deal more briefly with the case where  $C$  has two irreducible components meeting transversely, i.e.  $C = \{(xy - z^2)z = 0\}$ , showing that one can realize only two broad types of orbit data on this curve and then giving examples of each type.

The remainder of the paper is organized as follows. §1 provides background on plane cubics and quadratic transformations. It culminates in the proof of Theorem 1. §2 begins by considering when and how a quadratic transformation can be lifted to an automorphism  $\hat{f} : \hat{X} \circlearrowleft$ . It then discusses the nature of the associated operator  $\hat{f}^* : H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathbf{R})$ , which can be written down very explicitly and fairly simply in terms of the given orbit data. In §3 we seek automorphisms induced by quadratic transformations that properly fix irreducible cubics, and in §4 we treat the reducible case.

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## 1. QUADRATIC TRANSFORMATIONS FIXING A CUBIC

In this section, we recount some well-known facts about cubic curves and quadratic Cremona transformations in the plane. Then we characterize those quadratic transformations that 'properly' fix a given cubic. We point the reader to the recent article [DC] for a more thorough discussion of quadratic transformations.

**1.1. The group law on plane cubics.** Let  $C \subset \mathbf{P}^2$  be a *cubic curve*; that is,  $C$  is defined by a degree three homogeneous polynomial without repeated factors. The smooth part  $C_{reg}$  consists of at most three connected components, each isomorphic as a Riemann surface to the identity component  $\text{Pic}_0(C)$  of the picard group  $\text{Pic}(C)$ . That is, each is isomorphic to  $\mathbf{C}/\Gamma$  for some lattice  $\Gamma \subset \mathbf{C}$ . When  $C = C_{reg}$  is smooth  $\Gamma = \mathbf{Z} \oplus \tau\mathbf{Z}$  for some  $\tau \in \mathbf{C}$  with  $\text{Im } \tau > 0$ ; when the singularities of  $C$  are nodes  $\Gamma = \mathbf{Z}$ ; otherwise  $\Gamma$  is trivial.

In all cases, including  $\text{Pic}_0(C) \cong \mathbf{C}^*$ , we use ‘+’ to denote the group operation on  $\text{Pic}_0(C)$ . In what follows we will fix the identification  $C_{reg} \rightarrow \text{Pic}_0(C)$  and write  $p \sim q$  when  $p, q \in C_{reg}$  are identified with the same point in  $\text{Pic}_0(C)$ . For  $C$  irreducible, this is the same as saying  $p = q$ , but for reducible  $C$  it will allow us to state formulas relating points on different components of  $C_{reg}$ . The classical ‘group law’ for cubic curves says that we can (and will) choose our identification so that three points  $p_1, p_2, p_3 \in C_{reg}$  comprise the intersection of  $C$  with a line  $L \subset \mathbf{P}^2$  if and only if  $p_1 + p_2 + p_3 \sim 0$  (in  $\text{Pic}_0(C)$ ) and each irreducible component  $V \subset C$  contains precisely  $\deg V$  of the points  $p_j$ . More generally,  $3d$  points in  $C_{reg}$  comprise the intersection of  $C$  with a curve of degree  $d$  if and only if they sum to 0 and each component  $V \subset C_{reg}$  contains  $d \cdot \deg V$  of them.

In order to underline the explicit nature of the constructions we will make below, we point out that as long as  $C$  has at least one singularity, good identifications  $C_{reg} \rightarrow \text{Pic}_0(C)$  can be had quite simply. For instance, when  $C$  is cuspidal, we can choose coordinates on  $\mathbf{P}^2$  so that  $C_{reg} = \{y = x^3 : x \in \mathbf{C}\}$ , and define  $C_{reg} \rightarrow \mathbf{C}$  by  $(x, x^3) \mapsto x$ . Or if  $C$  is a union of a conic and a secant line, then we may assume  $C = \{z(z^2 - xy) = 0\}$  and identify points  $[1 : -t : 0] \in \{z = 0\}$  and  $[t^2 : 1 : t] \in \{z(z^2 - xy)\}$  with  $t \in \mathbf{C}^*$ .

We will say that  $T \in \text{Aut}(\mathbf{P}^2)$  *fixes* (or *leaves invariant*)  $C$  if  $T(C) = C$  as sets. Then we have for each connected component  $V \subset C_{reg}$  that  $T|_V : V \rightarrow T(V)$  is given by  $T(z) \sim a_V z + b_V$  for some *multiplier*  $a_V \in \mathbf{C}^*$ , and *translation*  $b_V \in \text{Pic}_0(C)$ . If  $V \cong \mathbf{C}/\Gamma$ , then necessarily  $\Gamma$  is invariant under multiplication by  $a_V$ . We call such  $a_V$  *admissible* for  $C$ . Explicitly, the admissible multipliers  $a \in \mathbf{C}^*$  are as follows.

- if  $C$  smooth and irreducible,  $a = \pm 1$  generically, but  $a = i^k$  when  $C = \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$ , and  $a = e^{\pm\pi ik/3}$  when  $C = \mathbf{C}/(\mathbf{Z} + e^{\pi i/3}\mathbf{Z})$ ;
- if  $C$  has nodal singularities,  $a = \pm 1$ ;
- in all other cases, arbitrary  $a \in \mathbf{C}^*$  are admissible.

**Proposition 1.1.** *Let  $T \in \text{Aut}(\mathbf{P}^2)$  be a linear transformation fixing a cubic curve  $C$ . Then*

- *The multiplier for  $T$  is the same on every irreducible component of  $C$ ;*
- $\sum_{V \subset C} (\deg V) b_V \sim 0$ .

*Conversely, given a multiplier  $a \in \mathbf{C}^*$  admissible for  $C$  and translations  $b_V \in \text{Pic}_0(c)$  satisfying the second condition, there exists a unique  $T \in \text{Aut}(\mathbf{P}^2)$  fixing each component  $V \subset C$  with multiplier  $a$  and translation  $b_V$ .*

In the irreducible case, the condition on the translations may be stated more simply by saying that the translation corresponds to an inflection point of  $C_{reg}$ . In the smooth case, this leaves nine possibilities for  $b$ , in the nodal three, and in the cuspidal only  $b = 0$ .

*Proof.* Let  $L \subset \mathbf{P}^2$  be a line intersecting  $C$  in three distinct points  $x, y, z$ . Then all three are in  $C_{reg}$  and hence  $x + y + z \sim 0$ . If  $T \in \text{Aut}(\mathbf{P}^2)$  fixes  $C$ , then  $T$  sends lines to lines and

each component of  $C_{reg}$  to another component of the same degree. Hence

$$a_x x + b_x + a_y y + b_y + a_z z + b_z \sim 0,$$

where  $a_x$  is the multiplier and  $b_x$  the translation for the restriction of  $T$  to the component of  $C_{reg}$  containing  $x$ . In particular, the left side is constant as the line  $L$  varies among lines through  $x$ . This implies that  $a_x = a_y = a_z$ . Therefore in fact  $b_x + b_y + b_z \sim 0$ . This proves the first assertion in the theorem.

Consider now the existence part of the second assertion. Note that if we are given an (abstract) automorphism  $T : C \rightarrow C$  satisfying the two conditions of the Proposition, then the above computations show that if  $x, y, z \in C_{reg}$  lie on a line  $L \not\subset C$ , then  $T^{-1}(x), T^{-1}(y), T^{-1}(z)$  also lie on a line ' $T^*L$ ' not contained in  $C$ . Since  $T$  is holomorphic and injective, and  $L \in (\mathbf{P}^2)^*$  depends holomorphically on  $x, y, z$ , we have that  $T^* : (\mathbf{P}^2)^* \setminus S \rightarrow (\mathbf{P}^2)^* \setminus S$  is an injective holomorphic map off the finite set  $S \subset (\mathbf{P}^2)^*$  of lines contained in  $C$ . Clearly  $T^*$  extends continuously to any  $L \subset C$  by setting  $T^*L$  equal to  $T^{-1}(L) \subset C$ . Hence  $T^* : (\mathbf{P}^2)^* \rightarrow (\mathbf{P}^2)^*$  is a well-defined automorphism. We therefore extend the abstract automorphism  $T : C \rightarrow C$  to the planar automorphism  $T : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  dual to  $T^*$ .

For uniqueness, we note that any extension of the given  $T : C \rightarrow C$  to  $\mathbf{P}^2$  must satisfy  $T^{-1}(L) = T^*L$ , where  $T^*$  is as in the previous paragraph. Hence  $T : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  must be the transformation dual to  $T^*$ .  $\square$

**1.2. Quadratic cremona transformations.** The most basic non-linear cremona (i.e. birational) transformation  $q : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  can be expressed in homogeneous coordinates as  $[x : y : z] \mapsto [yz : zx : xy]$ . Geometrically,  $q$  acts by blowing up the points  $[0 : 0 : 1]$ ,  $[0 : 1 : 0]$ ,  $[1 : 0 : 0]$  and then collapsing the lines  $\{x = 0\}$ ,  $\{y = 0\}$ ,  $\{z = 0\}$  that join them. Almost any other quadratic cremona transformation can be obtained from  $q$  by pre- and post- composing with linear transformations  $f = L \circ q \circ L'$ .

In fact, every quadratic transformation (we henceforth omit the word 'cremona')  $f$  can be obtained geometrically by blowing up three points  $p_1^+, p_2^+, p_3^+$  and collapsing three rational curves. We call the  $p_j^+$  *indeterminate* for  $f$  and let  $I(f)$  denote the set they comprise. We call the contracted curves *exceptional* for  $f$ . If  $f$  is a quadratic transformation, then so is  $f^{-1}$ , and we have  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ , where each  $p_j^-$  is the image of one of the exceptional curves for  $f$ . The indices 1, 2, 3 assigned to points in  $I(f)$  naturally determine an indexing of the points in  $I(f^{-1})$ . In the situation of the previous paragraph, this is given by declaring  $p_j^-$  to be the image of the exceptional line that does not contain  $p_j^+$ . In the sequel, however, we must allow our quadratic transformations to be degenerate, so we briefly review the three possibilities for the geometry of a quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ .

- *Generic case.* The points  $p_1^+, p_2^+, p_3^+ \in \mathbf{P}^2$  are distinct. They are all blown up (in any order) and the lines joining them are then contracted.
- *Generic degenerate case.* We have  $p_i^+ = p_j^+ \neq p_k^+$  for  $\{i, j, k\} = \{1, 2, 3\}$ . In this case, there is an exceptional line  $E_j^-$ , joining  $p_i^+$  and  $p_k^+$ , and another exceptional line  $E_k^-$  containing  $p_i^+$ . First,  $f$  blows up  $p_i^+$  and  $p_k^+$ , creating new rational curves  $E_i$  and  $E_k^+$ . Then  $f$  blows up the point  $E_k^- \cap E_i$  (which lies over  $p_j^+$ ). Next  $f$  contracts  $E_j^-$ ; finally  $f$  contracts  $E_i$  and  $E_k^-$ .

- *Degenerate degenerate case.* We have  $p_1^+ = p_2^+ = p_3^+$ . There is a single exceptional line  $E_k^- \subset \mathbf{P}^2$  containing  $p_i^+$ . The transformation blows up  $p_i^+$  creating a curve  $E_i$ , then blows up  $E_k^- \cap E_i$  creating  $E_j$ , and finally blows up some point on  $E_j$  different from  $E_j \cap E_k^-$  creating a curve  $E_k^+$ ; to descend back to  $\mathbf{P}^2$ ,  $f$  contracts  $E_k^-, E_j$  and  $E_i$  in order.

In the degenerate cases, we will readily abuse notation by treating e.g.  $p_k^+$  as a point in  $\mathbf{P}^2$  and also identifying it with the infinitely near point that is blown up to create  $E_k^+$ . In the first sense  $I(f)$  contains no more than three points, but in the second sense it always contains exactly three. The important thing is that in either sense, the points in  $I(f^{-1})$  are indexed so that  $p_k^-$  is the image of  $E_k^-$  after contraction. We note also that in each of the three cases, the geometry of  $f$  and  $f^{-1}$  is the same, so that  $p_j^+$  is infinitely near to  $p_i^+$  if and only if  $p_j^-$  is infinitely near to  $p_i^-$ , and  $\#I(f) = \#I(f^{-1})$  as sets in  $\mathbf{P}^2$ . In order to avoid tedious case-by-case exposition in this paper, we will generally give complete arguments only for the generic case where the points  $p_j^+$  are distinct, attending to details of the other cases only when they are conceptually different.

Given a curve  $C \subset \mathbf{P}^2$  and a quadratic transformation  $f$ , we define  $f(C) := \overline{f(C \setminus I(f))}$  to be the proper transform of  $C$  by  $f$ . When  $C \cap I(f) = \emptyset$ , we have that  $\deg f(C) = 2 \deg C$ . In general  $\deg f(C) = 2 \deg C - \sum_{p \in I(f)} \nu_p(C)$ , where  $\nu_p(C)$  is the multiplicity of  $C$  at  $p$ . Note that if  $p$  is infinitely near, appearing only in some modification  $\pi : X \rightarrow \mathbf{P}^2$ , then we take  $\nu_p(C)$  to be the multiplicity at  $p$  of the proper transform of  $C$  by  $\pi^{-1}$ .

We will say that  $C$  is *fixed* or *invariant* by  $f$  if  $f(C) = C$ . We will further say that  $C$  is *properly fixed* by  $f$  if additionally all points in  $I(f) \cap C$  and  $I(f^{-1}) \cap C$  are regular for  $C$ . In this case, we have that  $f$  permutes the singular points of  $C$ , preserves their type and restricts to a well-defined automorphism  $f : C_{reg} \rightarrow C_{reg}$ . It is a consequence of results in [DJS] that if  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  is a quadratic transformation properly fixing a curve  $C$ , then  $\deg C \leq 3$ , with equality if and only if  $I(f), I(f^{-1}) \subset C$ .

**1.3. Quadratic transformations fixing a cubic.** The starting point for our work is the following simple result, which gives us many quadratic transformations preserving a given cubic.

**Proposition 1.2.** *Let  $C \subset \mathbf{P}^2$  be a cubic curve. Then for any  $p_1^+, p_2^+, p_3^+ \in C_{reg}$ , there exists a quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  such that  $f(C) = C$  and  $I(f) = \{p_1^+, p_2^+, p_3^+\}$  if and only if  $\sum p_j^+ \not\sim 0$  and no line in  $C$  contains two of the  $p_j^+$ . The transformation  $f$  is completely determined by the points  $p_j^+$  and the induced automorphism  $f|_C$ .*

*Proof.* First we deal with existence of  $f$  in the case where the points  $p_j^+$  are all distinct. In this case, we let  $g$  be the map (defined up to post-composition by a planar automorphism) obtained by blowing up  $p_1^+, p_2^+, p_3^+$  and contracting the lines joining them. The condition  $p_1^+ + p_2^+ + p_3^+ \not\sim 0$  is equivalent to saying that the three exceptional lines are distinct. Comparing degrees  $\deg g(C) = 2 \deg C - 3 = 3$  shows that  $g(C)$  is also a cubic curve. That no line in  $C$  contains two of the  $p_j^+$  is equivalent to saying that no irreducible component of  $C$  is collapsed by  $g$ . Hence  $g$  induces a bijection between irreducible components of  $C$  and  $g(C)$ . Furthermore, each exceptional line intersects  $C$  in exactly one point  $p$  (counting multiplicity) that is not indeterminate for  $g$ . It follows that  $p$  is regular for  $C$  and  $f(p)$  is

regular for  $f(C)$ . As  $p_1^+, p_2^+, p_3^+$  are also all regular for  $C$ , we see that  $g(C)$  has the same singularities as  $C$ . That is,  $C$  and  $g(C)$  are isomorphic as abstract analytic sets. We can therefore find  $L \in \text{Aut}(\mathbf{P}^2)$  such that  $f := L \circ g$  leaves  $C$  invariant.

The case where one or more points in  $\{p_1, p_2, p_3\}$  coincide is similar, though the construction of  $g$  requires iterated blowing up and is therefore a bit more elaborate. The main thing is that one should always blow up points that lie on the proper transform of  $C$  under the previous blowups, so that  $\deg g(C) = 3$  goes through as before. The genericity assumptions concerning the  $p_j$  are again necessary and sufficient for the construction to succeed.

Suppose now that  $\tilde{f}$  is another quadratic transformation with  $I(\tilde{f}) = I(f)$  and  $\tilde{f}|_C = f|_C$ . Then  $f \circ \tilde{f}^{-1}$  is an automorphism of  $\mathbf{P}^2$  fixing  $C$  pointwise. So the uniqueness assertion in Proposition 1.1 implies that  $f \circ \tilde{f}^{-1} = \text{id}$  on all of  $\mathbf{P}^2$ . That is,  $f = \tilde{f}$  is unique.  $\square$

Let us consider the transformations  $f$  from Proposition 1.2 more closely. The group law on  $C$  gives us some helpful relationships between indeterminacy sets  $I(f)$ ,  $I(f^{-1})$  and the induced automorphism  $f|_{C_{reg}}$ . Note that as elsewhere, we describe the behavior of  $f|_{C_{reg}}$  in terms of the identification  $C_{reg} \rightarrow \text{Pic}_0(C)$ . This is essential since, among other things, we will derive equations relating points that lie on different components of  $C_{reg}$ .

**Proposition 1.3.** *Let  $C$ ,  $p_j^+ \in C_{reg}$ , and  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  be as in Proposition 1.2. For each component  $V \subset C_{reg}$ , let  $f|_V : V \rightarrow f(V)$  be given by  $f(z) \sim a_V z + b_V$ . Then the multiplier  $a = a_V$  for  $f|_V$  is independent of  $V$ . Also,*

- $\sum p_j^- \sim -\sum_{V \subset C} b_V \deg V$ ;
- $\sum p_j^+ \sim a^{-1} \sum b_V \deg V$ ;
- $p_j^- - a p_j^+ \sim b_j - \sum b_V \deg V$ , where  $b_j$  is the translation for  $f$  on the component containing  $p_j^+$ .

In particular  $\sum b_V \deg V \not\sim 0$ .

*Proof.* Given  $p_j^+ \in I(f)$ , let  $L$  be a non-exceptional line meeting  $C$  in three distinct points  $p_j^+, x, y$ . Then neither  $x$  nor  $y$  is indeterminate for  $f$  or singular for  $C$ . Thus  $L$  meets the exceptional component that maps to  $p_j^-$  in exactly one point, and we infer that  $f(x), f(y), p_j^- \in f(L)$ . As  $\deg f(L) = 2 \deg L - 1 = 1$ , we see that  $f(x) + f(y) + p_j^- \sim 0$ . That is,

$$a_x x + a_y y + b_x + b_y \sim -p_j^-,$$

where  $a_x, a_y$  and  $b_x, b_y$  are the multipliers and translations for  $f$  on the components of  $C_{reg}$  containing  $x$  and  $y$ . Since  $y \sim -p_j^+ - x$ , we see that the left side of this equality will be constant as  $x$  (i.e.  $L$ ) varies if and only if  $a_x = a_y$ . If  $C$  consists of a line and a conic, then we can take  $p_j^+$  on the conic and conclude that the multiplier for  $f|_C$  does not depend on the component. If  $C$  consists of three lines, we can reach the same conclusion by taking  $p_j^+$  from each component in turn. This proves that the multipliers for  $f|_C$  do not depend on components. The last item follows immediately.

To prove the remaining items, we choose a line  $L$  disjoint from  $I(f)$  that intersects  $C$  in three distinct points  $x, y, z$ . Then  $x + y + z \sim 0$  and  $L$  intersects each exceptional line in exactly one point (suitably interpreted when  $\#I(f) < 3$ ). Hence  $f(L)$  is a conic containing

$I(f^{-1})$ , and we deduce that

$$0 \sim f(x) + f(y) + f(z) + \sum p_j^- \sim a(x+y+z) + b_x + b_y + b_z + \sum p_j^- \sim b_x + b_y + b_z + \sum p_j^-.$$

Since each component  $V \subset C$  contains exactly  $\deg V$  of the points  $x, y, z$ , we obtain the desired formula for  $\sum p_j^-$ . The formula for  $\sum p_j^+$  is a consequence of the other two items.

