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Splittings and Cr-structures for manifolds with nonpositive sectional curvature

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Abstract. Let $\tilde{M}^n$ denote the universal covering space of a *compact* Riemannian manifold, M^n, with sectional curvature, $-1 \le K_{M^n} \le 0$. We show that a collection of deck transformations of $\tilde{M}^n$, satisfying certain (metric dependent) conditions, determines an open dense subset of M^n, at every point of which, there exists a local isometric splitting with nontrivial flat factor. Such a collection, which we call an abelian structure, also gives rise to an essentially canonical Cr-structure in the sense of Buyalo, i.e an atlas for an injective F-structure, for which additional conditions hold. It follows in particular that the minimal volume of M^n vanishes. We show that an abelian structure exists if the injectivity radius at all points of M^n is less than $\epsilon(n) > 0$. This yields a conjecture of Buyalo as well as a strengthened version of the conclusion of Gromov's "gap conjecture" in our special situation. In addition, we observe that abelian structures on nonpositively curved manifolds have certain stability properties under suitably controlled changes of metric.

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0. Introduction

Let $\tilde{M}^n$ be a simply connected complete Riemannian manifold with nonpositive sectional curvature, $K_{\tilde{M}^n} \leq 0$. For γ an isometry of $\tilde{M}^n$, let $\delta_\gamma(\tilde{x}) = \overline{\gamma(\tilde{x}), \tilde{x}}$, denote the displacement function of γ.

From now on, we consider elements, $\gamma \in \Gamma$, where Γ is the group of deck transformations of the universal covering space, $\tilde{M}^n$, of the compact Riemannian manifold M^n, with nonpositive sectional curvature, $K_{\tilde{M}^n} \leq 0$. For such γ, the set of points, $Min(\gamma)$, at which $\delta_\gamma(\tilde{x})$ takes its minimum, is nonempty and coincides with the union of the set of γ-invariant geodesics.

Let $\sigma : (-\infty, \infty) \to \tilde{M}^n$ be a geodesic. The assumption, $K_{M^n} \leq 0$, implies that the function $\delta_\gamma(\sigma(t))$ is *convex* and in particular, every sublevel set of δ_γ is a closed *convex* subset.

We recall that in general, a convex subset, C, of a Riemannian manifold is a totally geodesic submanifold, whose boundary, if nonempty, might not be smooth; see [CE]. Thus, the interior of C is nonempty if and only if C has top dimension.

If $\gamma_1, \dots, \gamma_k$ are mutually commuting deck transformations, then $\bigcap_{\gamma_i} Min(\gamma_i) \neq \emptyset$ and there is a *canonical* isometric splitting, $\bigcap_{\gamma_i} Min(\gamma_i) = D(\gamma_1, \dots, \gamma_k) \times \mathbb{R}^{k'}$, such that each γ_i acts by translation on the factor, $\mathbb{R}^{k'}$, and by the identity on $D(\gamma_1, \cdots, \gamma_k)$, and in addition, the quotient of $\mathbb{R}^{k'}$ by the projected action of the group of translations generated by $\gamma_1, \dots, \gamma_k$ is a torus; see [GW], [LY]. The set, $D(\gamma_1, \cdots, \gamma_k)$, might actually split off a nontrivial Euclidean factor. If for example, the curvature of M^n is strictly negative, then any such intersection of minimum set consists of a single geodesic line (i.e. $k' = 1$) and $D(\gamma_1, \dots, \gamma_k)$ is a point.

Definition 0.1. An *abelian structure*, $(A, \{Q_\phi\})$, on M^n is a collection of elements, $A \subset \Gamma$, with $1 \notin A$ and for each $\phi \in A$ a closed subset $\emptyset \neq Q_\phi \subset \tilde{M}^n$, such that the following conditions hold.

(0.1.1) $\gamma A \gamma^{-1} = A$, for all $\gamma \in \Gamma$.

(0.1.2) There exists $f : A \to [0, T]$, for some $T < \infty$, with $f(\phi) = f(\gamma\phi\gamma^{-1})$, for all $\phi \in A$, $\gamma \in \Gamma$, such that $Q_\phi = \{\tilde{x} \mid \delta_\phi(\tilde{x}) \leq f(\phi)\}$. Thus, $\gamma(Q_\phi) = Q_{\gamma\phi\gamma^{-1}}$.

(0.1.3) $\bigcup_{\phi \in A} Q_\phi = \tilde{M}^n$.

(0.1.4) If $Q_{\phi_1} \cap Q_{\phi_2} \neq \emptyset$, then ϕ_1, ϕ_2 commute.

From the discreteness of Γ, it follows that for any compact set, $K \subset \tilde{M}^n$, we have $Q_\phi \cap K \neq \emptyset$, for only finitely many of the sets Q_ϕ. In addition, the existence of compact fundamental domain for the discrete group, Γ, implies that there exists $N < \infty$, such for all $\tilde{x} \in \tilde{M}^n$, we have $\tilde{x} \in Q_\phi$, for at most N of the sets Q_ϕ.

The concept of abelian structure can be formulated in terms of the space, M^n, as follows.

For c a curve from $x_1 \in M^n$ to $x_2 \in M^n$, let I_c denote the canonical isomorphism, $I_c : \pi_1(M^n, x_1) \to \pi_1(M^n, x_2)$. We say that $\ell_1 \in \pi_1(M^n, x_1)$, $\ell_2 \in \pi_1(M^n, x_2)$ are conjugate, if $\ell_2 = I_c(\ell_1)$, for some c. We denote the conjugacy class of ℓ by $[\ell]$. If $\ell \in \pi_1(M^n, x)$, we write $L(\ell)$ for the length of the geodesic loop on x, of minimal length, representing ℓ.

Now consider $(A^*, \{Q^*_{[\ell]}\})$, where $A^* \subset \bigcup_{x \in M^n} \pi_1(M^n, x)$ is a collection of elements which does not contain $1 \in \pi_1(M^n, x)$, for any $x \in M^n$, and $Q^*_{[\ell]} \subset M^n$, for each $\ell \in A^*$. Assume that the following conditions hold.

(0.1.1)' A^* is a union of conjugacy classes.

(0.1.2)' There exists $f : A^* \to [0, T]$, for some $T < \infty$, with $f(\ell_1) = f(\ell_2)$, if ℓ_1, ℓ_2 are conjugate, such that $Q^*_{[\ell]} = \{x \mid L(\ell') \le f([\ell]),\ \text{for some } \ell' \in A^* \cap \pi_1(M^n, x),\ \text{with } \ell' \in [\ell]\}$.

(0.1.3)' $\bigcup_{\ell \in A^*} Q^*_{[\ell]} = M^n$.

(0.1.4)' If $\ell_1, \ell_2 \in A^* \cap \pi_1(M^n, x)$ satisfy $L(\ell_1) \le f([\ell_1])$, $L(\ell_2) \le f([\ell_2])$, then ℓ_1, ℓ_2 commute.

A collection, $(A^*, \{Q^*_{[\ell]}\})$, satisfying (0.1.1)'–(0.1.4)', determines a unique abelian structure. Conversely, an abelian structure, $(A, \{Q_\phi\})$, determines a unique collection, $(A^*, \{Q^*_{[\ell]}\})$, satisfying (0.1.1)'–(0.1.4)'.

If $\pi_1(M^n)$ contains a normal abelian subgroup, G, then for any metric on M^n there exists an abelian structure on M^n. To see this, observe that by compactness, there exists $T < \infty$, such that for all $\tilde{x} \in \tilde{M}^n$, there exists a set of generators, $G(\tilde{x})$, for G, such that $\delta_\phi(\tilde{x}) \le T$, for all $\phi \in G(\tilde{x})$. If we put $A = \bigcup_{\tilde{x}} G(\tilde{x})$, and $Q_\phi = \{\tilde{y} \mid \delta_\phi(\tilde{y}) \le T\}$, for $\phi \in A$, then $(A, \{Q_\phi\})$ defines an abelian structure.

On the other hand, it follows from Theorem 0.3 that $K_{M^n} < 0$ implies that there exists no abelian structure for the given metric.

Given an abelian structure, we denote by $A^\flat \subset A$, the subset of elements, ϕ, such that the interior of Q_ϕ is nonempty. Clearly, the collection, $(A^\flat, \{Q_\phi\})$, satisfies (0.1.1), (0.1.2), (0.1.4) (where now $\phi \in A^\flat$). In fact, (0.1.3) holds as well. In the following proposition (proved in Subsect. 1.a) the assumption, $K_{M^n} \le 0$, plays no role.

Proposition 0.2. *If $(A, \{Q_\phi\})$ is an abelian structure on M^n, then $(A^\flat, \{Q_\phi\})$ is an abelian structure on M^n.*

An element, γ is called *s-stable* if $Min(\gamma^i) = Min(\gamma)$, for all $1 \le i \le s$ and *stable* if it is s-stable for all s. We will show in Proposition 5.1 of Subsect. 5.b that γ is stable if the interior of $Min(\gamma)$ is nonempty. Let $\mathcal{A} \subset A^\flat$ denote the set of elements, ϕ, such that $Min(\phi)$ has nonempty interior. Thus, the set, $\mathcal{A}$, consists of stable elements. This fact plays a role in Sect. 5. (The stability condition also enters crucially in Theorem 0.6 below.)

Theorem 0.3. *Let M^n be a compact manifold with nonpositive curvature, $K_{M^n} \le 0$, and let $(A, \{Q_\phi\})$ be an abelian structure on M^n. Then:*

(0.3.1) The collection, $(\mathcal{A}, \{Min(\phi)\})$ *defines an abelian structure on* M^n*.*

(0.3.2) The elements, $\phi \in \mathcal{A}$ *are stable and the n-dimensional closed convex set,* $Min(\phi)$*, splits isometrically,* $Min(\phi) = D(\phi) \times \mathbb{R}^1$*, where* ϕ *acts by translation on the factor,* $\mathbb{R}^1$*, and by the identity on* $D(\phi)$*.*

The second part, (0.3.2), of Theorem 0.3, is just a restatement of facts which were mentioned above. If we grant that the collection, $(A, \{Min(\phi)\})$, defines an abelian structure, then (0.3.1) follows by applying Proposition 0.2 to this structure. Thus, to prove Theorem 0.3, it suffices to show that $(A, \{Min(\phi)\})$ defines an abelian structure; see Subsect. 1.c.

In view of (0.1.1), (0.1.2), the local splittings on an open dense subset of $\tilde{M}^n$ which are guaranteed by Theorem 0.3, induce a corresponding local splitting structure on an open dense subset of the manifold M^n. This splitting structure has additional properties which are not immediately apparent. These properties, discussed in Sect. 5 (and related in part to the stability of elements of $\mathcal{A}$) are relevant in proving the existence of the (essentially) canonical Cr-structure (in the sense of Buyalo) associated to the abelian structure A; see [Bu2]–[Bu5] for a detailed discussion of Cr-structures.

Theorem 0.4. *Let* M^n *be as in Theorem 0.3. Then* M^n *admits a* Cr*-structure which is compatible with the local splitting structure associated to the abelian structure,* $(\mathcal{A}, \{Min(\phi)\})$*.*

A Cr-structure is an atlas for an injective F-structure for which two additional conditions hold; see Sect. 4 for precise definitions of the concepts appearing in Theorem 0.4. Manifolds with Cr-structure can be viewed as generalized graph manifolds; compare [Gr1], [BGS], [Sc]. Nilmanifolds provide examples of 3-dimensional manifolds with pure injective F-structures which do not admit compatible Cr-structures.

From Theorem 0.4 and the collapsing construction of [CG1] for polarized (and hence, for injective) F-structures, we immediately obtain:

Theorem 0.5. *Let* M^n *be as in Theorem 0.3. Then the minimal volume of* M^n *vanishes.*

Let $\Gamma_\delta(\tilde{x})$ denote the subgroup generated by those $\gamma \in \Gamma$, for which $\delta_\gamma(\tilde{x}) \le \delta$. The Margulis Lemma provides constants, $\delta(n), i(n) > 0$, such that $\Gamma_{\delta(n)}(\tilde{x})$ has an abelian subgroup of index $\le i(n)$ (and is in fact, a Bieberbach group); see [Eb2], [GW], [LY]. An important application is the assertion (Margulis, Heintze) that if M^n is a compact manifold with $-1 \le K_{M^n} < 0$, then the injectivity radius at some point, $x \in M^n$, satisfies $Inj_x > \delta(n)/2$; see [BGS], p. 101.

The existence of flat tori with arbitrarily small injectivity radius already demonstrates that for compact manifolds with $-1 \le K \le 0$, the above assertion can fail, no matter how small $\delta(n)$ is chosen.

In the Theorem 0.6 below, the stability condition plays a crucial role. In this connection, we recall the following result whose proof is given on

p. 127 of [BGS]: For all $\phi \in \Gamma$, there exists $1 \leq j \leq 2s^{[\frac{n}{2}]}$, such that ϕ^j is s-stable.

Let $w(n)$ denote the smallest integer such that the index of the maximal normal abelian subgroup of any Bieberbach group of rank at most n, divides $w(n)$. Put $m(n) = 2w(n)^{[\frac{n}{2}]}$.

Let S denote the set of $w(n)$-stable elements, ϕ, for which the set, $Q_\phi = \{\tilde{x} \mid \delta_\phi(\tilde{x}) \leq \delta(n)\}$, is nonempty. Let $\mathcal{S} \subset S$ denote the subset of elements, ϕ, such that $Min(\phi)$ has nonempty interior.

Clearly, the collection, $(S, \{Q_\phi\})$, satisfies (0.1.1), (0.1.2). If we assume $K_{M^n} \leq 0$ and $inj_x \leq \delta(n)/m(n)$ for each $x \in M^n$, then from the above mentioned result of [BGS] concerning stability, it follows directly that (0.1.3) holds. Using the stability condition, we will show that (0.1.4) holds as well; see Sect. 2.