To finish, we observe that the hypothesis  $\sum p_j^+ \not\sim 0$  and the second item imply  $\sum b_V \not\sim 0$ .  $\square$

The identities in Proposition 1.3 are essentially the only ones that hold in general.

**Theorem 1.4.** *Let  $C \subset \mathbf{P}^2$  be a cubic curve,  $a \in \mathbf{C}^*$  be an admissible multiplier for  $C$  and  $b_V \in V$  be translations for each component  $V \subset C$ .*

- *If  $\{p_1^+, p_2^+, p_3^+\} \subset C$  satisfy the hypotheses of Proposition 1.2 and  $\sum p_j^+ \sim a^{-1} \sum b_V \deg V$ , then there exists a unique quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  properly fixing  $C$  with multiplier  $a$ , translations  $b_V$  and  $I(f) = \{p_1^+, p_2^+, p_3^+\}$ .*
- *If, alternatively,  $\{p_1^-, p_2^-, p_3^-\} \subset C$  satisfy the hypotheses of Proposition 1.2 and  $\sum p_j^- \sim -\sum b_V \deg V$ , then there exists a unique quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  properly fixing  $C$  with multiplier  $a$ , translations  $b_V$  and  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ .*

*Proof.* We prove only the first assertion. The second follows on applying the first to construct  $f^{-1}$ . By Proposition 1.2 there exists a quadratic transformation  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  fixing  $C$  with  $I(f) = \{p_1^+, p_2^+, p_3^+\}$ . Let  $\tilde{a}$  be the multiplier and  $\tilde{b}_V \in V$  the translations for  $f$ . Since admissible multipliers form a group, we have that  $a\tilde{a}^{-1} \in \mathbf{C}^*$  is admissible. By the last item in Proposition 1.3, we have that  $\sum (b_V - \tilde{b}_V) \deg V \sim 0$ . Hence by Proposition 1.1 there exists  $T \in \text{Aut}(\mathbf{P}^2)$  fixing  $C$  with multiplier  $a\tilde{a}^{-1}$  and translations  $b_V - \tilde{b}_V$ . Replacing  $f$  with  $T \circ f$ , we obtain a quadratic transformation with the desired properties.

For uniqueness, suppose that  $h$  is another quadratic transformations with these properties. Then since  $I(h) = I(g)$ , we have that  $T := h \circ g^{-1}$  is a planar automorphism with  $T|_C = \text{id}$ . It follows that  $T = \text{id}$  on  $\mathbf{P}^2$ . Hence  $h = g$ .  $\square$

We close this section with a remark concerning the final assertion from Proposition 1.2. If the cubic  $C$  is singular, then it is possible to write down algebraic formulas for those quadratic transformations  $f$  properly fixing  $C$  (see [Jac] for some of these). However, these tend to be quite long, and it seems to us preferable in many instances to take a more algorithmic point of view. Namely, if  $p \in \mathbf{P}^2$  is a point outside  $C$  and not lying on an exceptional curve, then for any  $p_j^+ \in I(f)$ , the line  $L$  joining  $p$  and  $p_j^+$  meets  $C_{\text{reg}}$  in two more points  $x$  and  $y$ . Additionally, the exceptional line that maps to  $p_j^-$  meets  $L$  in a point  $q$ . The image  $f(L)$  is therefore also a line, and it passes through  $f(x)$ ,  $f(y)$ , and  $f(q) = p_j^-$ . These last three points are determined by  $I(f)$  and  $f|_C$ . So we can find  $f(L)$  explicitly. Since  $f|_L : L \rightarrow f(L)$  is a map between copies of  $\mathbf{P}^1$ , and we know the images of three distinct points under  $f|_L$ , we can find an explicit formula for  $f|_L$ . This allows us to find  $f(p) = f|_L(p)$ .

## 2. AUTOMORPHISMS FROM QUADRATIC TRANSFORMATIONS

In this section, we consider the issue of when and how a quadratic transformation will lift to an automorphism on some blowup of  $\mathbf{P}^2$ . We also consider the linear pullback actions induced by such automorphisms. Several of the results here are assembled from other places and restated in a form that will be convenient for us.

**2.1. Lifting to automorphisms.** Let us first describe the precise situation and manner in which a quadratic transformation  $f$  can be lifted to an automorphism on a rational surface  $X$  obtained from  $\mathbf{P}^2$  be a sequence of blowups (see [BK2] and [DF] for more on this). Suppose that there exists  $n_1 \in \mathbf{N}$  and  $\sigma_1 \in \{1, 2, 3\}$  such that  $f^{n_1-1}(p_1^-) = p_{\sigma_1}^+$ . Relabeling the points  $p_j^-$  and changing the index  $\sigma_1$  if necessary, we may further assume that

- $n_1$  is minimal, i.e.  $f^j(p_1^-) \notin I(f) \cap I(f^{-1})$  for any  $0 < j < n_1 - 1$ , and  $p_1^- \in I(f)$  only if  $n_1 = 1$ ;
- $p_1^-$  is not infinitely near to some other point in  $I(f^{-1})$ ;
- $p_{\sigma_1}^+$  is not infinitely near to some other point in  $I(f)$ .

Then by blowing up the points  $p_1^-, \dots, f^{n_1-1}(p_1^-)$ , we obtain a rational surface  $X_1$  to which  $f$  lifts as a birational map  $f_1 : X_1 \rightarrow X_1$  with only two points (counting multiplicity)  $p_2^-, p_3^- \in I(f_1^{-1})$ . If then  $f_1^{n_2-1}(p_2^-) = p_{\sigma_2}^+$  for some  $n_2 \in \mathbf{N}$  and  $\sigma_2 \neq \sigma_1$ , then we can repeat this process obtaining a map  $f_2 : X_2 \rightarrow X_2$  with only one point  $p_3^- \in I(f_2^{-1})$ . If finally  $f_2^{n_3-1}(p_3^-) = p_{\sigma_3}^+$ , then we blow up along this last orbit segment and arrive at a genuine automorphism  $\hat{f} : X \rightarrow X$ . We call the integers  $n_1, n_2, n_3 \geq 1$  together with the permutation  $\sigma \in \Sigma_3$  the *orbit data* associated to  $f$ , noting that the surface  $X$  is completely determined by the orbit data and the points  $p_j^- \in I(f^{-1})$ . Conversely, we say that the quadratic transformation  $f$  *realizes* the orbit data  $n_1, n_2, n_3, \sigma$ . It follows from general theorems of Yomdin and Gromov (see e.g. [Can]) that the topological entropy of any automorphism  $\hat{f} : X \rightarrow X$  of a rational surface  $X$  is  $\log \lambda_1$ , where  $\lambda_1$  is the largest eigenvalue of the induced linear operator  $\hat{f}^* : H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathbf{R})$ . If  $\hat{f}$  is the lift of a quadratic transformation as in the previous paragraph, then it is not difficult to describe  $\hat{f}^*$  explicitly. Let  $H \in H^2(X, \mathbf{R})$  be the pullback to  $X$  of the class of a generic line in  $\mathbf{P}^2$ . Let  $E_{i,n} \in H^2(X)$ ,  $0 \leq n \leq n_i - 1$  be the class of the exceptional divisor<sup>2</sup> associated to the blowup of  $f^n(p_i^-)$ . Then  $H$  and the  $E_{i,n}$  give a basis for  $H^2(X, \mathbf{R})$  that is orthogonal with respect to intersection and normalized by  $H^2 = 1$ ,  $E_{i,n}^2 = -1$ . Under  $\hat{f}^*$  we have

$$\begin{aligned} H &\mapsto 2H - E_{1,n_1-1} - E_{2,n_2-1} - E_{3,n_3-1} \\ E_{i,n} &\mapsto E_{i,n-1}, \text{ for } 1 \leq n \leq n_i - 1; \end{aligned}$$

and under  $\hat{f}_* = (\hat{f}^*)^{-1}$  we have

$$\begin{aligned} H &\mapsto 2H - E_{1,0} - E_{2,0} - E_{3,0} \\ E_{i,n-1} &\mapsto E_{i,n}, \text{ for } 1 \leq n \leq n_i - 1; \end{aligned}$$

Hence we arrive at

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<sup>2</sup>Note that this will sometimes be reducible if there are infinitely near points blown up in constructing  $X$

**Proposition 2.1.** *With the notation above, we have  $\hat{f}^* = S \circ Q$ , where  $Q : H^2(X) \rightarrow H^2(X)$  is given by*

$$Q(H) = 2H - E_{1,0} - E_{2,0} - E_{3,0}, \quad Q(E_{i,0}) = H - \sum_{j \neq i} E_{j,0}, \quad Q(E_{i,n}) = E_{i,n} \text{ for } n > 0;$$

and  $S$  fixes  $H$  and permutes the  $E_{i,j}$  according to

$$E_{\sigma_i,0} \mapsto E_{i,n_i-1}, \quad E_{i,n} \mapsto E_{i,n-1} \text{ for } n < n_i - 1.$$

The characteristic polynomial  $P(\lambda)$  for  $\hat{f}^*$  has at most one root outside the unit circle, and if it exists this root is real and positive. Moreover, every root  $\lambda = a$  of  $P(\lambda)$  is Galois conjugate over  $\mathbf{Z}$  to its reciprocal  $a^{-1}$ .

*Proof.* The decomposition  $\hat{f}^* = S \circ Q$  follows from the discussion above. The assertion about roots outside the unit circle is well-known (see [Can]) and follows from the fact that the intersection form on  $H^2(X, \mathbf{R})$  has exactly one positive eigenvalue. Now if  $\lambda = e^{i\theta}$  is a root of  $P(\lambda)$  on the unit circle, then  $e^{i\theta}$  is Galois conjugate to  $\overline{e^{i\theta}} = (e^{i\theta})^{-1}$  because  $\hat{f}^*$  preserves integral cohomology classes. And if  $\lambda = a > 1$  is a root of  $P(\lambda)$ , then so is  $a^{-1}$ , because  $\hat{f}^*$  and  $\hat{f}_* = (\hat{f}^*)^{-1}$  are adjoint with respect to intersection, and therefore have the same characteristic polynomials. Since the product of the roots of the minimal polynomial for  $a^{-1}$  must be an integer, it follows that  $a$  and  $a^{-1}$  are Galois conjugate over  $\mathbf{Z}$ .  $\square$

Proposition 2.1 implies that the action  $\hat{f}^*$  (as well as the Lorenz space  $H^2(X, \mathbf{R})$ ) depends only on the orbit data associated to  $f$ . In fact, given any orbit data  $n_1, n_2, n_3, \sigma$ , whether or not it is realized by some quadratic transformation  $f$ , one can consider the (abstract) isometry

$$\hat{f}^* : V \mapsto V$$

of the Lorenz space  $V = \mathbf{R}H \oplus_{ij} \mathbf{R}E_{ij}$  defined by the equations preceding Proposition 2.1, and the characteristic polynomial of this isometry will still satisfy the conclusions of the proposition.