Theorem 0.6. *Let M^n be a compact manifold with nonpositive curvature, $-1 \leq K_{M^n} \leq 0$ and $inj_x \leq \delta(n)/m(n)$ for each $x \in M^n$. Then:*

(0.6.1) The collection, $(S, \{Q_\phi\})$, where $Q_\phi = \{\tilde{x} \mid \delta_\phi(\tilde{x}) \leq \delta(n)\}$, defines an abelian structure on M^n. Hence:

(0.6.2) The collection $(\mathcal{S}, \{Min(\phi)\})$, defines an abelian structure on M^n.

As an immediate corollary of Theorems 0.3 and 0.6, we obtain the following generalization of the result of Margulis and Heintze which was mentioned above.

Corollary 0.7. *Let M^n be a compact manifold with $-1 \leq K_{M^n} \leq 0$. If there is a point at which the Ricci curvature is negative, then the injectivity radius of M^n at some point is $> \delta(n)/m(n)$.*

From Theorems 0.3, 0.4 and 0.6 we get a strengthened version of Gromov's gap conjecture (for the minimal volume of sufficiently *injectivity radius* collapsed manifolds) under the strong additional assumption that the curvature of M^n is nonpositive; see [Gr2] and compare also [CG1], [CG2], [CR2], [Fu1], [Ro2].

Corollary 0.8. *Let M^n be a compact manifold with nonpositive curvature, $-1 \leq K_{M^n} \leq 0$ and $inj_x \leq \delta(n)/m(n)$ for each $x \in M^n$. Then the minimal volume of M^n vanishes.*

Theorems 0.3, 0.4, 0.6 can be also used to confirm a conjecture of Cheeger-Gromov in the special case in which the curvature satisfies $-1 \leq K \leq 0$. The conjecture asserts that if a compact $(4k-1)$-manifold admits a sequence of metrics whose volumes converge to zero and whose induced metrics on the universal covering space have uniformly bounded covering geometry, then the limit of the eta-invariants (in the sense of Atiyah-Patodi-Singer) exists and is rational. By [CG3], the limit does in fact exist and is independent of a particular collapsing sequence. It was shown in [Ro1] the conjecture is true for $k = 1$. We intend to discuss this elsewhere.

The following is another immediate corollary of Theorems 0.3, 0.4, 0.6.

Corollary 0.9. *Let M^n be as in Theorem 0.3. Then the fundamental group of M^n has an abelian subgroup of rank ≥ 2. In particular, this holds for M^n as in Theorem 0.6.*

To see that Theorems 0.3, 0.4 imply Corollary 0.9, observe that the conclusion is trivial unless all orbits of the injective F-structure have dimension 1. In this case, the collection of orbits determines a normal cyclic subgroup of $\pi_1(M^n)$ with generator ϕ. Let $[\phi, \gamma]$ denote the group generated by ϕ, γ. Then for all $\gamma \in \pi_1(M^n)$, the group $[\phi, \gamma]$, is solvable and hence, is a Bieberbach group which acts freely on $\mathbb{R}^k$, for some $k \leq 2$; see [GW], [LY]. If there exists γ, for which the lattice subgroup of the Bieberbach group, $[\phi, \gamma]$ has rank 2, then the assertion is trivial. Thus, we can assume that the lattice subgroup of the Bieberbach group, $[\phi, \gamma]$ has rank 1, for all $\gamma \in \pi_1(M^n)$. Since a Bieberbach group whose lattice has rank 1 is actually cyclic, and in particular, abelian, it follows that ϕ lies in the center of $\pi_1(M^n)$. By the center theorem of [GW], [LY], it follows that $\pi_1(M^n)$ has an abelian subgroup of rank ≥ 2, containing ϕ; a contradiction.

The fact that under the assumptions of Theorem 0.6, the fundamental group of M^n has an abelian subgroup of rank ≥ 2, is due to Buyalo; see [Bu1].

For $n = 3$, the existence of an injective F-structure on a sufficiently collapsed compact manifold of nonpositive curvature follows from [CG2]. For $n = 3, 4$, the existence of a Cr-structure and corresponding local splitting structures on such a manifold is proved in [Bu2], [Bu3], where it is conjectured that such exist in arbitrary dimensions. Buyalo's conjecture is a direct consequence of Theorems 0.3, 0.4 and 0.6. In the context of injective F-structures, similar results were asserted without proof in [Fu2]. In none of the above references does the notion of stability enter.

In [Bu4], it is shown that in dimension 3, a Cr-structure on a given manifold is unique. While it is not clear whether such a result holds in higher dimensions, we will observe that abelian structures on nonpositively curved manifolds do have certain rigidity properties under change of metric; see Sect. 3.

The rest of the paper is organized as follows:

In Sect. 1, we prove Proposition 0.2 and Theorem 0.3.

In Sect. 2, we prove Theorem 0.6.

In Sect. 3, we prove some rigidity properties under change of metric, of abelian structures on nonpositively curved manifolds.

In Sect. 4, we review the definitions pertaining to F-structures and Cr-structures.

In Sect. 5, we discuss some issues related to stability and prove Theorem 0.4.

In Sect. 6, we give an example which shows that on a sufficiently collapsed compact manifold of nonpositive curvature, the F-structure which arises from the general construction of [CG2], need not be *injective*.

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1. Proof of Proposition 0.2 and Theorem 0.3

a. Proof of Proposition 0.2

In this subsection, the assumption, $K_{M^n} \leq 0$ plays no role.

Proof of Proposition 0.2. It suffices to prove that if the collection, $(A, \{Q_\phi\})$, satisfies (0.1.1)–(0.1.4), then so does the subcollection, $(A^\flat, \{Q_\phi\})$.

Since, (0.1.1), (0.1.2), (0.1.4) are clear, it is enough to show that $A^\flat$ satisfies (0.1.3). In particular, it suffices to show that for all $\tilde{x} \in B_R(\tilde{y})$, a ball of finite radius (which we choose large enough so that $B_R(\tilde{y})$ contains a fundamental domain) we have $\tilde{x} \in Q_\phi$, for some $\phi \in A^\flat$.

Let $E \subset A$ denote the subset of elements, ϕ, such that $Q_\phi \cap B_R(\tilde{y}) \neq \emptyset$. Since Γ is discrete, it follows that E is a finite set. Since A satisfies (0.1.3), the nonempty open set, $B_R(\tilde{y})$, is contained in the finite union of closed sets, $\bigcup_{\phi \in E} Q_\phi$. But if a nonempty open set, U, is contained in a finite union, $K_1 \cup \cdots \cup K_N$ of closed sets, then U is actually contained in the union, say $K_1 \cup \cdots \cup K_{N'}$, of those K_i whose interiors are nonempty. (Otherwise, the nonempty open set, $U \setminus (K_1 \cup \cdots \cup K_{N'})$, is contained in a union of closed sets, $K_{N'+1} \cup \cdots \cup K_N$, all of whose interiors are empty. By an easy inductive argument, this is impossible.) This suffices to complete the proof. □

b. Convex sets and monotonicity of displacement functions

In this subsection, we recall some standard facts which will be used in the proof of Theorem 0.3; see [BO].

Let $\tilde{C} \subset \tilde{M}^n$ be a convex subset. Recall that $\tilde{C}$ is a totally geodesic submanifold with possibly nonempty boundary which might not be smooth; see [CE]. A geodesic ray, $\sigma : [0, \infty) \to \tilde{M}$, parameterized by arc length, is called *at least normal to* $\tilde{C}$ if $\sigma \cap \tilde{C} = \sigma(0)$ and the angle between $\sigma'(0)$ and any vector tangent to $\tilde{C}$ is at least $\pi/2$.

If $\tilde{x} \notin \tilde{C}$, then there is a unique point $\tilde{y} \in \tilde{C}$, closest to $\tilde{x}$. If $\sigma : [0, \infty) \to \tilde{M}^n$ is a geodesic ray, with $\sigma(0) = \tilde{y}$ and $\sigma(t_0) = \tilde{x}$, then σ is at least normal to $\tilde{C}$.

Proposition 1.1. *Let $\tilde{M}$ be a simply connected complete manifold with $K_{\tilde{M}} \leq 0$. Let $\tilde{C}$ be a convex subset, and let σ be a geodesic ray at least normal to $\tilde{C}$. If γ is an isometry of $\tilde{M}$ which preserves $\tilde{C}$, then:*

(1.1.1) The function, $\delta_\gamma(\sigma(t))$*, is convex and nondecreasing.*

(1.1.2) If $\sigma(0) \in Min(\gamma)$ *and* $\sigma(t_0) \notin Min(\gamma)$*, then for* $t \geq t_0$*, the function,* $\delta_\gamma(\sigma(t))$*, is strictly increasing and* $\lim_{t\to\infty} \delta_\gamma(\sigma(t)) = \infty$.

Proof. Since $\tilde{M}$ is a simply connected complete manifold with $K_{\tilde{M}} \leq 0$, it follows that $h(t) = \delta_\gamma(\sigma(t))$ is a smooth a convex function on $[0, \infty)$; see [CE]. Thus, it suffices to check that $h'(0) \geq 0$.

Since γ is an isometry preserving $\tilde{C}$, it follows that the geodesic ray, $\gamma(\sigma)$, is also at least normal to $\tilde{C}$. Thus, the first variation formula implies that $h'(0) \geq 0$. □

c. Proof of Theorem 0.3

Proof of Theorem 0.3. As mentioned after Theorem 0.3, by Proposition 0.2, it suffices to prove that $(A, \{Min(\phi)\})$ defines an abelian structure on M^n.

For $\tilde{x} \in \tilde{M}^n$, let $A(\tilde{x}) \subset A$ denote the subset of elements, ϕ, such that $\tilde{x} \in Q_\phi$. The discreteness of Γ implies that $\bigcup_{\tilde{x}\in C} A(\tilde{x})$ is a finite set, for any compact set C.

We will show that for all $\tilde{x} \in \tilde{M}^n$, there exists $\phi \in A(\tilde{x})$ such that $\tilde{x} \in Min(\phi)$.

We argue by contradiction.

Let $\tilde{x} \in \tilde{M}^n$ be such that $\tilde{x} \notin Min(\phi)$, for all $\phi \in A(\tilde{x})$, and in addition, the cardinality $\#(A(\tilde{x}))$, of the set, $A(\tilde{x})$, is minimal with respect to all $\tilde{x} \in \tilde{M}^n$ for which this property holds.

By (0.1.3), the set, $A(\tilde{x})$ is nonempty and by (0.1.4), the elements of $A(\tilde{x})$ are mutually commuting. Thus, $\bigcap_{\phi\in A(\tilde{x})} Min(\phi) \neq \emptyset$; see [GW], [LY]. Since, $\tilde{x} \notin \bigcap_{\phi\in A(\tilde{x})} Min(\phi)$, there is a unique point $\tilde{y} \in \bigcap_{\phi\in A(\tilde{x})} Min(\phi)$, closest to $\tilde{x}$. If $\sigma : [0, \infty) \to \tilde{M}^n$ is a geodesic ray, with $\sigma(0) = \tilde{y}$ and $\sigma(t_0) = \tilde{x}$, then σ is at least normal to $\bigcap_{\phi\in A(\tilde{x})} Min(\phi)$.

By Proposition 1.1, we have:

(1.2.1) $\sigma(t) \notin \bigcup_{\phi\in A(\sigma(t_0))} Min(\phi)$, for all $t \geq t_0$.

(1.2.2) $\lim_{t\to\infty} \delta_\phi(\sigma(t)) = \infty$, for all $\phi \in A(\sigma(t_0))$.

It follows from (1.2.2) and an obvious continuity argument that there is a largest value, t_1, with $t_0 \leq t_1 < \infty$, such that $A(\sigma(t_0)) \subset A(\sigma(t_1))$.

From the finiteness of the set, $\bigcup_{\tilde{y}\in B_1(\tilde{x})} A(\tilde{y})$, the fact that displacement functions, δ_ϕ, are continuous, and the minimality assumption on $\#(A(\sigma(t_0)))$, it follows that $A(\sigma(t_0))$ is a proper subset of $A(\sigma(t_1))$. Otherwise, we would have $A(\sigma(t_1)) = A(\sigma(t_0))$, and for $t > t_1$, with $t - t_1$ sufficiently small, $A(\sigma(t))) \subsetneq A(\sigma(t_1))$. Then $A(\sigma(t)) \subsetneq A(\sigma(t_0))$, which, together with (1.2.1), implies $\sigma(t) \notin \bigcup_{\phi\in A(\sigma(t))} Min(\phi)$. Since also $\#(A(\sigma(t)))) < \#(A(\sigma(t_0)))$, this contradicts the minimality assumption.

Let $\psi \in A(\sigma(t_1)) \setminus A(\sigma(t_0))$. By (0.1.4), ψ commutes with all elements of $A(\sigma(t_0))$. Thus, ψ leaves invariant the set, $\bigcap_{\phi\in A(\tilde{x})} Min(\phi)$. Hence, by

Proposition 1.1, the function, $\delta_\psi(\sigma(t))$, is nondecreasing on $[0, \infty)$. Since Q_ψ is a sublevel set of the displacement function, δ_ψ, it follows that $\psi \in A(\sigma(t))$, for all $0 \le t \le t_1$; a contradiction. □

2. Proof of Theorem 0.6

Proof of Theorem 0.6. Clearly, the collection, $(S, \{Q_\phi\})$, satisfies (0.1.1), (0.1.2).

For any $\tilde{x} \in \tilde{M}^n$, let $\gamma \neq 1$ be a deck transformation such that $\delta_\gamma(\tilde{x}) \le \delta(n)/m(n)$. By the result of [BGS], p.127, which was recalled prior to the statement of Theorem 0.6, there exists $1 \le i \le 2w(n)^{[\frac{n}{2}]}$ such that $\phi = \gamma^i$ is $w(n)$-stable. Then, $\delta_\phi(\tilde{x}) = \delta_{\gamma^i}(\tilde{x}) \le i \cdot \delta_\gamma(\tilde{x}) \le \delta(n)$. Thus, $\tilde{x} \in Q_\phi$ and $\phi \in S$ and it follows that $(S, \{Q_\phi\})$, satisfies (0.1.3).

Let $\tilde{x} \in Q_\phi \cap Q_\psi$, for some $\phi, \psi \in S$. Since $\Gamma_{\delta(n)}(\tilde{x})$ is a Bieberbach group of rank $\le n$, we have $\phi^{w(n)}\psi^{w(n)} = \psi^{w(n)}\phi^{w(n)}$. Hence, $Min(\phi^{w(n)}) \cap Min(\psi^{w(n)}) \neq \emptyset$. Moreover, $Min(\phi^{w(n)}) \cap Min(\psi^{w(n)})$ has a canonical isometric splitting, $D(\phi^{w(n)}, \psi^{w(n)}) \times \mathbb{R}^i$, where $i = 1$ or $i = 2$, such that $\phi^{w(n)}, \psi^{w(n)}$ each act on $\mathbb{R}^i$ by translation, $t_{\phi^{w(n)}}, t_{\psi^{w(n)}}$, and on $D(\phi^{w(n)}, \psi^{w(n)})$ by the identity; see [GW], [LY].

Since ϕ, ψ are $w(n)$-stable, $Min(\phi) = Min(\phi^{w(n)})$, $Min(\psi) = Min(\psi^{w(n)})$. Thus, the collection of all $\phi^{w(n)}$-axes coincides with the collection of all ϕ-axes and the collection of all $\psi^{w(n)}$-axes coincides with the collection of all ψ-axes. In addition, $D(\phi^{w(n)}) \times \mathbb{R} = D(\phi) \times \mathbb{R}$ and $D(\psi^{w(n)}) \times \mathbb{R} = D(\psi) \times \mathbb{R}$.

It follows that $Min(\phi) \cap Min(\psi) = Min(\phi^{w(n)}) \cap Min(\psi^{w(n)}) \neq \emptyset$ is a union of ϕ-axes and of ψ-axes. In particular, this set is invariant under ϕ, ψ. In addition, ϕ, ψ act on $\mathbb{R}^i$ by translations, $\frac{1}{w(n)} \cdot t_{\phi^{w(n)}}$, $\frac{1}{w(n)} \cdot t_{\psi^{w(n)}}$ and on $D(\phi^{w(n)}, \psi^{w(n)})$ by the identity.

Note that the action of a group of deck transformations on any invariant subset for this group is effective. Since on $D(\phi^{w(n)}, \psi^{w(n)}) \times \mathbb{R}^i$, the actions of ϕ and ψ commute, it follows that $\phi\psi = \psi\phi$. Hence, $(S, \{Q_\phi\})$, satisfies (0.1.4). This completes the proof of Theorem 0.6. □

3. Rigidity properties of abelian structures

The results of this section can be viewed as generalizations of Theorem 0.6.

Let M^n be a Riemannian manifold with metric g. In this section (and only this section) we will denote the collection of sets, $\{Q_\phi\}$, in (0.1.2), by $\{Q_{g,f,\phi}\}$.

In what follows, the statement, "ϕ is $w(n)$-stable, for all $\phi \in A$", will be taken to mean "either $\{\phi \in A\}$ consists of elements which are $w(n)$-stable with respect to g_1, or $\{\phi \in A\}$ consists of elements which are $w(n)$-stable with respect to g".

Theorem 3.1. *Let* $-1 \leq K_{M^n,g_1} \leq 0$*. Let* $K_{M^n,g} \leq 0$ *and let* $(A, \{Q_{g,f,\phi}\})$ *satisfy (0.1.1)–(0.1.3), where* $f \leq \delta(n)$ *and* ϕ *is* $w(n)$*-stable, for all* $\phi \in A$*.*

(3.1.1) If $Q_{g,f,\phi} \subset Q_{g_1,f,\phi}$*, for all* $\phi \in A$*, then the collections,* $(A, \{Q_{g,f,\phi}\})$*,* $(A, \{Q_{g_1,f,\phi}\})$*, define abelian structures.*

(3.1.2) In particular, (3.1.1) holds if $g_1 \leq g$*.*

Proof. Clearly, it suffices to verify that $\tilde{x} \in Q_{g_1,f,\phi} \cap Q_{g_1,f,\psi}$, implies $\phi\psi = \psi\phi$. As in Theorem 0.6, we have $\phi^{m(n)}\psi^{m(n)} = \psi^{m(n)}\phi^{m(n)}$. Since, ϕ, ψ are stable with respect to g or g_1, this implies $\phi\psi = \psi\phi$. □

A collection, $(A, \{Q_{g,f,\phi}\})$, satisfying (0.1.1)–(0.1.3) will be called *minimal*, if $B \subset A$ and $(B, \{Q_{g,f,\phi}\})$ satisfies (0.1.1)–(0.1.3) implies $B = A$. Note that A minimal implies $A = A^\flat$; see Proposition 0.2. If in addition, $K_{M^n,g} \leq 0$, it follows from Proposition 5.1 below, that ϕ is stable, for all $\phi \in A$.

Theorem 3.2. *Let* $-1 \leq K_{M^n,g_1} \leq 0$*. Let* $K_{M^n,g} \leq 0$ *and let* $(A, \{Q_{g,f,\phi}\})$ *satisfy (0.1.1)–(0.1.3), where* $f \leq \delta(n)$ *and* ϕ *is* $w(n)$*-stable, for all* $\phi \in A$*.*

(3.2.1) If $\emptyset \neq Q_{g_1,\lambda f,\phi} \subset Q_{g,f,\phi} \subset Q_{g_1,f,\phi}$*, for some* $\lambda \leq 1$ *and all* $\phi \in A$*, then* $(A, \{Q_{g,f,\phi}\})$ *is an abelian structure, if and only if* $(A, \{Q_{g_1,f,\phi}\})$ *is an abelian structure, if and only if* $(A, \{Q_{g,\lambda f,\phi}\})$ *is an abelian structure. Moreover, any one of these structures is minimal if and only if all are minimal.*

(3.2.2) In particular, (3.2.1) holds if $g_1 \leq g \leq \lambda^{-2} g_1$*.*

Proof. Clearly, it suffices to check that if $B \subset A$ and $(B, \{Q_{g_1,f,\phi}\})$ defines an abelian structure, then so does $(B, \{Q_{g_1,\lambda f,\phi}\})$. Since $Min_{g_1}(\phi) \subset Q_{g_1,\lambda f,\phi}$ and we assume $Q_{g_1,\lambda f,\phi} \neq \emptyset$, for all ϕ, this follows from Theorems 0.3, 3.1. □

From Theorem 3.2 we immediately obtain:

Theorem 3.3. *Let* $-1 \leq K_{g_1} \leq 0$*,* $inj_x \leq \lambda\delta(n)/m(n)$*, for some* $\lambda \leq 1$ *and all* $\tilde{x} \in M^n$*. If* $K_{M^n,g} \leq 0$ *and* $\eta^{-2}\lambda^2 g_1 \leq g \leq \eta^{-2} g_1$*, for some* $0 < \eta \leq \lambda$*, then the conclusions of Theorem 3.2 hold with A the set of m(n)-stable elements,* ϕ*, such that* $\min(\delta_\phi) \leq \lambda\delta(n)$*,* $f = \delta(n)$*, and the metric, g, of Theorem 3.2, replaced by* $\eta^2 g$*.*

Note that the existence of a local splitting structure for the metric, $\eta^2 g$ (which follows from Theorems 0.3, 3.3) implies the existence of such a structure for the metric g as well.

If for g_1 in Theorem 3.3, we have $inj_x < \delta(n)/m(n)$, then we can choose $\lambda < 1$. In this case, Theorem 3.3 asserts that metrics, g, with $K_{M^n,g} \leq 0$ which are sufficiently close to g_1, possess a significant degree of rigidity.

It is easy to construct 3-dimensional examples with associated abelian structures with $Q_\phi = Min(\phi)$, for all ϕ, such that there exist ϕ, ψ, with $\dim(Min(\phi) \cap Min(\psi)) = 1$, for one of g, g_1 and $\dim(Min(\phi) \cap Min(\psi)) = 2$, for the other.

4. Injective F-structures and Cr-structures

Let M^n be a smooth manifold.

A *chart* is a pair, $(\mathcal{U}, \mathfrak{f})$, where $\mathcal{U} \subset M^n$ is open and $\mathfrak{f}$ is a sheaf of Lie algebras of vector fields on $\mathcal{U}$, with the property that there exists a finite normal covering, $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$, such that at each point of $\mathcal{U}$, the stalk of the pullback sheaf, $\tilde{\mathfrak{f}}$, is generated by the values of global sections and the space of global sections is the infinitesimal generator of the effective action of some torus T^k.

The orbits of the action of T^k on $\tilde{\mathcal{U}}$ are invariant under the action of the group of covering transformations. Thus, their images under the map, π, induce a partition of $\mathcal{U}$ into disjoint subsets which are also called *orbits*. A subset, $V \subset \mathcal{U}$, is called *saturated*, if it is the union of orbits.

An *atlas* is given by a collection of charts, $\{(\mathcal{U}_i, \mathfrak{f}_i)\}$, such that:

(4.1.1) $\{\mathcal{U}_i\}$ is a locally finite cover of M^n.

(4.1.2) For all $x \in M^n$, if $\mathcal{U}_1, \dots, \mathcal{U}_N$ are those $\mathcal{U}_i$ which contain x, then after possible reordering, we have $\mathfrak{f}_1 \subset \mathfrak{f}_2 \subset \cdots \subset \mathfrak{f}_N$.

(4.1.3) For $\mathcal{U}_1, \dots, \mathcal{U}_N$ as in (4.1.2), the set, $\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_N$, is saturated for $\mathfrak{f}_N$.

A sheaf of Lie algebras, $\mathfrak{f}$, on M^n is called an *F-structure* if there exists an atlas, $\{(\mathcal{U}_i, \mathfrak{f}_i)\}$, such that for all $x \in M^n$ and $(\mathcal{U}_N, \mathfrak{f}_N)$ as (4.1.2), there is an open set $\mathcal{U}$, with $x \in \mathcal{U} \subset \mathcal{U}_N$, such that $\mathfrak{f} \,|\, U = \mathfrak{f}_N \,|\, U$, and $\mathcal{U}$ is saturated for $\mathfrak{f}_N$; compare [CG1], [CG2].

An F-structure is called *polarized* if for all x, as above, the dimension of every orbit is equal to $\dim(\mathfrak{f}_N)$.

An F-structure is called *injective* if the fundamental group of every orbit injects into that of M^n. Clearly, an injective F-structure is polarized, but the converse need not hold.

A Cr-structure is an atlas for an injective F-structure for which the following additional conditions hold.

(4.2.1) Let $\mathcal{U}_1, \dots, \mathcal{U}_N$ be as in (4.1.2). Then after possible reordering, $\dim(\mathfrak{f}_1) < \dim(\mathfrak{f}_2) < \cdots < \dim(\mathfrak{f}_N)$.

(4.2.2) For all $\mathcal{U}_i$, the finite normal covering, $\tilde{\mathcal{U}}_i$, can be chosen such that for some V_i, there is a diffeomorphism, $\Phi_i : \tilde{\mathcal{U}}_i \to V_i \times T^{k(i)}$, for which the induced actions of the group of covering transformations and of the torus, $T^{k(i)}$, preserve the product structure, and the induced action of $T^{k(i)}$ is by the identity on the factor, V_i, and by left (equivalently, right) translation on the factor $T^{k(i)}$.

A *local splitting structure* on a manifold, M^n, is an open dense subset, $U \subset M^n$, and for each point $x \in U$, an open neighborhood, $U(x)$, with an isometric splitting, $B_{\epsilon(x)}(0) \times Z(x)$, where $B_{\epsilon(x)}(0) \subset \mathbb{R}^{k(x)}$. According to Theorem 0.3, given an abelian structure on a compact manifold of nonpositive curvature, there is a canonically associated local splitting structure.

A local splitting structure and an F-structure are said to be *compatible*, if for all x, every factor, $B_{\epsilon(x)} \times z$, is contained in an orbit of the F-structure (whose dimension might be strictly greater than $k(x)$). A local splitting structure and a Cr-structure are *compatible* if the local splitting structure is compatible with the F-structure underlying the Cr-structure.

5. Proof of Theorem 0.4

a. Outline of the construction

In this section, we construct the (essentially canonical) Cr-structure associated to an abelian structure, $(\mathcal{A}, \{Min(\phi)\})$, on M^n, with $K_{M^n} \leq 0$. This Cr-structure is compatible with the local splitting structure associated to $(\mathcal{A}, \{Min(\phi)\})$.

In Subsects. 5.b, 5.c, we discuss properties of stable elements which are required in order to obtain condition (4.2.1); see (5.4.2) (and (5.6.2)).

In Subsect. 5.d, in which assumption, $K_{M^n} \leq 0$, plays no role, we construct a canonical Γ-invariant decomposition, $\tilde{M}^n = \bigcup_{\mathcal{C}} N(\mathcal{C})$, of $\tilde{M}^n$ into disjoint subsets. Here, $\mathcal{C} \subset \mathcal{A}$, is of the form $\mathcal{C} = \mathcal{A}(\tilde{x})$, for some $\tilde{x} \in \tilde{M}^n$. When suitably modified, the sets, $N(\mathcal{C})$, give rise to the charts of the desired Cr-structure. (Typically, the sets, $N(\mathcal{C})$, are not open.)

In Subsect. 5.e, we modify the sets, $N(\mathcal{C})$, to obtain open sets, $W(\mathcal{C})$, which form a Γ-invariant cover of $\tilde{M}^n$. (In Subsect. 5.i, we consider sets, $\hat{W}(\mathcal{C})$, which are slight modifications of the sets $W(\mathcal{C})$.)