We observe in passing that if  $\sigma$  is the identity permutation, then the permutation  $S$  in the theorem decomposes into three cycles

$$S = (E_{1,n_1-1} \dots E_{1,0})(E_{1,n_2-1} \dots E_{2,0})(E_{3,n_3-1} \dots E_{3,0});$$

if  $\sigma$  is an involution, swapping e.g. 1 and 2, then  $S$  decomposes into two cycles

$$S = (E_{1,n_1-1} \dots E_{1,0} E_{2,n_2-1} \dots E_{2,0})(E_{3,n_3-1} \dots E_{3,0});$$

and if  $\sigma = (123)$  is cyclic, then  $S$  is cyclic

$$S = (E_{1,n_i-1} \dots E_{1,0} E_{2,n_1-1} \dots E_{2,0} E_{3,n_2-1} \dots E_{3,0}).$$

Bedford and Kim [BK2] have computed  $P(\lambda)$  explicitly for any orbit data  $n_1, n_2, n_3, \sigma$ , and their formula will be useful to us below (see the fortuitous coincidence in the proof of Theorem 3.5). Specifically, they show that  $P(\lambda) = \lambda^{1+\sum n_j} p(1/\lambda) + (-1)^{\text{ord } \sigma} p(\lambda)$ , where

$$(1) \quad p(\lambda) = 1 - 2\lambda + \sum_{j=\sigma_j} \lambda^{1+n_j} + \sum_{j \neq \sigma_j} \lambda^{n_j} (1 - \lambda).$$

**2.2. Some general observations.** The following fact is folklore among people working in complex dynamics. We include the proof for the reader's convenience.

**Proposition 2.2.** *Let  $X$  be a rational surface obtained by blowing up  $n \leq 9$  points in  $\mathbf{P}^2$  and  $f : X \rightarrow X$  be an automorphism. Then the topological entropy of  $f$  vanishes. If  $n \leq 8$ , then  $f^k$  descends to a linear map of  $\mathbf{P}^2$  for some  $k \in \mathbf{N}$ .*

*Proof.* Suppose that  $f$  has positive entropy  $\log \lambda > 0$ . Then there exists [Can] a non-trivial real cohomology class  $\theta \in H^2(X, \mathbf{R})$  with  $f^*\theta = \lambda\theta$  and  $\theta^2 = 0$ . Moreover,  $f_*K_X = f^*K_X = K_X$ , where  $K_X$  is the class of a canonical divisor on  $X$ . Intersecting  $K_X$  and  $\theta$ , we see that

$$\lambda^{-1} \langle \theta, K_X \rangle = \langle f^*\theta, K_X \rangle = \langle \theta, f_*K_X \rangle = \langle \theta, K_X \rangle.$$

Hence  $\langle \theta, K_X \rangle = 0$ . Since the intersection form on  $X$  has only one positive eigenvalue, and  $K_X^2 \geq 0$  for  $n \leq 9$  we infer that  $\theta = cK_X$  for some  $c < 0$ . But then  $f^*\theta = \theta \neq \lambda\theta$ . This contradiction shows that  $f$  has zero entropy.

If  $n \leq 8$ , then in fact  $K_X^2 > 0$ . Thus the intersection form is strictly negative on the orthogonal complement  $H \subset H^2(X, \mathbf{R})$  of  $K_X$ . Since  $H$  is finite dimensional and invariant under  $f^*$ , and  $f^*$  preserves  $H^2(X, \mathbf{Z})$ , it follows that  $f^*$  has finite order on  $H$ . Hence  $f^{k*} = \text{id}$  for some  $k \in \mathbf{N}$ . In particular,  $f^k$  preserves each of the exceptional divisors in  $X$  that correspond to the  $n \leq 8$  points blown up in  $\mathbf{P}^2$ . It follows that  $f^k$  descends to a well-defined automorphism of  $\mathbf{P}^2$ .  $\square$

**Corollary 2.3.** *Suppose that  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  is a quadratic transformation that properly fixes a cubic curve  $C \subset \mathbf{P}^2$  and lifts to an automorphism  $\hat{f}$  of some modification  $X \rightarrow \mathbf{P}^2$ . If the multiplier of  $f|_C$  is  $-1$  and  $f$  fixes each irreducible component of  $C$ , then  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  is linear. Similarly, if  $f$  fixes each irreducible component of  $C$  and the multiplier of  $f|_C$  is a primitive cube root of unity, then the topological entropy of  $\hat{f}$  vanishes.*

*Proof.* Suppose  $f$  realizes orbit data  $n_1, n_2, n_3 \geq 1$ ,  $\sigma \in \Sigma_3$ . If the multiplier of  $f$  is  $-1$  and  $f^2(V) = V$  for each irreducible  $V \subset C$ , then it follows that  $f^2|_C = \text{id}$ . Hence  $n_j = 1$  or  $2$  for each  $j$ , and the surface  $X$  may be created by blowing up at most six points in  $\mathbf{P}^2$ . The first assertion follows from Proposition 2.2. If the multiplier of  $f$  is a primitive cube root of unity, then  $f^3$  fixes  $C$  component-wise, and the same argument shows that  $X$  may be constructed by blowing up at most 9 points in  $\mathbf{P}^2$ . The second assertion likewise follows.  $\square$

**Theorem 2.4.** *Let  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  be a quadratic transformation properly fixing a cubic curve  $C \subset \mathbf{P}^2$ . Suppose that  $f$  permutes the irreducible components of  $C$  transitively and that  $f|_C$  has multiplier 1. Let  $X$  be the rational surface obtained by blowing up all points (with multiplicity) in  $I(f)$ ,  $I(f^{-1})$  and  $f(I(f^{-1}))$ . Then  $f$  lifts to an automorphism  $\hat{f} : X \rightarrow X$  with an invariant elliptic fibration.*

Of course, the topological entropy must vanish for the map in this theorem. A more detailed analysis shows that either  $f^2 = \text{id}$ , or  $\|\hat{f}^{n*}\|$  grows quadratically with  $n$  and the invariant elliptic fibration is unique (see [PS, Can, McM]) for more about this phenomenon.

*Proof.* The hypothesis that  $f$  cycles the components  $V \subset C$  and Proposition 1.1 allow us to conjugate by a linear transformation  $T$  that restricts to a translation on each  $V$  in order to arrange that the translation  $b = b_V$  for  $f|_V$  is independent of  $V$ .

From Proposition 1.3, we obtain that  $p_j^- \sim p_j^+ - 2b$  for each  $p_j^-$  in  $I(f)$ . Hence  $f^2(p_j^-) \sim p_j^+$ . In fact, if  $V \subset C$  is the component containing  $p_j^+$ , then it follows that  $p_j^- \in f(V)$  when  $C$  has three irreducible components and  $p_j^- \in V$  when  $C$  has two components. In any case, we find that  $f^2(p_j^-) \in V$ , so that  $f^2(p_j^-) = p_j^+$ . Since  $3b \not\sim 0$ , it follows that  $f(p_j^-) \neq p_j^+$ . If  $p_j^- = p_j^+$  for some  $j$ , then in fact  $2b \sim 0$  and  $p_j^- = p_j^+$  for all  $j$ . Hence  $f$  is conjugate to the ‘standard’ quadratic transformation  $q$ , and the theorem is trivial. Henceforth, we assume  $p_j^- \neq p_j^+$ .

Suppose further for the moment that there are no pairs of indices  $j \neq k$  such that  $p_j^- = p_k^+$  or  $f(p_j^-) = p_k^+$ . Then we may blow up the points  $p_j^-, f(p_j^-), p_j^+$  for each  $j$  to obtain a rational surface  $X$  to which  $f$  lifts as an automorphism. Furthermore,  $\sum p_j^- + \sum (p_j^- + b) + \sum p_j^+ = -3b + 0 + 3b \sim 0$ . Finally, one finds by comparing degrees that regardless of the number of components  $V \subset C$ , each  $V$  contains precisely  $3 \deg V$  of the points blown up. Hence there is a pencil of cubic curves that contains  $C$  and whose basepoints are precisely the ones blown up. Each curve  $C'$  in the pencil intersects each exceptional curve for  $f$  precisely once and contains each point in  $I(f)$  with multiplicity one. Comparing degrees, we see that  $f(C')$  is another cubic curve containing all the basepoints. We conclude that the pencil lifts to an invariant elliptic fibration of  $X$ .

Now if it happens that  $p_j^- = p_k^+$  or  $f(p_j^-) = p_k^+$  for one or more pairs of indices  $j \neq k$ , then we can reach the same conclusion as before, except that constructing  $X$  will require iterated blowing up, the precise nature of which depends on which special case we are in. The important thing is that since  $2b, 3b \not\sim 0$ , one always has to blow up nine evenly distributed points in  $C_{reg}$  that sum to zero in  $\text{Pic}_0(X)$ .  $\square$

**Proposition 2.5.** *Let  $P$  be the characteristic polynomial for orbit data  $n_1, n_2, n_3, \sigma$ . If  $n_j = 1$  for some  $j = \sigma(j)$  that is fixed by  $\sigma$ , then all roots of  $P$  lie on the unit circle<sup>3</sup>.*

*Proof.* Suppose e.g. that  $j = 1$  and that  $P$  has a root  $\lambda$  with magnitude different from 1. Recalling the discussion after Proposition 2.1, we let  $\hat{f}^* : V \rightarrow V$  be the ‘abstract Lorenz isometry’ associated to the data  $1, n_2, n_3, \sigma$ . Then  $f^*v = \lambda v$  for some  $v \in V$ .