In Subsect. 5.f, we construct torus actions on finite normal covering spaces of the sets $N(\mathcal{C})$. Although these finite coverings, are canonical, it is not clear that they satisfy condition (4.2.2).

In Subsect. 5.g we show that, the sets, $N(\mathcal{C})$, have finite normal coverings, for which the conditions of (4.2.2) hold. (Since the sets, $N(\mathcal{C})$, are not open, we must eventually extend this result to the sets $\hat{W}(\mathcal{C})$.)

In Subsect. 5.h, we extend the local abelian actions to the sets, $\hat{W}(\mathcal{C})$, in such a way that the extended actions are compatible with the local splitting structure in the appropriate sense.

In Subsect. 5.i, we modify the extended actions so that (after replacing the $\hat{W}(\mathcal{C})$ by slightly smaller sets, still denoted $\hat{W}(\mathcal{C})$) they are compatible on nonempty intersections. This yields an injective F-structure which is compatible with the local splitting structure.

In Subsect. 5.j, in which the proof is concluded, we observe that our injective F-structure has an atlas which is a Cr-structure. It follows directly from Subsects. 5.b, 5.c, and the definition of the sets, $W(\mathcal{C})$, that (4.2.1) holds. Using the results of Subsect. 5.g, we show that (4.2.2) holds as well.

b. Stable elements

In this subsection, we prove Proposition 5.1, which concerns sets of stable elements. The proof is similar to that of the result of [BGS], p. 127, which was employed in the proof of Theorem 0.6; see Sect. 2.

Proposition 5.1. *Let $\phi \in \Gamma$ satisfy* $\dim(Min(\phi)) = n$. *Then ϕ is stable.*

Proof. Let $Min(\phi) = D \times \mathbb{R}^1$ and $Min(\phi^i) = D_i \times \mathbb{R}^1$. Then, $Min(\phi) \subset Min(\phi^i)$ and $D \subset D_i$. Moreover, ϕ is an isometry of $Min(\phi^i)$ which preserves the splittings.

Since D, D_i each have dimension $n-1$, it follows that the interior of D is a nonempty open subset of the interior of D_i. Therefore, since ϕ acts by the identity on D, ϕ acts by the identity on D_i as well. This easily implies that $Min(\phi^i) \subset Min(\phi)$ (see [GW], [LY]) which suffices to complete the proof. □

c. Minimal sets and stable elements

Let $G \subset \Gamma$ be a subset of mutually commuting elements. Put $Min(G) = \bigcap_{g \in G} Min(g) \neq \emptyset$. Let $\Theta(G)$ denote the subgroup generated by G. The first assertion of the following lemma is well known ([GW], [LY]).

Lemma 5.2. *(5.2.1) $Min(G) = Min(\Theta(G))$.*

(5.2.2) If every element of G is s-stable and $\Theta' \subset \Theta(G)$ is a subgroup of index $\leq s$, then $Min(\Theta') = Min(\Theta(G))$.

Proof. Clearly, to prove (5.2.1), it suffices to show $Min(G) \subset Min(\Theta(G))$. If $\psi \in \Theta(G)$, then $\psi = \phi_1^{a_1}\phi_2^{a_2}\cdots\phi_N^{a_N}$, for certain elements, $\phi_1, \ldots, \phi_N \in G$ and integers, $a_1, \ldots, a_n$. Each element, ϕ_i, preserves $Min(G)$ and the canonical splitting, $Min(G) = D \times \mathbb{R}^{k(G)}$, and acts by translation on the factor, $\mathbb{R}^{k(G)}$, and by the identity on the factor D (where $k(G)$ is the rank of $\Theta(G)$). From the representation, $\psi = \phi_1^{a_1}\phi_2^{a_2}\cdots\phi_N^{a_N}$, it follows directly that ψ has these properties as well. Thus, $Min(G)$ is a union of ψ-axes i.e. ψ-invariant geodesics. This implies $Min(G) \subset Min(\psi)$ and $Min(G) \subset Min(\Theta(G))$; see [GW], [LY].

Put $G^i = \bigcup_{g \in G} g^i$. If $i \leq s$ and every element of G is s-stable, it follows that $Min(G^i) = Min(G)$. If Θ' has index i, then $G^i \subset \Theta'$, which implies $Min(\Theta') \subset Min(G^i) = Min(G)$. Since by (5.2.1), we have $Min(G) = Min(\Theta(G))$, we get (5.2.2). □

As in Subsect. 1.b, we denote by $\mathcal{A}(\tilde{x})$, the subset of the elements, ϕ, such that $\tilde{x} \in Min(\phi)$. We will denote by $\mathcal{C} \subset \mathcal{A}$, a subset such that $\mathcal{C} = \mathcal{A}(\tilde{x})$, for some $\tilde{x}$. By Theorem 0.3, each such set, $\mathcal{C}$, is nonempty. We put $Min(\mathcal{C}) = \bigcap_{\phi \in \mathcal{C}} Min(\phi)$. Equivalently, if $\mathcal{C} = \mathcal{A}(\tilde{x})$ then $Min(\mathcal{C}) = \{\tilde{y} \mid \mathcal{A}(\tilde{x}) = \mathcal{C} \subset \mathcal{A}(\tilde{y})\}$.

One checks easily that

(5.3) $\mathcal{C}_1 \subset \mathcal{C}_2$ if and only if $Min(\mathcal{C}_1) \supset Min(\mathcal{C}_2)$.

Our next result, the main result of this subsection, plays a role in showing that the atlas for the injective F-structure which we will construct is actually a Cr-structure; see (4.2.1), (5.9.2).

Proposition 5.4. *(5.4.1) If $\Theta(\mathcal{C}_1) \cap \Theta(\mathcal{C}_2)$ has finite index in both $\Theta(\mathcal{C}_1)$ and $\Theta(\mathcal{C}_2)$ then $\mathcal{C}_1 = \mathcal{C}_2$.*

(5.4.2) In particular, if $\mathcal{C}_1 \subset \mathcal{C}_2$, then either $\mathcal{C}_1 = \mathcal{C}_2$, or $rank(\Theta(\mathcal{C}_1)) < rank(\Theta(\mathcal{C}_2))$.

Proof. Let $\mathcal{C}_i = \mathcal{C}(\tilde{x}_i)$, $i = 1, 2$. To see (5.4.1), note that by Proposition 5.1 and Lemma 5.2, we have $Min(\mathcal{C}_1) = Min(\mathcal{C}_2)$. Thus, our assertion follows from (5.3).

Clearly, (5.4.1) implies (5.4.2). □

d. A canonical decomposition of M^n

Let $K_{M^n} \leq 0$ and consider an abelian structure $(\mathcal{A}, \{Min(\phi)\})$. Although the constructions of this subsection apply equally well to arbitrary abelian structures on Riemannian manifolds whose curvature is not constrained, we make the above assumptions (solely) in order to avoid switching notation in subsequent subsections.

Put $N(\mathcal{C}) = \{\tilde{y} \,|\, \mathcal{A}(\tilde{y}) = \mathcal{C}\}$. Thus, the collection of sets, $\{N(\mathcal{C})\}$, coincides with the set of equivalence classes of points of M^n, under the relation: $\tilde{x} \sim \tilde{y}$, if and only if $\mathcal{A}(\tilde{x}) = \mathcal{A}(\tilde{y})$. Clearly, $\tilde{M}^n = \bigcup_{\mathcal{C}} N(\mathcal{C})$.

The sets, $N(\mathcal{C})$, have the following basic properties (where the bar denotes closure).

(5.5.1) $N(\mathcal{C}_1) \cap N(\mathcal{C}_2) \neq \emptyset$, implies $\mathcal{C}_1 = \mathcal{C}_2$.

(5.5.2) $\overline{N(\mathcal{C}_1)} \cap N(\mathcal{C}_2) \neq \emptyset$, implies $\mathcal{C}_1 \subset \mathcal{C}_2$.

(5.5.3) $\tilde{x} \in \overline{N(\mathcal{C}_1)} \cap \overline{N(\mathcal{C}_2)}$, implies $\tilde{x} \in N(\mathcal{C}_3)$, for some $\mathcal{C}_3 \supset \mathcal{C}_1 \cup \mathcal{C}_2$.

Let $\#(\mathcal{C})$ denote the cardinality of the set $\mathcal{C}$. From (5.5.3), we conclude the following.

(5.5.4) If $\#(\mathcal{C}_1) \leq \#(\mathcal{C}_2) = i$ and $\mathcal{C}_1 \not\subset \mathcal{C}_2$, then $\overline{N(\mathcal{C}_1)} \cap \overline{N(\mathcal{C}_2)} \subset \bigcup_{\#(\mathcal{C})>i} N(\mathcal{C})$.

By using (5.5.1)–(5.5.3), it follows that for all i, the set, $\bigcup_{\#(\mathcal{C})>i} N(\mathcal{C})$, which appears in (5.5.4), is closed.

It follows from (5.5.1) that $\tilde{M}^n$ is the *disjoint union* of sets of the form $N(\mathcal{C})$. This decomposition is preserved by the action of the group, Γ, of deck transformations.

Note that the interiors of some (but not all) $N(\mathcal{C})$ might be empty.

Distinct sets, $\pi(N(\mathcal{C}))$, are disjoint (and typically are not open). Moreover, as noted above, the interior of a set, $\pi(N(\mathcal{C}))$, might be empty. In order to obtain a Cr-structure on M^n, the sets, $N(\mathcal{C})$, must be replaced by suitable open sets $W(\mathcal{C})$. The issue here is *not* one of convention; even in the context of polarized F-structures, absent the relevant open sets, the collapsing construction of [CG1] would not apply; compare Theorem 0.5.

e. The open sets $W(\mathcal{C})$

In this subsection, for each set, $\mathcal{C}$, we construct an open set $W(\mathcal{C})$. To this end, we begin with some observations concerning splittings.

For the splitting, $Min(\mathcal{C}) = D(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$, the $\mathbb{R}^{k(\mathcal{C})}$ factor through a point, $\tilde{x}$, is called a $\mathcal{C}$-*axis* and denoted $F(\mathcal{C}, \tilde{x})$. Recall that by definition, the projection onto the factor, $\mathbb{R}^{k(\mathcal{C})}$, of the group, $\Theta(\mathcal{C})$, is a cocompact group of translations.

If $\mathcal{C}_1 \subset \mathcal{C}_2$, then $Min(\mathcal{C}_2) \subset Min(\mathcal{C}_1)$, and the splitting of $Min(\mathcal{C}_1)$ restricts to a splitting $Min(\mathcal{C}_2) = D(\mathcal{C}_1, \mathcal{C}_2) \times \mathbb{R}^{k(\mathcal{C}_1)}$. Moreover, the splittings of $Min(\mathcal{C}_1)$ and $Min(\mathcal{C}_2)$ are *compatible* in the following sense.

(5.6) If $\mathcal{C}_1 \subset \mathcal{C}_2$, then on $Min(\mathcal{C}_2)$, each $\mathcal{C}_2$-axis is a union of $\mathcal{C}_1$-axes.

Since $N(\mathcal{C}) = Min(\mathcal{C}) \setminus (\bigcup_{\mathcal{C}' \supsetneq \mathcal{C}} Min(\mathcal{C}'))$, it follows from (5.6) that the splitting of $Min(\mathcal{C})$ restricts to a splitting of $N(\mathcal{C}) = \underline{D}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$.

The sets, $W(\mathcal{C})$, will be of the form, $W(\mathcal{C}) = \bigcup_\ell U_\ell(\mathcal{C})$, where each set, $U_\ell(\mathcal{C})$, is a small tubular neighborhood of a $\mathcal{C}$-axis; the role of the sets, $U_\ell(\mathcal{C})$, is essentially technical. We will verify:

(5.7.1) The collections, $\{W(\mathcal{C})\}, \{U_\ell(\mathcal{C})\}$, are Γ-invariant open covers of $\tilde{M}^n$.

(5.7.2) If $\mathcal{C}_1 \neq \mathcal{C}_2, \#(\mathcal{C}_1) \leq \#(\mathcal{C}_2)$ and $W(\mathcal{C}_1) \cap W(\mathcal{C}_2) \neq \emptyset$, then $\mathcal{C}_1 \subset \mathcal{C}_2$. In particular, $rank(\Theta(\mathcal{C}_1)) < rank(\Theta(\mathcal{C}_2))$.

Note that (5.7.2) corresponds to (4.2.1) in the definition of Cr-structure; see also (5.4.2).

From the existence of a compact fundamental domain for the action of Γ, together with the fact that the collection of sets, $\{Q_\phi\}$, is Γ-invariant and locally finite, it follows that there exists $\underline{N} < \infty$, such that $\#(\mathcal{C}) \leq \underline{N}$, for all $\mathcal{C}$, and $\#(\mathcal{C}) = \underline{N}$, for some $\mathcal{C}$.

Put $N_i = \bigcup_{\#(\mathcal{C})=i} N(\mathcal{C})$. For all i, the set, $N_i \cup \cdots \cup N_{\underline{N}}$ is closed. Moreover, $\tilde{M}^n = N_1 \cup \cdots \cup N_{\underline{N}}$.

Let $T_r(X)$ denote the set of all points whose distance from X is less than r.

For every collection of positive numbers, $0 < \delta_1 \leq \cdots \leq \delta_{\underline{N}}$, we define sets, $K_i(\delta_{i+1}, \ldots, \delta_{\underline{N}})$, where $1 \leq i \leq \underline{N}$. From now on we just write K_i for such a set, suppressing the dependence on $\delta_{i+1}, \ldots, \delta_{\underline{N}}$. In the sequel, we will only consider those $0 < \delta_1 \leq \cdots \leq \delta_{\underline{N}}$ satisfying a series of inductively defined constraints of the form, $0 < \delta_{\underline{N}} \leq c_{\underline{N}}(\tilde{M}^n, \Gamma)$, and in addition, $0 < \delta_i \le c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \dots, \delta_{\underline{N}})$, for all $1 \le i \le \underline{N} - 1$. In Subsect. 5.i we will add another constraint $0 < \delta_{\underline{N}} \le c'_{\underline{N}}(\tilde{M}^n, \Gamma)$ and in Subsect. 5.j (for the purpose of verifying (4.2.2)) we will add a final inductively defined constraint. Any specific choice of $0 < \delta_1 \le \cdots \le \delta_{\underline{N}}$ satisfying the above constraints will suffice for our purposes.

Put $K_{\underline{N}} = N_{\underline{N}}$ and for $1 \le i < \underline{N}$, define by induction, $K_i = N_i \setminus (T_{\delta_{i+1}}(K_{i+1}) \cup \cdots \cup T_{\delta_{\underline{N}}}(K_{\underline{N}}))$. It follows directly from the definition that $K_i \subset N_i \subset K_i \cup T_{\delta_{i+1}}(K_{i+1}) \cup \cdots \cup T_{\delta_{\underline{N}}}(K_{\underline{N}})$. Thus,

(5.8.1) $N_i \subset T_{\delta_i}(K_i) \cup \cdots \cup T_{\delta_{\underline{N}}}(K_{\underline{N}})$,

(5.8.2) $N_i \cup \cdots \cup N_{\underline{N}} \subset T_{\delta_i}(K_i) \cup \cdots \cup T_{\delta_{\underline{N}}}(K_{\underline{N}})$,

(5.8.3) $\tilde{M}^n \subset T_{\delta_1}(K_1) \cup \cdots \cup T_{\delta_{\underline{N}}}(K_{\underline{N}})$.

Set $K(\mathcal{C}) = N(\mathcal{C}) \setminus \bigcup_{j > \#(\mathcal{C})} T_{\delta_j}(K_j) \subset N(\mathcal{C})$. We have $K_i = \bigcup_{\#(\mathcal{C})=i} K(\mathcal{C})$ and $T_{\delta_i}(K_i) = \bigcup_{\#(\mathcal{C})=i} T_{\delta_i}(K(\mathcal{C}))$. As a consequence of (5.5.2), (5.8.2), the sets, $K(\mathcal{C})$, are closed and by local finiteness of the $K(\mathcal{C})$, the sets, K_i, are closed as well. Since the sets, N_i, K_i and $K(\mathcal{C})$ are Γ-invariant, it follows that $\{T_{\delta_i}(K(\mathcal{C}))\}$ is a Γ-invariant open cover of $\tilde{M}^n$.

Let $\mathcal{C}_1 \neq \mathcal{C}_2$, with $\#(\mathcal{C}_1) \le \#(\mathcal{C}_2)$. There are three mutually exclusive possibilities:

Case 1. There does not exist $\mathcal{C}_3$ such that $\mathcal{C}_1 \cup \mathcal{C}_2 \subset \mathcal{C}_3$.

Case 2. There exists $\mathcal{C}_3$ such that $\mathcal{C}_1 \cup \mathcal{C}_2 \subset \mathcal{C}_3$ and in addition, $\#(\mathcal{C}_2) < \#(\mathcal{C}_3)$, for all such $\mathcal{C}_3$.

Case 3. $\mathcal{C}_1 \subset \mathcal{C}_2$.

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be as in Case 1. By (5.5.3), it follows that $\overline{N(\mathcal{C}_1)} \cap \overline{N(\mathcal{C}_2)} = \emptyset$. From the existence of a compact fundamental domain for the action of Γ, together with the fact that the collection of sets, $\{Q_\phi\}$, is Γ-invariant and locally finite, it follows that there exists $c_{\underline{N}}(\tilde{M}^n, \Gamma)$, such that for $\mathcal{C}_1$, $\mathcal{C}_2$ as in Case 1, the distance from $N(\mathcal{C}_1)$ to $N(\mathcal{C}_2)$ is at least $7c_{\underline{N}}(\tilde{M}^n, \Gamma)$. (Observe that if $\mathcal{C}_1 \not\subset \mathcal{C}_2$ and $\#(\mathcal{C}_2) = \underline{N}$, then we are necessarily in Case 1.)

Suppose, by induction, that constants, $c_j(\tilde{M}^n, \Gamma, \delta_{j+1}, \dots, \delta_{\underline{N}}) > 0$, have been specified, for all j, with $i < j \le \underline{N} - 1$, such that if $\mathcal{C}_1$, $\mathcal{C}_2$ are as in Case 2, with $\#(\mathcal{C}_2) = j$, then the distance from $N(\mathcal{C}_1)$ to $K(\mathcal{C}_2)$ is at least $7c_j(\tilde{M}^n, \Gamma, \delta_{j+1}, \dots, \delta_{\underline{N}}) > 0$. From (5.5.4), (5.8.2) and the definition of $K(\mathcal{C})$, it follows that there exists $c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \cdots, \delta_{\underline{N}}) > 0$, such that if $\mathcal{C}_1$, $\mathcal{C}_2$ are as in Case 2, with $\#(\mathcal{C}_2) = i$, then the distance from $N(\mathcal{C}_1)$ to $K(\mathcal{C}_2)$ is at least $7c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \cdots, \delta_{\underline{N}}) > 0$.

From now on we only consider $0 < \delta_1 \le \cdots \le \delta_{\underline{N}}$ such that $0 < \delta_{\underline{N}} \le c_{\underline{N}}(\tilde{M}^n, \Gamma)$, and in addition, $0 < \delta_i \le c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \dots, \delta_{\underline{N}})$, for all $1 \le i \le \underline{N} - 1$. In fact it would suffice to make the specific choice $\delta_i = c_i$, for all $1 \le i \le \underline{N}$.

Since, $K(\mathcal{C}) \subset N(\mathcal{C})$, we have that in Case 1, the distance from $K(\mathcal{C}_1)$ to $K(\mathcal{C}_2)$ is at least $7c_{\underline{N}}(\tilde{M}^n, \Gamma) \ge 7\delta_{\underline{N}} > 0$. Also, since $K(\mathcal{C}_1) \subset N(\mathcal{C}_1)$, it follows that in Case 2, with $\#(\mathcal{C}_1) \leq \#(\mathcal{C}_2) = i$, the distance from $K(\mathcal{C}_1)$ to $K(\mathcal{C}_2)$ is at least $7c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \ldots, \delta_{\underline{N}}) \geq 7\delta_i > 0$.

In particular, either $\mathcal{C}_1 \subset \mathcal{C}_2$, or $K(\mathcal{C}_1), K(\mathcal{C}_2)$ lie at distance at least $7\delta_i > 0$, where $\#(\mathcal{C}_1) \leq \#(\mathcal{C}_2) = i$. This separation property is what is required in the sequel.

Note that from the definition of $K(\mathcal{C})$ and induction, it follows that each set, $K(\mathcal{C})$, is a union of $\mathcal{C}$-axes.

For each set, $\mathcal{C}$, choose a maximal subset, $\{\tilde{x}_\ell\} = \{\tilde{x}_\ell(\mathcal{C})\} \subset K(\mathcal{C})$, such that any two members of the collection of $\mathcal{C}$-axes, $\{F(\mathcal{C}, \tilde{x}_\ell)\}$, are at mutual distance at least $\delta_{\#(\mathcal{C})}$, and in addition, for all $\gamma \in \Gamma$, the collection associated to $\gamma(\mathcal{C})\gamma^{-1}$, is $\{\gamma(x_\ell(\mathcal{C}))\}$. Since $\mathcal{C}$-axes are factors of isometric splittings, it follows that the collection of $\mathcal{C}$-axes, $\{F(\mathcal{C}, \tilde{x}_\ell)\}$, is $\delta_{\#(\mathcal{C})}$-dense in $K(\mathcal{C})$.

Put $U_\ell(\mathcal{C}) = T_{3\delta_{\#(\mathcal{C})}}(F(\mathcal{C}, \tilde{x}_\ell))$ and $W(\mathcal{C}) = \bigcup_\ell U_\ell(\mathcal{C})$. Then $T_{2\delta_{\#(\mathcal{C})}}(K(\mathcal{C})) \subset W(\mathcal{C}) \subset T_{3\delta_{\#(\mathcal{C})}}(K(\mathcal{C}))$ and it follows that the collection, $\{W(\mathcal{C})\}$, is a Γ-invariant open cover of $\tilde{M}^n$. (Indeed, the smaller sets, $\bigcup_\ell T_{2\delta_{\#(\mathcal{C})}}(F(\mathcal{C}, \tilde{x}_\ell))$, form a Γ-invariant open cover as well.)

Finally, it follows from the preceding discussion that if $\#(\mathcal{C}_1) \leq \#(\mathcal{C}_2) \leq i$, then either the distance between $W(\mathcal{C}_1), W(\mathcal{C}_2)$ is at least $\delta_i > 0$, or $\mathcal{C}_1 \subset \mathcal{C}_2$ (and $W(\mathcal{C}_1) \cap W(\mathcal{C}_2) \neq \emptyset$). Thus, (5.7.2) holds.

f. Injective torus actions on finite coverings of the sets $\pi(N(\mathcal{C}))$

Let $\pi : \tilde{M}^n \to M^n$ denote the natural projection. In view of the discussion at the beginning of Subsect. 5.e, it is clear that the splitting of $N(\mathcal{C})$ induces a decomposition of $\pi(N(\mathcal{C}))$, as a disjoint union of compact flat manifolds. Each such flat manifold, is the quotient of a $\mathcal{C}$-axis by the action of its stabilizer in Γ, in which the group, $\Theta(\mathcal{C})$, which acts by translations, has finite index. The union over all $\mathcal{C}$ of these decompositions induces a corresponding decomposition of M^n into compact flat manifolds, perhaps of different dimensions for different sets, $\pi(N(\mathcal{C}))$.

Let $\phi \in \mathcal{C}$, for some $\mathcal{C}$ and let $Min(\phi) = D(\phi) \times \mathbb{R}^1$ denote the canonical splitting. Let $H(\phi)$ denote the 1-parameter group of isometries of $Min(\phi)$ which act by translation on the factor, $\mathbb{R}$, and by the identity on $D(\phi)$. Thus, $\phi \in H(\phi)$. It follows easily from (5.6), that for all, $\mathcal{C}$, the collection of groups, $H(\phi)$, where $\phi \in \mathcal{C}$, generate an abelian group, $H(\mathcal{C})$, of isometries of $\bigcap_{\phi \in \mathcal{C}} Min(\phi)$ which act by translation on the factor, $\mathbb{R}^{k(\mathcal{C})}$, and by the identity on $D(\mathcal{C})$.

Clearly, the set, $N(\mathcal{C})$, is invariant under the action of $H(\mathcal{C})$ and the infinitesimal generator, $\mathfrak{h}(\mathcal{C})$, of $H(\mathcal{C})$, induces a sheaf of Killing fields, $\pi_*(\mathfrak{h}(\mathcal{C}))$, on $\pi(N(\mathcal{C}))$.

We summarize our observations in the following theorem.

Theorem 5.9. *(5.9.1) M^n is the disjoint union of sets, $\pi(N(\mathcal{C}))$, such that each set, $N(\mathcal{C})$ splits isometrically, $N(\mathcal{C}) = \underline{D}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$.*

(5.9.2) Each set, $\pi(N(\mathcal{C}))$, is the disjoint union of compact flat manifolds, each of which is the quotient of a $\mathcal{C}$-axis by its stabilizer.

(5.9.3) The sheaf of Killing fields, $\pi_*(\mathfrak{h}(\mathcal{C}))$, on $\pi(N(\mathcal{C}))$, has rank $k(\mathcal{C})$ at all points of $\pi(N(\mathcal{C}))$.

(5.9.4) The sheaf of Killing fields, $\pi_*(\mathfrak{h}(\mathcal{C}))$, induces a pure injective F-structure on the interior of $\pi(N(\mathcal{C}))$. In particular, for every orbit in the interior of $\pi(N(\mathcal{C}))$, the action of $\pi_*(\mathfrak{h}(\mathcal{C}))$ lifts to the infinitesimal generator of a torus action on a finite normal covering of some tubular neighborhood (which can be taken to be the holonomy covering of the orbit).

g. The product structure on special finite coverings of the sets $\pi(K(\mathcal{C}))$

Let $\Gamma(\mathcal{C})$ denote the subgroup of Γ which preserves $N(\mathcal{C})$. Since $\mathcal{C}$ is characterized uniquely by the condition, $\mathcal{C} = \mathcal{A}(\tilde{x})$, for all $\tilde{x} \in N(\mathcal{C})$, it follows that $\Gamma(\mathcal{C})$ coincides with the normalizer of $\mathcal{C}$ in Γ. Moreover, if $\gamma \notin \Gamma(\mathcal{C})$, then $\gamma(N(\mathcal{C})) \cap N(\mathcal{C}) = \emptyset$.

Let the closed set, $K(\mathcal{C}) \subset N(\mathcal{C})$, be as in Subsect. 5.e. As noted there, the collection $\{K(\mathcal{C})\}$ is Γ-invariant. Since $\mathcal{C}$ is also characterized uniquely by the condition, $\mathcal{C} = \mathcal{A}(\tilde{x})$, for all $\tilde{x} \in K(\mathcal{C})$, it follows that the stabilizer of $K(\mathcal{C})$ is also $\Gamma(\mathcal{C})$ and if $\gamma \notin \Gamma(\mathcal{C})$, then $\gamma(K(\mathcal{C})) \cap K(\mathcal{C}) = \emptyset$. Since $\tilde{M}^n/\Gamma$ is compact, it follows by a standard argument that $K(\mathcal{C})/\Gamma(\mathcal{C})$ is compact as well.

Let $c(\mathcal{C}) \subset \Gamma(\mathcal{C})$ denote the subgroup of elements which commute with every element of $\mathcal{C}$. Note that the index of $c(\mathcal{C})$ in $\Gamma(\mathcal{C})$ is at most $\#(\mathcal{C})! < \infty$.