Using the fact that  $f_*$  is both inverse and adjoint to  $f^*$ , we find

$$\langle v, v \rangle = \langle v, \hat{f}_* \hat{f}^* v \rangle = \langle \hat{f}^* v, \hat{f}^* v \rangle = |\lambda|^2 \langle v, v \rangle.$$

Thus  $\langle v, v \rangle = 0$ . Now it follows from Proposition 2.1 that  $\hat{f}_*(H - E_{1,0}) = H - E_{1,0}$ . Thus

$$\langle H - E_{1,0}, v \rangle = \langle \hat{f}_*(H - E_{1,0}), v \rangle = \langle H - E_{1,0}, \hat{f}^* v \rangle = \lambda \langle H - E_{1,0}, v \rangle.$$

We infer that  $\langle H - E_{1,0}, v \rangle = 0$ . Since  $H - E_{1,0}$  also has vanishing self-intersection, and the intersection form has exactly one positive eigenvalue, it follows that  $v$  is a multiple of  $H - E_{1,0}$ . Hence  $\lambda = 1$  contrary to assumption.  $\square$

<sup>3</sup>Since  $P$  is monic with integer coefficients, a theorem of Kronecker tells us that all roots are roots of unity.

## 3. IRREDUCIBLE CUBICS

**Corollary 3.1.** *Suppose that  $f$  is a quadratic transformation properly fixing a nodal irreducible cubic curve  $C$ . If  $f$  lifts to an automorphism on some modification  $X \rightarrow \mathbf{P}^2$ , then the topological entropy of  $f$  vanishes.*

*Proof.* Since  $\text{Pic}_0(C) \cong \mathbf{C}^*$ , the multiplier of  $f|_{C_{\text{reg}}}$  is  $\pm 1$ . Since  $C$  is irreducible, the assertion follows from Corollary 2.3 and Theorem 2.4.  $\square$

**Corollary 3.2.** *Suppose that  $f$  is a quadratic transformation properly fixing a smooth irreducible cubic curve  $C$ . If  $f$  has positive entropy and lifts to an automorphism of some modification  $X \rightarrow \mathbf{P}^2$ , then either*

- $C \cong \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$  and the multiplier for  $f|_C$  is  $\pm i$ ; or
- $C \cong \mathbf{C}/(\mathbf{Z} + e^{\pi i/3}\mathbf{Z})$  and the multiplier for  $f|_C$  is a primitive cube root of  $-1$ .

*Proof.* If we are not in one of the two cases described in the conclusion, then the multiplier for  $f|_C$  must be a square or cube root of 1. From Corollary 2.3 and Theorem 2.4, we deduce that if  $f$  lifts to an automorphism, then the entropy of  $f$  is zero.  $\square$

**Example 3.3.** *Suppose  $C \cong \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$ . Then remarkably, there are quadratic transformations properly fixing  $C$  and lifting to automorphisms with positive entropy. For example, Theorem 1.4 gives us a quadratic transformation  $f$  properly fixing  $C$  with  $I(f) = \{p_1^+, p_2^+, p_3^+\} = \{i/9, 4i/9, 7i/9\}$  and such that  $f|_C$  is given by  $z \mapsto iz + 5/9$ . Applying Proposition 1.3, we find that  $p_1^- = ip_1^+ - 2b = 7/9$ , and similarly  $p_2^- = 4/9$ ,  $p_3^- = 1/9$ .*

*Iterating  $f$  gives*

$$p_1^- = 7/9 \mapsto 7i/9 + 5/9 \mapsto -7/9 + 5i/9 \mapsto 7i/9 = p_3^+.$$

*Similarly  $f^3(p_2^-) = p_1^+$  and  $f^3(p_3^-) = p_2^+$ . In summary,  $f$  realizes the orbit data  $\sigma : 1 \mapsto 3 \mapsto 2$ ,  $n_1 = n_2 = n_3 = 4$ .*

*On blowing up the twelve points  $f^k(p_j^-)$ ,  $0 \leq k \leq 3$ ,  $1 \leq j \leq 3$ , we obtain an automorphism  $\hat{f} : X \rightarrow X$ . By (1), the characteristic polynomial for  $\hat{f}^*$  is  $P(\lambda) = \lambda^{13} - 2\lambda^{12} + 3\lambda^9 - 3\lambda^8 + 3\lambda^5 - 3\lambda^4 + 2\lambda - 1$ , which has largest root  $\lambda_1 = 1.722\dots$ . Hence  $\hat{f}$  has entropy  $\log \lambda > 0$ .*

*We make two further observations about this example. The restriction of  $\hat{f} : X \rightarrow X$  to (the proper transform of)  $C$  is periodic with period 4. Hence  $\hat{f}^4$  is an example of a positive entropy automorphism of a rational surface that fixes a smooth elliptic curve pointwise. Secondly, since  $C$  has negative self-intersection  $C^2 = 9 - 12$  in  $X$ , one can contract  $C$  to obtain an automorphism  $\check{f} : \check{X} \rightarrow \check{X}$  with positive entropy on a normal surface with a simple elliptic singularity.*

On the other hand, as Eric Riedl points out, not all orbit data that looks plausible (i.e.  $n_j \leq 4$ ) for the ‘square’ torus is actually realizable.

**Example 3.4.** *Let  $C = \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$  again, and consider the orbit data  $n_1 = n_2 = n_3 = 4$ ,  $\sigma = \text{id}$ . If  $f$  properly fixes  $C$  and realizes this data, then we have  $f|_C : z \mapsto iz + b$  for some  $b \in C$ , and  $(f|_C)^3(p_j^-) = p_j^+$ . Since  $(f|_C)^4 = \text{id}$ , this is equivalent to  $f|_C(p_j^+) = p_j^-$ . Hence Proposition 1.3 implies  $ap_j^+ + b = p_j^- = ap_j^+ - 2b$ , which gives  $3b = 0$ , contrary to the last assertion in the proposition.*

The final irreducible case occurs when  $C$  has a cusp, and in this one it is much easier to construct automorphisms. In order to state our result, let us make a convenient definition. Suppose we are given orbit data  $n_1, n_2, n_3 \geq 1$  and a quadratic transformation  $f$  properly fixing  $C$ . We will say that  $f$  *tentatively realizes* the orbit data if  $(f|_{C_{reg}})^{n_j-1}(p_j^-) = p_{\sigma_j}^+$  for each  $n_j$ . We stress that this does not mean that  $f$  *realizes* the orbit data in the fashion described in §2. For instance, one might find that  $f^{n-1}(p_1^-) = p_{\sigma_1^+}$  for some  $n < n_1$  so that  $f$  actually realizes the orbit data  $n, n_2, n_3, \sigma$  instead of  $n_1, n_2, n_3, \sigma$ . Tentative realization is, however, a necessary precondition for realization.

**Theorem 3.5.** *Let  $C$  be a cuspidal cubic curve,  $n_1, n_2, n_3 \geq 1$  and  $\sigma \in \Sigma_3$  be orbit data. If  $f$  is a quadratic transformation properly fixing  $C$  that tentatively realizes this orbit data, then the multiplier for  $f|_{C_{reg}}$  is a root of the corresponding characteristic polynomial  $P(\lambda)$ . Conversely, there exists a tentative realization  $f$  for each root  $\lambda = a$  of  $P(\lambda)$  that is not a root of unity, and  $f$  is unique up to conjugacy by linear transformations preserving  $C$ .*

*Proof.* Since  $a \neq 1$  by hypothesis, the restriction  $f|_{C_{reg}}$  is given by  $f(z) \sim az + b$  which has a unique fixed point  $z_0 = b/(1-a)$ . We write  $\tilde{z} = z - z_0$  for any point  $z \in C_{reg}$ . Hence  $\widetilde{f^k(z)} = a^k \tilde{z}$ . Proposition 1.1 and the fact that all  $a \in \mathbf{C}^*$  are admissible for  $C$  allow us to conjugate by  $T \in \text{Aut}(\mathbf{P}^2)$  to arrange that  $z_0 \sim \frac{1}{3}$ . The first and third items in Proposition 1.3 then become

- $\sum \tilde{p}_j^- = a - 2$ ;
- $\tilde{p}_j^- = a\tilde{p}_j^+ + a - 1$ , for  $j = 1, 2, 3$ .

Therefore if the points  $p_j^- \in C_{reg}$  satisfy the first of these conditions, Theorem 1.4 gives us a quadratic transformation  $f$  that properly fixes  $C$  with multiplier  $a$  and  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ . Necessarily, the points  $p_j^+ \in I(f) = \{p_1^+, p_2^+, p_3^+\}$  satisfy the second condition.

Now  $f$  tentatively realizes the given orbit data if and only if  $a^{n_j-1}\tilde{p}_j^- = \tilde{p}_{\sigma_j}^+$  for  $j = 1, 2, 3$ . If  $\sigma$  is the identity permutation, then in light of the second condition above, this is equivalent to

$$(2) \quad \tilde{p}_j^- = \frac{a-1}{1-a^{n_j}}, \quad j = 1, 2, 3.$$

The first condition in turn gives  $\sum_j \frac{1}{1-a^{n_j}} = \frac{a-2}{a-1}$ . One verifies readily that this is equivalent<sup>4</sup> to  $P(a) = 0$ , where  $P$  is the characteristic polynomial for the orbit data  $n_1, n_2, n_3, \text{id}$ . This proves the theorem when  $\sigma = \text{id}$ .

The cases where  $\sigma$  is an involution or  $\sigma$  is cyclic are similar. If  $\sigma$  is the involution swapping e.g. indices 1 and 2, then one finds that

$$(3) \quad \tilde{p}_1^- = \frac{(a-1)(1+a^{n_2})}{1-a^{n_1+n_2}}, \quad \tilde{p}_2^- = \frac{(a-1)(1+a^{n_1})}{1-a^{n_1+n_2}}, \quad \tilde{p}_3^- = \frac{a-1}{1-a^{n_3}},$$

---

<sup>4</sup>This fortunate coincidence is largely accounted for in §7 of [McM] whose arguments show that the multiplier  $a$  for a tentative realization must be a root of  $P(\lambda)$  and conversely that each root of  $P(\lambda)$ , disregarding multiplicity, gives rise to at least one tentative realization.

where  $a$  is a root of the characteristic polynomial associated to  $n_1, n_2, n_3, \sigma$ . And if  $\sigma$  is the cyclic permutation  $\sigma : 1 \mapsto 2 \mapsto 3$ , then

$$(4) \quad \tilde{p}_1^- = \frac{(a-1)(1+a^{n_3}+a^{n_2+n_3})}{1-a^{n_1+n_2+n_3}}, \quad \tilde{p}_2^- = \frac{(a-1)(1+a^{n_1}+a^{n_3+n_1})}{1-a^{n_1+n_2+n_3}}, \quad \tilde{p}_3^- = \frac{(a-1)(1+a^{n_2}+a^{n_1+n_2})}{1-a^{n_1+n_2+n_3}}.$$

□

As it turns out, most of the tentative realizations given by Theorem 3.5 actually do realize the given orbit data.