In order to eventually establish (4.2.2), which is part of the definition of the concept of Cr-structure (see Subsect. 5.j) matters would be simplest if every subgroup of finite index of $\Gamma(\mathcal{C})$ were known to be finitely generated; compare the role of finite generation in the proof of Proposition 5.11. Since $\Gamma(\mathcal{C})$ acts cocompactly on $K(\mathcal{C})$, this would follow from a standard argument, if the sets, $K(\mathcal{C})$, were known to be connected. Although, the connectivity properties of the $K(\mathcal{C})$ are not clear, the following (elementary) Proposition 5.10 will enable us to construct a $\Gamma(\mathcal{C})$-invariant decomposition of $K(\mathcal{C})$, for which each piece is connected in a weak sense and in addition, the pieces are well separated from one another. This, is enough to enable us to apply a version of the above mentioned argument to the individual pieces of $K(\mathcal{C})$. For the quotient space, $M^n = \tilde{M}^n/\Gamma$, the consequences are the same as would hold if $K(\mathcal{C})$ were known to be connected.

Let Y be a metric space for which closed balls of finite radius are compact. Fix $y \in Y$. Let Ω denote a discrete, fixed point free, group of isometries of Y, such that for some closed ball, $\overline{B_r(y)}$, we have $\Omega(\overline{B_r(y)}) = Y$. Let Ω_u denote those elements, $\omega \in \Omega$, such that $\omega(y) \in \overline{B_u(y)}$. Since Ω is discrete, Ω_u is finite. Let $[\Omega_u]$ denote the group generated by Ω_u and put $Y_{r,s} = [\Omega_{s+r}](\overline{B_r(y)})$. For $Z \subset Y$, let $T_r(Z)$ denote the set of all points of distance $< r$ from Z.

Proposition 5.10. *Let* $0 < r \le s$*. Then:*

(5.10.1) $\overline{T_{s-r}(Y_{r,s})} = Y_{r,s}$*.*

(5.10.2) If $\omega(Y_{r,s}) \cap Y_{r,s} \neq \emptyset$*, for some* $\omega \in \Omega$*, then* $\omega \in [\Omega_{r+s}]$*. In particular,* $[\Omega_{r+s}]$ *is the stabilizer of* $Y_{r,s}$*.*

(5.10.3) Every subgroup of finite index of $[\Omega_{s+r}]$ *is finitely generated.*

(5.10.4) There is a natural homeomorphism, $Y_{r,s}/[\Omega_{r+s}] = Y/\Omega$*. Hence, every subgroup,* $\Lambda \subset [\Omega_{r+s}]$*, defines a possibly disconnected covering space,* $Y_{r,s}/\Lambda$*, of the possibly disconnected space* $Y_{r,s}/[\Omega_{r+s}] = Y/\Omega$*. The order of this covering is the index of* Λ *in* $[\Omega_{r+s}]$*. The covering is normal if and only if* Λ *is normal in* $[\Omega_{r+s}]$*.*

Proof. If $y' \in Y_{r,s}$, there exists $\omega' \in [\Omega_{r+s}]$, such that $\overline{y', \omega'(y)} \le r$. Let $\Omega_{r+s} = \omega_1, \ldots, \omega_{N(r,s)}$. Then $\overline{B_{s-r}(y')} \subset \overline{B_s(\omega'(y))} \subset \bigcup_j \omega'\omega_j(\overline{B_r(y)})$, which suffices to prove (5.10.1).

If $y' \in Y_{r,s}$ and $\omega(y') \in Y_{r,s}$, then as above, we have $\overline{\omega\omega'(y), \omega''(y)} \le 2r$, for some $\omega', \omega'' \in [\Omega_{r+s}]$. Hence, $(\omega'')^{-1}\omega\omega' \in \Omega_{2r} \subset \Omega_{r+s}$, which implies $\omega \in [\Omega_{r+s}]$. This gives (5.10.2).

If $\Lambda \subset [\Omega_{r+s}]$ is a subgroup of finite index, it follows that $\Lambda(\overline{B_R(y)}) \cap Y_{r,s} = Y_{r,s}$, for some $0 < R < \infty$. From (5.10.1), (5.10.2), with Y, $Y_{r,s}$ replaced by $Y_{r,s}$, $Y_{R,R+s}$ and Ω, Ω_{r+s} replaced by Λ, $\Lambda \cap \Omega_{2R+s}$, we get $\Lambda = [\Lambda \cap \Omega_{2R+s}]$. Since $[\Lambda \cap \Omega_{2R+s}]$ is finitely generated, this gives (5.10.3).

There is a natural map from the compact space, $Y_{r,s}/[\Omega_{r+s}]$ to Y/Ω. Since $\Omega(Y_{r,s}) = Y$ this map is surjective, and by (5.10.2), it is injective as well. Hence it is a homeomorphism. In particular, every subgroup, $\Lambda \subset [\Omega_{r+s}]$, defines a possibly disconnected covering space, $Y_{r,s}/\Lambda$, of the possibly disconnected space $Y_{r,s}/[\Omega_{r+s}] = Y/\Omega$. The remaining assertions of (5.10.4) follow trivially. □

Recall that $K(\mathcal{C})/\Gamma(\mathcal{C})$ is compact. Let $\tilde{x} \in K(\mathcal{C})$ and let $0 < d < \infty$ be such that $\Gamma(\mathcal{C})(\overline{B_d(\tilde{x})}) \cap K(\mathcal{C}) = K(\mathcal{C})$.

Take $\Omega = \Gamma(\mathcal{C})$, $Y = K(\mathcal{C})$, $y = \tilde{x} \in K(\mathcal{C})$.

As noted at the beginning of this subsection, $c(\mathcal{C})$ has finite index in $\Gamma(\mathcal{C})$. Thus, if $\Gamma^{\#}(\mathcal{C}) \subset c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$ is a subgroup of finite index, it follows from (5.10.3) that $\Gamma^{\#}(\mathcal{C})$ is finitely generated and $K(\mathcal{C})_{d,d+1}/\Gamma^{\#}(\mathcal{C})$ is a finite covering of $K(\mathcal{C})/\Gamma(\mathcal{C})$. Moreover, if $\Gamma^{\#}(\mathcal{C}) \subset c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$ is a normal subgroup, then $K(\mathcal{C})_{d,d+1}/\Gamma^{\#}(\mathcal{C})$ is a normal covering of $K(\mathcal{C})/\Gamma(\mathcal{C})$.

Let $K(\mathcal{C}) = \underline{\hat{D}}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$ denote the splitting of $K(\mathcal{C})$ induced by the splitting $N(\mathcal{C}) = \underline{D}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$.

The proof of the following Proposition 5.11 is very similar to that of the corresponding result in 3.4 of [Bu2], which is based on [Eb1]. Proposition 5.11 corresponds to the verification of the counterpart of (4.2.2), for the closed sets $K(\mathcal{C})$.

Proposition 5.11. *There is a normal subgroup,* $\Gamma^{\#}(\mathcal{C}) \subset c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$*, of finite index, such that the finite normal covering,* $K(\mathcal{C})_{d,d+1}/\Gamma^{\#}(\mathcal{C})$*, of* $K(\mathcal{C})_{d,d+1}/[\Gamma(\mathcal{C})_{2d+1}] = K(\mathcal{C})/\Gamma(\mathcal{C})$*, satisfies the following:*

(5.11.1) There is a space, $\underline{D}^*(\mathcal{C})$*, which is the quotient of* $\hat{\underline{D}}(\mathcal{C})$ *by a discrete fixed point free group of isometries, and a homeomorphism,* $\Phi : K(\mathcal{C})_{d,d+1}/\Gamma^{\#}(\mathcal{C}) \to \underline{D}^*(\mathcal{C}) \times T^{k(\mathcal{C})}$*, whose restriction to the interior of* $K(\mathcal{C})_{d,d+1}/\Gamma^{\#}(\mathcal{C})$ *is a diffeomorphism.*

(5.11.2) The action induced by Φ*, of* $[\Gamma(\mathcal{C})_{2d+1}]/\Gamma^{\#}(\mathcal{C})$ *on* $\underline{D}^*(\mathcal{C}) \times T^{k(\mathcal{C})}$*, preserves the product structure,* $\underline{D}^*(\mathcal{C}) \times T^{k(\mathcal{C})}$*. Moreover, the action induced by* Φ *of the lifted action of* $\pi_*(\mathfrak{h}(\mathcal{C}))$ *on* $\underline{D}^*(\mathcal{C}) \times T^{k(\mathcal{C})}$*, preserves the product structure and is by the identity on the factor,* $\underline{D}^*(\mathcal{C})$ *and by left (equivalently right) translation on the factor* $T^{k(\mathcal{C})}$*.*

Proof. Since the action of each $\gamma \in \Gamma(\mathcal{C})$ preserves the splitting, $K(\mathcal{C}) = \hat{\underline{D}}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$, it follows that $\gamma = (p_1(\gamma), p_2(\gamma))$, with $p_1(\gamma)$ an isometry of $\underline{D}(\mathcal{C})$ and $p_2(\gamma)$ an isometry of $\mathbb{R}^{k(\mathcal{C})}$. We denote by π_1, π_2, the projections of $K(\mathcal{C})$ onto the factors, $\hat{\underline{D}}(\mathcal{C})$, $\mathbb{R}^{k(\mathcal{C})}$, respectively.

We have, $\gamma\Theta(\mathcal{C})\gamma^{-1} = \Theta(\mathcal{C})$ and hence, $p_2(\gamma)p_2(\Theta(\mathcal{C}))p_2(\gamma)^{-1} = p_2(\Theta(\mathcal{C}))$. In addition, since $p_2(\Theta(\mathcal{C}))$ is a cocompact group of translations, $c(\mathcal{C})$ consists of those elements, $\gamma \in \Gamma(\mathcal{C})$, for which $p_2(\gamma)$ is a translation. Note that no power of $p_2(\gamma)$ need lie in $p_2(\Theta(\mathcal{C}))$.

Let G be the subgroup of $p_2(c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}])$ consisting of those elements, g, such that $i \cdot g \in p_2(\Theta(\mathcal{C}))$, for some $i \neq 0$. Let F be a maximal subgroup of $p_2(c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}])$ such that if $0 \neq f \in F$, then $i \cdot f \notin p_2(\Theta(\mathcal{C}))$, for all $i \neq 0$. We have $p_2(c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]) = F \oplus G$.

For $\lambda \in \mathbb{Z}_+$, let $\lambda^{-1}p_2(\Theta(\mathcal{C}))$, denote the subgroup of translations, t, such that $i \cdot t \in p_2(\Theta(\mathcal{C}))$, for some $i \leq \lambda$.

Since as noted above, $c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$ is finitely generated, we get $G \subset \underline{\lambda}^{-1}p_2(\Theta(\mathcal{C}))$, for some $\underline{\lambda} < \infty$. Thus, the group $F \oplus p_2(\Theta(\mathcal{C}))$ has finite index in $p_2(c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}])$, and hence, in $p_2([\Gamma(\mathcal{C})_{2d+1}])$ as well. Thus, $p_2^{-1}(F \oplus p_2(\Theta(\mathcal{C}))) \cap c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$ is of finite index in $c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$.

Let $\Gamma^{\#}(\mathcal{C})$ denote the intersection of all of the conjugacy classes in $[\Gamma(\mathcal{C})_{2d+1}]$, of $p_2^{-1}(F \oplus p_2(\Theta(\mathcal{C}))) \cap c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$. Since there are only finitely many such classes and $p_2^{-1}(F \oplus p_2(\Theta(\mathcal{C}))) \cap c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$ is of finite index in $c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$, it follows that $\Gamma^{\#}(\mathcal{C})$ is a normal subgroup of finite index of $c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]$.

We claim that $p_1(\Gamma^{\#}(\mathcal{C}))$ is a discrete fixed point free group of isometries of $\hat{\underline{D}}(\mathcal{C})$. To see this, first suppose that there exists a sequence, $\{\gamma_i\}$, with $p_1(\gamma_i) \to 1$, such that $p_1(\gamma_i) \neq p_1(\gamma_j)$ for all $i \neq j$, By the cocompactness of $p_2(\Theta(\mathcal{C}))$, there exists a sequence, $\{\psi_i\}$, with $\psi_i \in \Theta(\mathcal{C})$, such that the sequence, $\{p_2(\gamma_i\psi_i^{-1})\}$, converges. Since the action of $p_1(\Theta(\mathcal{C}))$ is trivial, $p_1(\gamma_i^{-1}\gamma_{i+1}\psi_{i+1}^{-1}\psi_i) \neq 1$, and by taking i sufficiently large, the minimum of the displacement function of the element, $\gamma_i^{-1}\gamma_{i+1}\psi_{i+1}^{-1}\psi_i$, can be made arbitrarily small. This contradicts the discreteness of Γ.

If $\gamma \in c(\mathcal{C})$ and $p_1(\gamma)$ has a fixed point, $\pi_1(\tilde{x})$, it follows that $p_2(\gamma) \in G$. Otherwise, there exist sequences, $\{n_i\}$ and $\{\psi_i\}$, with $\psi_i \in \Theta(\mathcal{C})$, such that $\{p_2(\gamma^{n_i}\psi_i^{-1})(\pi_2(\tilde{x}))\}$ converges, and hence, the minimum of the displacement function of $\gamma^{n_i - n_{i+1}}\psi_{i+1}^{-1}\psi_i$ can be made arbitrarily small by taking i sufficiently large. If $p_2(\gamma) \notin G$, then $p_2(\gamma^{n_i-n_{i+1}}\psi_{i+1}^{-1}\psi_i) \neq 1$, and this contradicts the discreteness of Γ. From the definition of $\Gamma^\#(\mathcal{C})$, together with $p_2(c(\mathcal{C}) \cap [\Gamma(\mathcal{C})_{2d+1}]) = F \oplus G$, it now follows that if $\gamma \in \Gamma^\#(\mathcal{C})$ and $p_1(\gamma)(\pi_1(\tilde{x})) = \pi_1(\tilde{x})$, then there exists $\sigma \in \Theta(\mathcal{C})$, with $p_2(\sigma) = p_2(\gamma)$. Since $p_1(\sigma) = 1$, we get $\sigma^{-1}\gamma(\tilde{x}) = \tilde{x}$. However, Γ is fixed point free, so $\sigma = \gamma$, and hence, $p_1(\gamma) = 1$. Therefore, if $p_1(\gamma) \neq 1$, then $p_1(\gamma)$ acts without fixed points. Thus, the group, $p_1(\Gamma^\#(\mathcal{C}))$, is a discrete fixed point free group of isometries of $\hat{\underline{D}}(\mathcal{C})$.