**Theorem 3.6.** *Suppose in Theorem 3.5 that  $a$  is a root of  $P(\lambda)$  that is not equal to a root of unity, and let  $f$  be the tentative realization corresponding to  $a$  of the given orbit data  $n_1, n_2, n_3, \sigma$ . Then  $f$  realizes the orbit data if and only if we are not in one of the following two cases*

- $\sigma \neq \text{id}$  and  $n_1 = n_2 = n_3$ ;
- $\sigma$  is an involution swapping indices  $i$  and  $j$  such that  $n_i = n_j$ .

*Proof.* The tentative realization  $f$  will necessarily realize *some* orbit data. The problem occurs when the orbit of some point  $p_j^-$  intersects  $I(f)$  too soon and/or at the wrong point so that the orbit data that is realized differs from the given data.

That is, we have  $f^{n-1}(p_j^-) = p_{\sigma_i}^+$  for some  $i, j$  and some positive  $n \in \mathbf{N}$ , where  $i \neq j$  and/or  $n < n_i$ . Using the notation from the proof of Theorem 3.5, this becomes

$$(5) \quad a^n \tilde{p}_j^- = \tilde{p}_{\sigma_i}^+ = a^{n_i} \tilde{p}_i^-$$

In particular, we may suppose  $i \neq j$  since  $a$  is not a root of unity. Since  $\tilde{p}_i^-, \tilde{p}_j^-$  are given by rational expressions (over  $\mathbf{Z}$ ) in  $a$ , (5) amounts to a polynomial equation satisfied by  $a$ . But  $a$  is a root of the characteristic polynomial for the orbit data and therefore by Proposition 2.1 Galois conjugate to  $a^{-1}$ . Hence (5) remains true if we replace  $a$  by  $a^{-1}$  throughout.

Assume for now that  $\sigma = \text{id}$  or that  $\sigma$  exchanges two indices. Replacing  $a$  by  $a^{-1}$  in the formula for  $\tilde{p}_j^-$  amounts to replacing  $\tilde{p}_j^-$  by  $\tilde{p}_{\sigma_j}^+ = a^{n_j-1} \tilde{p}_j^-$ . One can verify this directly using the formulas (2), (3). However, this follows also on general principle from the fact that (given the normalization  $z_0 \sim 1/3$ ) there is a unique tentative realization  $g$  of the orbit data  $n_1, n_2, n_3, \sigma$  corresponding to the multiplier  $a^{-1}$ . Since  $\sigma = \sigma^{-1}$ , one can relabel indices  $j \mapsto \sigma(j)$  and see that  $f^{-1}$  gives such a realization. Hence  $g = f^{-1}$ . The upshot is that  $a$  must satisfy the second equation  $a^{-n+n_j} \tilde{p}_j^- = \tilde{p}_{\sigma_i}^-$ . Combined with (5) this implies that  $a^{n_i+n_j-2n} = 1$ . Since by hypothesis  $a$  is not a root of unity, it follows that  $n_i + n_j = 2n$ .

Suppose  $n_i \neq n_j$ , e.g.  $n_i < n_j$ . Then we may write  $n_i = n - k$ ,  $n_j = n + k$  for some  $k > 0$ . Thus the orbit of  $p_j^-$  contains that of  $p_i^-$  as follows:

$$p_j^-, \dots, f^k(p_j^-) = p_i^-, \dots, f^{n_j-k-1}(p_j^-) = p_{\sigma_i}^+, \dots, f^{n_j-1}(p_j^-).$$

Hence in the blowing up procedure used to lift the birational map  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  to an automorphism  $\hat{f} : X \rightarrow X$ , the orbit segment  $p_i^-, \dots, p_{\sigma_i}^-$  is blown up *before* the segment  $p_j^-, \dots, p_{\sigma_j}^+$ . Hence despite the coincidence (5),  $f$  still realizes the given orbit data.

If instead  $n_i = n_j = n$ , then (5) implies  $p_i^- = p_j^-$ . Without loss of generality, we may assume that  $p_i^-$  is infinite near to  $p_j^-$ . Then symmetry of  $f$  and  $f^{-1}$  implies that  $p_i^+$  is infinitely near to  $p_j^+$ , whereas  $n_i = n_j$  implies that  $p_{\sigma_i}^+$  is infinitely near to  $p_{\sigma_j}^+$ . Hence, under our assumption that  $\sigma$  is the identity or a transposition,  $f$  realizes the given orbit data if and only if  $\sigma = \text{id}$ .

Turning to the remaining case, where  $\sigma : 1 \mapsto 2 \mapsto 3$  is cyclic, we begin again with (5). Without loss of generality, we further suppose that  $j = 1, i = 2$ . Then (4) and (5) give us  $a^n(1 + a^{n_3} + a^{n_2+n_3}) = a^{n_2}(1 + a^{n_1} + a^{n_1+n_3})$ . Replacing  $a$  with  $a^{-1}$  in this equation also gives  $a^{n_1}(1 + a^{n_2} + a^{n_2+n_3}) = a^n(1 + a^{n_3} + a^{n_1+n_3})$ . Adding the two equations and simplifying, we obtain that  $(a^{n+n_3} - 1)(a^{n_1} - a^{n_2}) = 0$ . Since  $a$  is not a root of unity, we infer that either  $n = -n_3$  or  $n_1 = n_2$ .

In the first case, we substitute for  $a^{-n_3}$  for  $a^n$  in (5) and find that  $(a^{n_3} + 1)(a^{n_1+n_2+n_3} - 1) = 0$ , which is impossible because  $a$  is not a root of unity and  $n_1, n_2, n_3 \geq 1$ . In the second case, when  $n_1 = n_2$ , we rewrite (5) as

$$a^n \tilde{p}_1^- = a \tilde{p}_{\sigma_2}^+ = a \tilde{p}_3^+ = \tilde{p}_3^- + 1 - a.$$

Substituting our formulas (4) for  $\tilde{p}_1^-$  and  $\tilde{p}_3^-$ , we obtain that  $a^n(1 + a^{n_3} + a^{n_3+n_2}) = a^{n_2}(1 + a^{n_3} + a^{n_1+n_3})$ . Using  $n_1 = n_2$ , we obtain that either  $1 + a^{n_3} + a^{n_1+n_3} = 0$  or (since  $a$  is not a root of unity)  $n = n_2$ . In the first case, we replace  $a$  with  $a^{-1}$  and deduce finally that  $n_1 = n_3$ . In the second case, we return to (5) and find that  $\tilde{p}_1^- = \tilde{p}_2^-$ , which again gives  $n_1 = n_3$ . Regardless, we arrive at the condition  $n_1 = n_2 = n_3$ . From here we obtain a contradiction following the same logic used to rule out the possibility that  $n_i = n_j$  when  $\sigma$  transposes  $i$  and  $j$ .  $\square$

#### 4. REDUCIBLE CUBICS

We now deal briefly with the cases where the cubic curve  $C$  is reducible with only one singularity—i.e.  $C$  consists of three distinct lines through a single point, or  $C$  consists of a smooth conic and one of its tangent lines. In either case, the components of  $C_{\text{reg}}$  are copies of  $\mathbf{C}$ , and the story is much the same as it is for cuspidal cubics. The only additional complication is that a quadratic transformation cannot realize given orbit data unless the permutation it induces on the components of  $C$  is compatible with the permutation  $\sigma$  in the orbit data.

**Theorem 4.1.** *Let  $C$  be the plane cubic consisting of three lines meeting at a single point. Let  $n_1, n_2, n_3 \in \mathbf{N}$ ,  $\sigma \in \Sigma_3$  be orbit data whose characteristic polynomial  $P(\lambda)$  has a root outside the unit circle. Then the orbit data can be realized by a quadratic transformation  $f$  that properly fixes  $C$  if and only if one of the following is true:*

- $\sigma = \text{id}$ ;
- $\sigma$  is cyclic and either all  $n_j \equiv 0 \pmod{3}$  or all  $n_j \equiv 2 \pmod{3}$ ;
- $\sigma$  is a transposition (say  $\sigma$  interchanges 1 and 2) and either  $n_1$  and  $n_2$  are odd, or no two  $n_j$  are the same mod 3 and  $n_3 \equiv 0 \pmod{3}$ .

*If one of these holds, we can arrange that  $f|_{C_{\text{reg}}}$  has multiplier  $a$  where  $a$  is any root of  $P$  that is not a root of unity. The choice of  $a$  determines  $f$  uniquely up to linear conjugacy.*

*Proof.* We only sketch the argument. Let  $V_j \subset C_{reg}$  denote the component containing  $p_j^+$ . Since  $a \neq 1$ , the restriction  $f|_{V_j}$  has a unique ‘fixed point’  $z_j \sim f(z_j)$ . Using Proposition 1.1 we may conjugate by an element of  $\text{Aut}(\mathbf{P}^2)$  to arrange that  $z_j = \frac{1}{3(a-1)}$  for all  $j = 1, 2, 3$ . Hence  $f(z) \sim a(z - z_j) + z_j$  has the same expression on each  $V_j$ .

Given orbit data whose characteristic polynomial  $P$  has a root  $a$  that is not a root of unity, we can repeat the arguments used to prove Theorem 3.5 to prove that there exists a quadratic transformation  $f$  properly fixing  $C$  such that the multiplier of  $f|_{C_{reg}}$  is  $a$  and  $f^{n_j-1}(p_j^-) \sim p_j^+$  for each  $j = 1, 2, 3$ . Indeed given  $a$  and the fixed points  $z_j$ ,  $f$  is determined up to permutation of the  $V_j$ . Let us write  $f(V_j) = V_{s_j}$  where  $s \in \Sigma_3$ .

Now each  $V_j$  contains one point of indeterminacy—say  $p_j^+ \in V_j$ ; and  $p_j^-$  therefore lies in  $f(V_j) = V_{s_j}$ . Therefore if  $\sigma = \text{id}$ , we also choose  $s = \text{id}$ , and then  $f^{n_j-1}(p_j^-) \sim p_j^+$  implies  $f^{n_j-1}(p_j^-) = p_j^+$ . Hence  $f$  realizes the given orbit data.

If  $\sigma$  is cyclic (say  $\sigma : 1 \mapsto 2 \mapsto 3$ ), then certainly  $f$  must permute the  $V_j$  transitively. That is,  $s$  must also be cyclic. If  $s = \sigma$ , then we have  $p_j^- \in V_{\sigma_j}$ . Hence  $f^{n_j-1}(p_j^-)$  lies in  $V_j$  if and only if  $n \equiv 0 \pmod{3}$ . That is, when  $s = \sigma$  then  $f$  realizes the given orbit data if and only if each  $n_j \equiv 0 \pmod{3}$ . To realize orbit data for which  $n_j \equiv 2 \pmod{3}$ , one may check that it is similarly necessary and sufficient that  $s = \sigma^{-1}$ . We note that the exceptional cases from Theorem 3.6 need not concern us here, because different points of indeterminacy lie in different components of  $C_{reg}$  and cannot therefore coincide.

The case where  $\sigma$  is a transposition can be analyzed similarly. The case where  $n_1$  and  $n_2$  are odd can be realized by a quadratic transformation  $f$  that swaps  $V_1$  and  $V_2$  while fixing  $V_3$ . The other case can be achieved by letting  $f$  permute the  $V_j$  cyclically.  $\square$

When  $C$  is the union of a smooth conic with one of its tangent lines, one has a result similar to Theorem 4.1. However, in this situation it will always be the case that the conic portion of  $C$  contains more than one point of indeterminacy. Since such points of indeterminacy might coincide, it is necessary to hypothesize away exceptional cases like those in Theorem 3.6. The upshot is that the analogue of Theorem 4.1 for  $C$  equal to a conic and a tangent line is somewhat messy to state. Since it is not conceptually different, we omit it.

**4.1. Reducible cubics with nodal singularities.** Finally, we consider reducible cubics with more than one singularity. As above, we devote more attention to the case of a cubic with three irreducible components.

**Theorem 4.2.** *Suppose  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  is a quadratic transformation that properly fixes  $C = \{xyz = 0\}$  and lifts to an automorphism with positive entropy on some blowup of  $\mathbf{P}^2$ . Then  $f$  fixes  $C_{reg}$  component-wise and  $f|_{C_{reg}}$  has multiplier 1. Hence  $f$  realizes orbit data of the form  $n_1, n_2, n_3 \geq 1$ ,  $\sigma = \text{id}$ .*

*Proof.* Since  $\text{Pic}_0(C) \cong \mathbf{C}^*$ , the multiplier of  $f|_{C_{reg}}$  is  $\pm 1$ . We claim that the multiplier of  $f$  is  $-1$  if and only if  $f$  swaps two components of  $C_{reg}$  and preserves the other. Indeed, if  $f$  fixes  $\{z = 0\}$  while swapping  $\{x = 0\}$  and  $\{y = 0\}$ , then in particular,  $f$  interchanges the points  $[0, 1, 0]$  and  $[1, 0, 0]$ . Hence the multiplier of  $f|_{C_{reg}}$ , which is the same as that of  $f|_{\{z=0\}}$ , is  $-1$ . Similarly, if  $f$  fixes all three components of  $C_{reg}$ , then it also fixes all three singularities of  $C$ , and we infer that  $f$  has multiplier  $+1$ . Finally, if  $f$  cycles the components

of  $C_{reg}$ , then  $f^3$  fixes  $C_{reg}$  component-wise, and we infer again that the multiplier of  $f|_{C_{reg}}$ , which is the same as that of  $f^3|_{C_{reg}}$ , is  $+1$ . This proves our claim.

Suppose now that the multiplier is  $-1$  and, without loss of generality, that  $f$  fixes the component  $V \subset C_{reg}$  containing  $p_1^\pm$ . Hence  $f^2|_V = \text{id}$  and  $\sigma_1 = 1$ . It follows that  $n_1 = 1$  or  $n_1 = 2$ . If  $n_1 = 2$ , then on the one hand, we have  $p_1^+ \sim -p_1^+ + b_1$ , where  $b_1$  is the translation for  $f|_V$ . And on the other hand, we have from Proposition 1.3 that  $p_1^- \sim -p_1^+ - b_2 - b_3$  where  $b_2, b_3 \in \mathbf{C}^*$  are the translations on the other two components of  $C_{reg}$ . We infer that  $b_1 + b_2 + b_3 \sim 0$ , contradicting the final assertion of Proposition 1.3. So  $n_1 = 1$ . From Proposition 2.5, it follows that the automorphism induced by  $f$  has entropy 0, contrary to hypothesis.

Hence the multiplier for  $f|_{C_{reg}}$  is  $+1$ . If  $f$  permutes the components of  $C_{reg}$  cyclically, then Proposition 2.2 and Theorem 2.4 imply that  $f$  lifts to an automorphism with zero entropy, again counter to our hypothesis. We conclude that  $f$  fixes  $C$  component-wise.  $\square$

Having just ruled out many types of orbit data on  $C = \{xyz = 0\}$ , we consider whether the remaining cases may be realized. Let  $n_1, n_2, n_3 \geq 1$ ,  $\sigma = \text{id}$  be orbit data and  $f$  be a quadratic transformation that fixes  $C$  component-wise with multiplier 1. Then we have  $f(p) \sim p + b_j$ , on the component containing  $p_j^\pm$ . Proposition 1.3 tells us that  $p_j^- \sim p_j^+ + b_j - b$  where  $b \sim b_1 + b_2 + b_3$ ; and  $f$  tentatively realizes the given orbit data if  $p_j^+ \sim p_j^- + (n_j - 1)b_j$ . We infer  $n_j b_j \sim b$  for  $j = 1, 2, 3$ .

Note that these equations hold relative to the group structure on  $\mathbf{C}/\mathbf{Z}$  and are therefore to be understood ‘mod 1’: in the universal cover  $\mathbf{C}$  of  $\mathbf{C}/\mathbf{Z}$ , we have  $n_j b_j = b + m_j$  for some  $m_j \in \mathbf{Z}$ . Solving for  $b_j$  and summing over  $j$  gives

$$b \left( 1 - \sum \frac{1}{n_j} \right) = \sum \frac{m_j}{n_j},$$

which implies

$$(6) \quad b_j = \frac{m_j}{n_j} + \frac{1}{n_j} \frac{m_1 n_2 n_3 + m_2 n_3 n_1 + m_3 n_1 n_2}{n_1 n_2 n_3 - n_1 n_2 - n_2 n_3 - n_3 n_1}.$$

On the other hand, it is not difficult to see from Theorem 1.4 that if  $m_1, m_2, m_3 \in \mathbf{Z}$  is any choice of integers, then we get a tentative realization of our orbit data.

**Proposition 4.3.** *Let  $C = \{xyz = 0\}$  and  $n_1, n_2, n_3, \sigma = \text{id}$  be orbit data. Then this data may be tentatively realized by a quadratic transformation  $f$  properly fixing  $C$  if and only if  $n_1 n_2 n_3 \neq n_1 n_2 + n_2 n_3 + n_3 n_1$ . Any such  $f$  has translations  $b_j$ ,  $j = 1, 2, 3$  given by equation (6). Conversely, any choice of  $m_1, m_2, m_3 \in \mathbf{Z}$  in (6) determines a tentative realization  $f$  that is unique up to linear conjugacy.*

*Proof.* The above discussion shows that the restrictions on  $f$  are necessary and sufficient for  $f$  to tentatively realize the orbit data. We need only argue that there actually exists a quadratic transformation  $f$  that satisfies the restrictions. For this we rely on Theorem 1.4. Note that the above discussion also shows that while the conditions  $f^{n_j-1}(p_j^-) = p_j^+$  constrain the translations  $b_j$ , they do not (otherwise) constrain the points  $p_j^\pm$ . Hence we need only adhere to the conditions in Theorem 1.4, choosing  $p_j^+$  so that  $\sum p_j^+ \sim b$  and then

$p_j^- \sim p_j^+ + b_j - b$ . From Proposition 1.1, we see in fact that we can always conjugate by a linear transformation to obtain  $p_1^+ \sim p_2^+ \sim 0$  and  $p_3^+ \sim b$ .  $\square$

Since the points of indeterminacy for  $f$  lie in different components of  $C$ , the only way the transformations  $f$  in the proposition can fail to realize the given orbit data is if  $f^k(p_j^-) = p_j^+$  for some  $0 \leq k \leq n_j - 2$ . This happens if and only if  $f^\ell p_j^- = p_j^-$ , i.e.  $\ell b_j \in \mathbf{Z}$ , for some  $0 < \ell < n_j - 2$ .

**Theorem 4.4.** *Let  $C = \{xyz = 0\}$  and consider orbit data of the form  $n_1 \geq n_2 \geq n_3 \geq 2$ ,  $\sigma = \text{id}$  for which the corresponding characteristic polynomial has a root outside the unit circle. Then there exists a quadratic transformation properly fixing  $C$  and realizing this orbit data if and only if we are not in one of the following cases.*

- $n_2 + n_3 \leq 6$ ;
- $n_3 = 2$ , and  $n_1 = n_2 = 5$  or  $n_1 = n_2 = 6$ ;
- $n_1 = n_2 = n_3 = 4$ .

*Proof.* If a quadratic transformation  $f$  realizes orbit data  $n_1 \geq n_2 \geq n_3$ , then it must be one of the tentative realizations from Proposition 4.3. By Proposition 2.5 we may assume  $n_3 \geq 2$ . If  $n_2 = n_3 = 2$ , we have  $n_1 n_2 n_3 - n_1 n_2 - n_2 n_3 - n_3 n_1 = 0$  contrary, so by Proposition 4.3, we may assume  $n_2 \geq 3$ .