For $\gamma \in \Gamma^\#(\mathcal{C})$, put $p_2(\gamma) = (f(\gamma), \theta(\gamma))$, corresponding to the decomposition, $p_2(\Gamma^\#(\mathcal{C})) = F \oplus p_2(\Theta(\mathcal{C}))$.

For all $0 \leq s \leq 1$, define an action, ρ_s, of $\Gamma^\#(\mathcal{C})$ on $K(\mathcal{C}) = \underline{D}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})}$, by $\rho_s(\gamma) = (p_1(\gamma), s \cdot f(\gamma), \theta(\gamma))$. The family, ρ_s, provides a deformation of the original action of $\Gamma^\#(\mathcal{C})$, corresponding to $s = 1$, to the action $\rho_0(\gamma) = (p_1(\gamma), 0, \theta(\gamma))$. Since $1 \neq p_1(\gamma)$ acts without fixed point, for all $1 \neq \gamma \in \Gamma^\#(\mathcal{C})$, this deformation is through actions which are fixed point free and preserve the product structure.

For each $0 \leq s \leq 1$, the action of $\mathbb{R}^{k(\mathcal{C})}$ induces the structure of a flat principle $\mathbb{T}^{k(\mathcal{C})}$-bundle on $(\hat{\underline{D}}(\mathcal{C}) \times \mathbb{R}^{k(\mathcal{C})})/\rho_s(\Gamma^\#(\mathcal{C}))$, with base space $\underline{D}^*(\mathcal{C}) = \hat{\underline{D}}(\mathcal{C})/p_1(\Gamma^\#(\mathcal{C}))$. For $s = 0$, this bundle is just the trivial globally flat bundle, since $\rho_0(\Gamma^\#(\mathcal{C}))$ is the direct product of its projections under p_1, p_2. Thus, it follows by a standard argument, that for each $0 \leq s \leq 1$, the corresponding bundle has a section, or equivalently, a trivialization. With no loss of generality, this trivialization can be taken to be smooth over the interior of $\underline{D}^*(\mathcal{C})$. Clearly, this suffices □

h. Extending the actions

Let $\phi \in \mathcal{C}$, for some $\mathcal{C}$ and let $D(\phi) \times \mathbb{R}$ denote the canonical splitting of $Min(\phi)$. Let $F(\phi, \tilde{x})$ denote the ϕ-axis, $d \times \mathbb{R}$, through $\tilde{x}$ and let $v \neq 0$ be a vector tangent to $F(\phi, \tilde{x})$. If $\tilde{x}$ is an *interior* point of $Min(\phi)$, we have $R(\cdot\,, \cdot)v = 0$, where $R(\cdot\,, \cdot)$ denotes the curvature tensor. It follows by continuity that this relation holds for *arbitrary* points of $Min(\phi)$.

In connection with the following proposition (as well as Propositions 5.14, 5.15) note that the dimension of $Min(\mathcal{C})$ might be strictly less than n.

Proposition 5.12. *If $\tilde{x} \in Min(\mathcal{C})$, then the restriction of the tangent bundle of M^n to the flat totally geodesic submanifold, $F(\mathcal{C}, \tilde{x})$, is globally flat.*

Proof. The tangent space to the $\mathcal{C}$-axis, $F(\mathcal{C}, \tilde{x})$, is spanned by vectors, v, tangent to ϕ-axes, $F(\phi, \tilde{x})$, where $\phi \in \mathcal{C}$. From this we get $R(\cdot\,,\,\cdot)v = 0$, for *all* v tangent to $F(\mathcal{C}, \tilde{x})$, which, together with the Jacobi identity, implies $R(v_1, v_2) = 0$, for all v_1, v_2 tangent to $F(\mathcal{C}, \tilde{x})$. Since $F(\mathcal{C}, \tilde{x})$ is simply connected, the proposition follows. □

Let $\phi \in \mathcal{C}$ and let $h \in H(\phi)$. It is clear that for $\tilde{x}$ an *interior* point of $Min(\phi)$, the differential, $dh : \tilde{M}^n_{\tilde{x}} \to \tilde{M}^n_{h(\tilde{x})}$, is given by parallel translation, P_τ, along the geodesic segment, τ, from $\tilde{x}$ to $h(\tilde{x})$. Let $exp_{F(\mathcal{C},\tilde{x})}$, denote the exponential map of the normal bundle to the $\mathcal{C}$-axis $F(\mathcal{C}, \tilde{x})$. Then, for such $\tilde{x}$ and arbitrary $\tilde{y} \in Min(\phi)$, we have

(5.13) $h(\tilde{y}) = \exp_{F(\mathcal{C},h(\tilde{x}))} \circ P_\tau \circ (\exp_{F(\mathcal{C},\tilde{x})})^{-1}(\tilde{y})$.

As above, it follows by continuity, that in fact, (5.13) holds for *arbitrary* $\tilde{x} \in Min(\mathcal{C})$. The right-hand side of (5.13) is defined for arbitrary $\tilde{y} \in \tilde{M}^n$. Thus, for all $\tilde{x} \in Min(\mathcal{C})$, (5.13) defines a smooth 1-parameter group, $\hat{H}(\mathcal{C}, \phi, \tilde{x})$, of diffeomorphisms of $\tilde{M}^n$, whose restriction to $Min(\phi)$ is the 1-parameter group $H(\phi)$. Moreover, if $\hat{h} \in \hat{H}(\mathcal{C}, \phi, \tilde{x})$ corresponds to an integral value, $i \in \mathbb{Z}$, of the parameter, then $\hat{h} = \phi^i$.

Proposition 5.14. *Let $\tilde{x} \in Min(\mathcal{C})$ for some $\mathcal{C}$ and let $\phi_1, \phi_2 \in \mathcal{C}$. If $\hat{h}_1 \in \hat{H}(\mathcal{C}, \phi_1, \tilde{x})$, $\hat{h}_2 \in \hat{H}(\mathcal{C}, \phi_2, \tilde{x})$, then $\hat{h}_1 \cdot \hat{h}_2 = \hat{h}_2 \cdot \hat{h}_1$.*

Proof. Let τ_1, τ_2 denote the minimal geodesic segments from $\tilde{x}$ to the points, $\hat{h}_1(\tilde{x}), \hat{h}_2(\tilde{x})$, respectively. We have $\hat{h}_1 \cdot \hat{h}_2(\tilde{x}) = \hat{h}_2 \cdot \hat{h}_1(\tilde{x})$ and by Proposition 5.12, $P_{\hat{h}_1(\tau_2)\cup\tau_1} = P_{\hat{h}_2(\tau_1)\cup\tau_2}$. From this and (5.13), our claim follows. □

From Proposition 5.14, we obtain for all $\tilde{x} \in Min(\mathcal{C})$, an abelian group, $\hat{H}(\mathcal{C}, \tilde{x})$, of diffeomorphisms of $\tilde{M}^n$, whose restriction to $Min(\mathcal{C})$ is the group of isometries, $H(\mathcal{C})$. We have $rank(\Theta(\mathcal{C})) = rank(\hat{H}(\mathcal{C}, \tilde{x}))$. In addition, although $rank(\Theta(\mathcal{C}))$ might be strictly less than $\#(\mathcal{C})$, it is clear that the dimension of every orbit is equal to $rank(\Theta(\mathcal{C}))$.

For all $\mathcal{C}$, choose $\tilde{x}_{\mathcal{C}} \in N(\mathcal{C})$, such that the collection $\{\tilde{x}_{\mathcal{C}}\}$, is Γ-equivariant. We put $\hat{H}(\mathcal{C}, \phi) = \hat{H}(\mathcal{C}, \phi, \tilde{x}_{\mathcal{C}})$ and $\hat{H}(\mathcal{C}) = \hat{H}(\mathcal{C}, \tilde{x}_{\mathcal{C}})$.

i. Compatibility of local actions; construction of the injective F-structure

For h in the 1-parameter group, $H(\phi)$, we now write $h(t)$, where $h(1) = \phi$. We also write the corresponding element of $\hat{H}(\mathcal{C}, \phi)$ as $\hat{h}(\mathcal{C}, t)$, where $\hat{h}(\mathcal{C}, 1) = \phi$. If for all $-\infty < t < \infty$, we let $[t]$ denote the greatest integer $\le t$, then $\hat{h}(\mathcal{C}, t) = \hat{h}(\mathcal{C}, s)\phi^{[t]}$, where $0 \le s = t - [t] < 1$.

An arbitrary element, $\hat{h}(\mathcal{C}) \in \hat{H}(\mathcal{C})$, corresponding to $h \in H(\mathcal{C})$, can be written as a product, $\hat{h}(\mathcal{C}) = \hat{h}_1(\mathcal{C}, s_1) \cdots \hat{h}_j(\mathcal{C}, s_j)\theta$, where $0 \le s_1, \ldots, s_j \le 1$ and $\theta \in \Theta(\mathcal{C})$.

If $\mathcal{C}_1 \subset \mathcal{C}_2$, then we can regard $H(\mathcal{C}_1) \subset H(\mathcal{C}_2)$. We denote by $\hat{H}(\mathcal{C}_1, \mathcal{C}_2) \subset \hat{H}(\mathcal{C}_2)$, the subgroup corresponding to $H(\mathcal{C}_1)$. Thus, $\hat{H}(\mathcal{C}_1) = \hat{H}(\mathcal{C}_1, \mathcal{C}_1)$. On $Min(\mathcal{C}_1)$, we have $\hat{H}(\mathcal{C}_1) = \hat{H}(\mathcal{C}_1, \mathcal{C}_2) = H(\mathcal{C}_1)$. Although it is possible that $\dim(Min(\mathcal{C}_1)) < n$, none-the-less, the following holds.

Proposition 5.15. *If $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\hat{H}(\mathcal{C}_1)$, $\hat{H}(\mathcal{C}_1, \mathcal{C}_2)$ agree to infinite order on $Min(\mathcal{C}_1)$. In particular, $\hat{H}(\mathcal{C}_1)$, $\hat{H}(\mathcal{C}_1, \mathcal{C}_2)$ are close to order, $o(d)$, on any tubular neighborhood, $T_d(Min(\mathcal{C}_1))$.*

Proof. Let $\phi \in \mathcal{C}_1$ and $\mathcal{C}_1 \subset \mathcal{C}_2$. Since on the set, $Min(\phi)$, *whose interior is nonempty*, we have $\hat{h}(\mathcal{C}_1, s) = \hat{h}(\mathcal{C}_2, s)$, it follows in particular, that the diffeomorphisms, $\hat{h}(\mathcal{C}_1, s), \hat{h}(\mathcal{C}_2, s)$, agree to infinite order on the set, $Min(\mathcal{C}_1)$.

More generally, let $h \in H(\mathcal{C}_1) \subset H(\mathcal{C}_2)$. Then we have corresponding representations, $\hat{h}(\mathcal{C}_1) = \hat{h}_1(\mathcal{C}_1, s_1) \cdots \hat{h}_j(\mathcal{C}_1, s_j)\theta$, $\hat{h}(\mathcal{C}_2) = \hat{h}_1(\mathcal{C}_2, s_1) \cdots \hat{h}_j(\mathcal{C}_2, s_j)\theta$. If we write, $\hat{h}(\mathcal{C}_1) = \hat{h}(\mathcal{C}_2)[(\hat{h}(\mathcal{C}_2))^{-1}\hat{h}(\mathcal{C}_1)] \cdots$, our assertions easily follow. □

Let $\hat{U}_\ell(\mathcal{C})$ denote the smallest $\hat{H}(\mathcal{C})$ invariant set containing $U_\ell(\mathcal{C})$. Put $\bigcup_\ell \hat{U}_\ell(\mathcal{C}) = \hat{W}(\mathcal{C})$. From Proposition 5.15, together with the existence of a compact fundamental domain for Γ, it follows that $\hat{U}_\ell(\mathcal{C}) \subset T_{o(\delta_{\#(\mathcal{C})})}(U_\ell(\mathcal{C}))$ (where $\delta_{\#(\mathcal{C})}$ is as in Subsect. 5.e). Assume $\hat{W}(\mathcal{C}_1) \cap \hat{U}_{\ell_2}(\mathcal{C}_2) \neq \emptyset$. Then on this set, the action of the group, $\hat{H}(\mathcal{C}_1)$, is uniformly $o(\delta_{\#(\mathcal{C}_1)})$-close in the C^∞ topology to the action of the corresponding subgroup $\hat{H}(\mathcal{C}_1, \mathcal{C}_2)$. As usual, from Γ-invariance, local finiteness and the existence of a compact fundamental domain, it follows that the quantities which are $o(\delta_{\#(\mathcal{C}_1)})$, can be chosen uniformly small with respect to $\delta_{\#(\mathcal{C}_1)}$, for all pairs, $\mathcal{C}_1 \subset \mathcal{C}_2$ as above. Therefore, by choosing $\delta_{\underline{N}}$ sufficiently small, these quantities can be made as small as we like with respect to $\delta_{\#(\mathcal{C}_1)}$.