Now if  $n_2 = 3, n_3 = 2$ , equation (6) gives

$$b_1 = \frac{m_1 + 2m_2 + 3m_3}{n_1 - 6}$$

Hence  $\ell b_1 \in \mathbf{Z}$  for  $\ell = n_1 - 6 \leq n_1 - 2$ . That is, every tentative realization of the orbit data  $n_1, 3, 2, \text{id}$  fails to actually realize this data. The same argument rules out orbit data with  $n_2 = 4, n_3 = 2$  or  $n_2 = n_3 = 3$ .

We are left with three remaining bad cases. The data  $n_1 = n_2 = 5$  and  $n_3 = 2$  is ruled out in the same way as the previous cases. Suppose  $n_1 = n_2 = n_3 = 4$ . This time (6) tells us that for any tentative realization, the translations are given by

$$b_j = \frac{m_j + (m_1 + m_2 + m_3)}{4},$$

where  $m_1, m_2, m_3 \in \mathbf{Z}$ . Thus the numerator will be even for some  $j$ , which implies  $(n_j - 2)b_j = 2b_j \in \mathbf{Z}$ . Hence the data is not realized. Similar arguments rule out the data  $n_1 = n_2 = 6, n_3 = 2$ .

Turning to the good cases, we first assume  $n_2 > n_1 \geq 4$ . We set  $m_1 = 1, m_2 = m_3 = 0$  and take  $f$  to be the tentative realization from Proposition 4.3. Then (6) gives

$$0 < b_1 = \frac{n_2 n_3 - n_2 - n_3}{n_1(n_2 n_3 - n_2 + n_3 - n_2 n_3) - n_2 n_3} = \frac{1}{n_1 - \frac{1}{1 - n_2^{-1} - n_3^{-1}}} < \frac{1}{n_1 - 2}.$$

Hence  $0 < \ell b_1 < 1$  for all  $0 < \ell \leq n_1 - 2$ . Similarly, we find for  $j = 2, 3$  that  $0 < \ell b_j < 1$  for all  $0 < \ell < n_j - 2$ . We conclude that  $f$  actually realizes the given orbit data.

The same argument works when  $n_1 > n_2 = n_3 = 4$  except that we set  $m_2 = 1$  and  $m_1 = m_3 = 0$  in choosing  $f$ ; it works for  $n_2 > n_3 = 3$  if we set  $m_1 = 1, m_2 = 0, m_3 = -1$ ; it works for  $n_1 > n_2 \geq 5$  and  $n_1 \neq n_2$  if we set  $m_1 = 1, m_2 = -1$ .

The final case we need to consider is  $n_3 = 2$  and  $n_1 = n_2 \geq 7$ . This time we set  $m_1 = 1$ ,  $m_2 = m_3 = 0$ . It follows that  $0 < \ell b_2 < 1$  for all  $0 < \ell \leq n_2 - 2$ . It also follows that  $b_3 \notin \mathbf{Z}$ . For  $b_1$ , however, things are a bit more delicate. One shows here that  $0 < \ell b_1 < 1$  for all  $0 < \ell \leq n_1 - 3$  but  $1 < (n_1 - 3)b_1 < 2$ . Regardless, the data is realizable.  $\square$

Of course, each realization  $f$  given by Theorem 4.4 lifts to an automorphism  $\hat{f} : X \rightarrow X$  on the rational surface  $X$  obtained by blowing up orbit segments  $p_j^-, \dots, f^{n_j-1}(p_j^-)$ . These automorphisms are broadly similar to those in Examples 3.3. That is, some iterate  $\hat{f}^k$  restricts to the identity on the proper transform  $\hat{C}$  of  $C$  in  $X$ . And in a different direction, the intersection form is negative definite for divisors supported on  $\hat{C}$ , so by Grauert's theorem [BHPVdV, page 91] one can collapse  $\hat{C}$  to a point and obtain a normal surface  $Y$  with a cusp singularity to which  $\hat{f}$  descends as an automorphism.

The other reducible cubic curve with nodal singularities is the one with two components  $C = \{z(xy - z^2) = 0\}$ . As with  $\{xyz = 0\}$ , there are infinitely many sets of orbit data that can be realized by quadratic transformations fixing  $C$  and also infinitely many that cannot be realized. Rather than give the complete story, we make some broad observations and give examples indicating the range of possibilities.

**Theorem 4.5.** *Suppose that  $C = \{z(xy - z^2)\}$  is the reducible cubic with two singularities. If  $f$  is a quadratic transformation realizing orbit data  $n_1, n_2, n_3, \sigma$  whose characteristic polynomial has a root outside the unit circle, then  $f$  fixes  $C$  component-wise and  $f|_{C_{reg}}$  has multiplier 1. Moreover, either*

- $\sigma$  is a transposition; or
- $\sigma = \text{id}$  and two of the  $n_j$  are equal.

*Proof.* The admissible multipliers are  $\pm 1$  for  $f|_{C_{reg}}$ . Let  $b, c \in \mathbf{C}^*$  denote the translations of  $f$  on  $\{xy - z^2\}$  and  $\{z = 0\}$ , respectively.

Suppose that the multiplier is  $-1$ . Then by Corollary 2.3,  $f$  switches the two components of  $C_{reg}$ . Then  $f^2(p) \sim p + (b - c)$  on the conic  $\{xy - z^2\}$  and  $f^2(p) \sim p + (c - b)$  on  $\{z = 0\}$ . Moreover, degree considerations force all points  $p_j^\pm$  of indeterminacy for  $f$  and  $f^{-1}$  to lie on this conic. Hence from Proposition 1.3 we have  $p_j^- + p_j^+ \sim b - c$  for  $j = 1, 2, 3$ ; and  $\sum p_j^- \sim \sum p_j^+ \sim -2b - c$ . Combining all the formulas gives

$$-3(b + c) \sim \sum (p_j^+ + p_j^-) \sim -2b - 4c,$$

which implies that  $b - c = 0$ . Hence  $f^2 = \text{id}$  on  $C$ . It follows that  $f$  can only realize orbit data for which all orbit lengths satisfy  $n_j \leq 2$ . Proposition 2.2 now implies that all roots of the characteristic polynomial have magnitude 1, contrary to hypothesis.

We can assume therefore that  $f|_{C_{reg}}$  has multiplier  $+1$ . Theorem 2.4 implies that  $f$  fixes  $C$  component-wise. Comparing degrees, we find that  $\{xy - z^2\}$  contains two points, say  $p_1^+, p_2^+$ , of  $I(f)$  and  $\{z = 0\}$  contains  $p_3^+$ . Since the components map to themselves, it follows that  $p_1^-, p_2^- \in \{xy = z^2\}$  and  $p_3^- \in \{z = 0\}$ . Proposition 1.3 gives

$$p_1^- - p_1^+ \sim p_2^- - p_2^+ \sim -b - c, \quad p_3^+ - p_3^- \sim -2b, \quad \sum p_j^- \sim -2b - c.$$

The permutation  $\sigma$  in the orbit data must fix the index 3. Hence either  $\sigma = \text{id}$  or  $\sigma$  switches the indices 1 and 2. Suppose we are in the former case. Then for  $j = 1, 2$ , we have  $p_j^+ - p_j^- \sim (n_j - 1)b$ . Combining this with the formulas above gives  $(n_j - 1)b \sim c$  and hence  $(n_2 - n_1)b \sim 0$ . So if  $n_2 \neq n_1$ , we see that  $b \sim m/n$ , where  $0 < n < \max\{n_1 - 1, n_2 - 1\}$  and  $0 \leq m < n$  are integers. So if, say,  $n_2 \geq n_1$ , we find  $f^{n_2 - n_1}(p_2^-) \sim p_2^+$  and therefore  $f$  does not realize the given orbit data. It follows that  $n_2 = n_1$ .  $\square$

**Example 4.6.** *We can realize the orbit data  $n_1 = n_2 = 5$ ,  $n_3 = 4$ ,  $\sigma = \text{id}$  on  $C = \{(xy - z^2)z = 0\}$  as follows. Choose  $p_1^-, p_2^- \in \{xy = z^2\}$  so that  $p_1^- \sim 0 \in \mathbf{C}/\mathbf{Z}$ ,  $p_2^- \sim i$ , and  $p_3^- \in \{z = 0\}$  so that  $p_3^- \sim -i - 5/7$ . Then from Theorem 1.4 we obtain a quadratic transformation  $f$  with  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$  that properly fixes each component of  $C$ , acting on  $\{xy = z^2\}$  by  $f(p) \sim p + 1/7$  and on  $\{z = 0\}$  by  $f(p) \sim p + 3/7$ . Also, from Proposition 1.3 we obtain that the points in  $I(f)$  satisfy  $p_3^+ = p_3^- + 2/7$ , and that for  $j = 1, 2$ ,  $p_j^+ \sim p_j^- + 4/7$ . Since for each  $j$ , the points  $p_j^+$  and  $p_j^-$  lie in the same component of  $C$ , we infer that  $f^3(p_3^-) = p_3^+$  and that for  $j = 1, 2$ ,  $f^4(p_j^-) = p_j^+$ . Hence  $f$  tentatively realizes the given orbit data. Since as one verifies directly, all 14 points  $p_j^-, \dots, f^{n_j-1}(p_j^-)$ ,  $j = 1, 2, 3$  are distinct, we conclude that  $f$  realizes the given orbit data.*

**Example 4.7.** *Let  $p_1^-, p_2^- \in \{xy = z^2\}$  be given by  $p_1^- \sim 8/13$ ,  $p_2^- \sim 0$ , and  $p_3^- \in \{z = 0\}$  by  $p_3^- \sim 12/13$ . Then from Theorem 1.4, we get a unique quadratic transformation  $f$  with  $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$  that properly fixes each component of  $C$ , acting by  $f(p) \sim p + 3/13$  on  $\{xy = z^2\}$  and by  $f(p) \sim p + 1/13$  on  $\{z = 0\}$ . The points in  $I(f)$  are given by  $p_1^+ \sim 12/13$ ,  $p_2^+ \sim 4/13$ ,  $p_3^+ \sim 5/13$ . From this information, one verifies that  $f$  realizes the orbit data  $n_1 = 3$ ,  $n_2 = 4$ ,  $n_3 = 7$ ,  $\sigma = (12)$ .*

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