In [CG2], F-structures of positive rank were constructed on a sufficiently collapsed manifolds of bounded curvature. The only essential respect in which the present situation differs from that considered in [CG2] is the following. In our case, for given $\mathcal{C}$, the radius, $3\delta_{\#(\mathcal{C})}$, of a tube, $U_\ell(\mathcal{C})$, depends on $\#(\mathcal{C})$. Thus, in contrast to the situation considered in [CG2], in which the ratios of radii of all tubes were *apriori* bounded (away from 0 and ∞) here we have a collection of tubes such that for say $\#(\mathcal{C}_1) < \#(\mathcal{C}_2)$, the ratio of radii, $\delta_{\#(\mathcal{C}_1)}/\delta_{\#(\mathcal{C}_2)}$, has no *apriori* positive lower bound. However, subject to minor technical changes, the construction can still be carried out as in [CG2]. The necessary changes are explained below. For the remaining details of the construction, we refer to [CG2].

Consider a pair, $\mathcal{C}_2 \subset \mathcal{C}_3$. As in Lemmas 1.4, 1.5 of [CG2], we can modify the action of $\hat{H}(\mathcal{C}_3)$, by conjugating it by a suitable diffeomorphism, so that after slightly shrinking the sets, $\hat{W}(\mathcal{C}_2)$, $\hat{W}(\mathcal{C}_3)$, the conjugated action of $\hat{H}(\mathcal{C}_2, \mathcal{C}_3)$ agrees with the action of $\hat{H}(\mathcal{C}_2)$ on the common domain of definition of these actions.

The construction of the diffeomorphism mentioned in the previous paragraph is completely analogous to that of the corresponding diffeomorphism, ϕ, of Lemma 1.5 of [CG2]; see also the discussion in the next paragraph. Recall, that in [CG2], the stability theorem of [GK] for compact group actions, was applied to the relevant torus actions on the (finite normal) holonomy coverings of tubular neighborhoods of the orbits. The stability theorem gives a map, Ψ, such that the actions are related by conjugation by Ψ on their common domains. Then, a diffeomorphism, (denoted here by Λ) is constructed from Ψ with the aid of a cutoff function (denoted here by f). In our case, the stability theorem is applied on the (finite normal) holonomy coverings of the sets, $\pi(\hat{U}_{\ell_k}(\mathfrak{C}_k))$, with the quotient torus actions induced by the subgroups, $\hat{H}(\mathfrak{C}_j)$, $\hat{H}(\mathfrak{C}_k)$.

Since we will require a certain specific property of the diffeomorphism, Λ, we recall its construction. To simplify the notation, we consider a model case.

For $i = 0, 1$, let ρ_i be a smooth action of a compact Lie group, G, on an open set W_i, such that:

(5.16.1) For some Riemannian metric, $< \, , \, >$, on W_0, the action, ρ_0, is isometric.

Let $W_i'' \subset\subset W_i' \subset\subset W_i$, be ρ_i-invariant open subsets, $i = 0, 1$. Assume that there exists, V, with $W_0' \cap W_1' \subset\subset V \subset\subset W_0 \cap W_1$, and an imbedding, $\Psi : V \to W_0 \cap W_1$, such that $\rho_0(g)(\Psi(w))$, $\Psi(\rho_1(g)(w))$, are defined for all $g \in G, w \in W_0' \cap W_1'$ and for such g, w, we have $\rho_0(g)(\Psi(w)) = \Psi(\rho_1(g)(w))$.

Let Ψ be so close to the inclusion map that for all $w \in W_0' \cap W_1'$, there is a unique geodesic segment (with respect to the metric $< \, , \, >$) $\sigma_w : [0, \ell(w)] \to W_0 \cap W_1$, from w to $\Psi(w)$. Let $f : W_0' \cap W_1' \to [0, 1]$, be a smooth function such that $f \,|\, \Psi^{-1}(W_0'') \cap W_1'' \equiv 1$, and in addition, f vanishes identically in a neighborhood of $\partial(W_0' \cap W_1')$.

For $w \in W_0' \cap W_1'$, put $\Lambda(w) = \sigma_w(f(w)\ell(w))$. Assume that Ψ is sufficiently C^1-close to the inclusion map so that $\Lambda : W_0' \cap W_1' \to W_0' \cap W_1'$, is an imbedding. Extend the domain of Λ by putting $\Lambda(w) = w$, for $w \in W_0'' \setminus W_1'$. So extended, $\Lambda : (W_0' \cap W_1') \cup W_0'' \to (W_0' \cap W_1') \cup W_0''$.

Assume in addition:

(5.16.2) The function, f, is ρ_0-invariant.

Define an action, $\rho_0^\flat$, with domain, $\Lambda^{-1}(W_0'')$, by $\rho_0^\flat = \Lambda^{-1}\rho_0\Lambda$. Note that $\Lambda^{-1}(W_0') \subset W_0$. We have $\rho_0^\flat \,|\, \Lambda^{-1}(W_0'') \cap W_1'' = \rho_1 \,|\, \Lambda^{-1}(W_0'') \cap W_1''$ and $\rho_0^\flat \,|\, (\Lambda^{-1}(W_0'') \setminus W_1') = \rho_0 \,|\, (\Lambda^{-1}(W_0'') \setminus W_1')$. Thus, there exists a smooth action, ρ, of the group, G, on the set, $\Lambda^{-1}(W_0'') \cup W_1''$, such that $\rho \,|\, \Lambda^{-1}(W_0'') = \rho_0^\flat \,|\, \Lambda^{-1}(W_0'')$ and $\rho \,|\, W_1'' = \rho_1 \,|\, W_1''$.

Moreover:

(5.16.3) If $\rho_0(g)(w) = \rho_1(g)(w)$, for some $g \in G, w \in \Lambda^{-1}(W_0'')$, then $\rho(g)(w) = \rho_0(g)(w) = \rho_1(g)(w)$.

To see (5.16.3), note that it suffices to assume $w \in W_0' \cap W_1'$. From the assumption, (5.16.1), that $\rho_0(g)$ is an isometry, together with $\rho_1(g)(w) = \rho_0(g)(w)$, it follows easily that $\rho(g)(\sigma_w(f(w)\ell(w))) = \sigma_{\rho_0(g)(w)}(f(w)\ell(\rho_0(w)))$. Since by (5.16.2), we also have $f(w) = f(\rho_0(g)(w))$, we get (5.16.3).

We now return to the construction of the F-structure. The fact that we are dealing with a collection of tubes, the ratios of whose radii are not apriori bounded, necessitates our performing *all* modifications in actions on the scale δ_i, before performing *any* modifications on the scale δ_{i+1}. Note that on the scale, δ_i, the actions change by an amount $o(\delta_i)$. Thus, if we take $\delta_{\underline{N}}$ sufficiently small, then for all i, after the changes on the scales, $\delta_1, \cdots, \delta_i$, have been performed, the relevant actions will still be close enough for the modifications on the scale, δ_{i+1}, to be performed as well. (Viewed from the perspective of the ranks of the subgroups, $\hat{H}(\mathcal{C})$, we are forced to perform the modifications in the order *opposite* to the one chosen in [CG2].)

Consider all pairs of integers, (j, k), where $1 \leq j \leq k \leq \underline{N}$, with the ordering, $(j', k') < (j, k)$ if either $j' < j$ or $j' = j$ and $k' < k$. We run through such pairs in ascending order. Assume by induction, that at a given stage, (j, k), for all $\mathcal{C}_2' \subset \mathcal{C}_3'$, with $(\#(\mathcal{C}_2'), \#(\mathcal{C}_3')) < (j, k)$, the previously modified actions of $\hat{H}(\mathcal{C}_2')$, $\hat{H}(\mathcal{C}_2', \mathcal{C}_3')$, agree on their common domains of definition.

For a given pair, $\mathcal{C}_2 \subset \mathcal{C}_3$, with $(\#(\mathcal{C}_2), \#(\mathcal{C}_3)) = (j, k)$, by conjugating the torus action induced by $\hat{H}(\mathcal{C}_3)$, by a suitable map, Λ, as above, we modify the action of $\hat{H}(\mathcal{C}_3)$, in such a way that on their new (slightly smaller) domains of definition, the actions of $\hat{H}(\mathcal{C}_2)$, $\hat{H}(\mathcal{C}_2, \mathcal{C}_3)$, agree.

It follows easily from the existence of the local isometric splittings on the sets, $N(\mathcal{C})$, that by choosing $\delta_{\underline{N}} \leq c'_{\underline{N}}(\tilde{M}^n, \Gamma)$, for some sufficiently small constant, $c'_{\underline{N}}(\tilde{M}^n, \Gamma) > 0$, and averaging over the torus action induced by $\hat{H}(\mathcal{C}_3)$, we can arrange that (5.16.1), (5.16.2) hold. Thus, (5.16.3) holds as well. (The constant, $c'_{\underline{N}}(\tilde{M}^n, \Gamma)$, is the one referred to in the paragraph preceding (5.8.1)–(5.8.3).)

Let $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3$. To complete the inductive construction, it suffices to check that the modification in the action of $\hat{H}(\mathcal{C}_3)$, leaves unchanged the action of the subgroup $\hat{H}(\mathcal{C}_1, \mathcal{C}_3)$, at all points of the new domain of the action of $\hat{H}(\mathcal{C}_3)$. Hence, the actions of $\hat{H}(\mathcal{C}_1)$ and $\hat{H}(\mathcal{C}_1, \mathcal{C}_3)$, continue to agree on their common domain of definition.

Observe that the modified action of $\hat{H}(\mathcal{C}_3)$ differs from the previous action only on a subset of the intersection of the original domains of $\hat{H}(\mathcal{C}_2)$ and $\hat{H}(\mathcal{C}_3)$. Consider the subset of such points which are also in the domain of $\hat{H}(\mathcal{C}_1)$. By the induction assumption, at these points, prior to the modification of the action of $\hat{H}(\mathcal{C}_3)$, the actions of $\hat{H}(\mathcal{C}_1)$, $\hat{H}(\mathcal{C}_1, \mathcal{C}_2)$, agreed, as did the actions of $\hat{H}(\mathcal{C}_1)$, $\hat{H}(\mathcal{C}_1, \mathcal{C}_3)$. Hence, the actions of $\hat{H}(\mathcal{C}_1, \mathcal{C}_2)$, $\hat{H}(\mathcal{C}_1, \mathcal{C}_3)$ agreed as well. Thus, it follows from (5.16.3), that at points of its new (slightly smaller) domain, the action of the subgroup, $\hat{H}(\mathcal{C}_1, \mathcal{C}_3)$ does not change under the modification process.

It is clear that the collection of mutually compatible actions obtained as above, determines an injective F-structure on M^n. (For the remaining details of the construction, see [CG2].)

The conjugacy map, Λ, constructed in [GK], has the property that $\Lambda(\tilde{x}) = \tilde{x}$, for those points, $\tilde{x}$, for which the two actions agree. Since for $\mathcal{C}_1 \subset \mathcal{C}_2$, we have $\hat{H}(\mathcal{C}_1) = \hat{H}(\mathcal{C}_1, \mathcal{C}_2) = H(\mathcal{C}_1)$ on $Min(\mathcal{C}_1)$, it follows that the F-structure we have obtained is compatible with the local splitting structure associated to the abelian structure $(\mathcal{A}, \{Min(\phi\})$.

j. Construction of the Cr-structure; conclusion of the proof of Theorem 0.4

It follows from (5.7.2) that the atlas, $\hat{W}(\mathcal{C})/\Gamma(\mathcal{C})$, for the injective F-structure obtained in Subsect. 5.i satisfies (4.2.1). We will complete the proof of Theorem 0.4 by showing that if the quantities, δ_i in Subsect. 5.e are subjected to an additional inductively defined constraint, then (4.2.2) holds as well.

Let $\hat{W}_1(\mathcal{C})$ denote the union of components of $\hat{W}(\mathcal{C})$ which have non-empty intersection with $K(\mathcal{C})_{d,d+1}$. Since $\hat{W}(\mathcal{C})$ and $K(\mathcal{C})_{d,d+1}$ are $[\Gamma(\mathcal{C})_{2d+1}]$-invariant, $\hat{W}_1(\mathcal{C})$ is also $[\Gamma(\mathcal{C})_{2d+1}]$-invariant. From (5.10.4), we get $K(\mathcal{C})_{d,d+1}/[\Gamma(\mathcal{C})_{2d+1}] = K(\mathcal{C})/\Gamma(\mathcal{C})$. Thus, from (5.10.1), it follows that $\hat{W}_1(\mathcal{C})/[\Gamma(\mathcal{C})_{2d+1}] = \hat{W}(\mathcal{C})/\Gamma(\mathcal{C})$. Hence, it suffices to check (4.2.2) for $\hat{W}_1(\mathcal{C})/[\Gamma(C)_{2d+1}]$.

For the sets, $K(\mathcal{C})$, of Subsect. 5.e, we now write $K(\mathcal{C})(\delta_{i+1}, \ldots, \delta_{\underline{N}})$, where $\#(\mathcal{C}) = i$, to indicate the dependence on the specific choice of constants, $\delta_{i+1}, \ldots, \delta_{\underline{N}}$.

The constants, δ_i, were chosen to satisfy $0 < \delta_i \leq c_i(\tilde{M}^n, \Gamma, \delta_{i+1}, \ldots, \delta_{\underline{N}})$. We now make the stronger assumption, $0 < \delta_i \leq c_i(\tilde{M}^n, \Gamma, \alpha\delta_{i+1}, \ldots, \alpha\delta_{\underline{N}})$, where as described below, $\alpha > 0$ will be chosen sufficiently small.

Put $K_1(\mathcal{C}) = K(\mathcal{C})(\alpha\delta_{i+1}, \ldots, \alpha\delta_{\underline{N}})_{d,d+1} \supset K(\mathcal{C})_{d,d+1}$. By applying Proposition 5.11 to the set $K_1(\mathcal{C})$, we obtain a finite covering, $K_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$, of $K_1(\mathcal{C})/[\Gamma(\mathcal{C})_{2d+1}]$, such that (5.11.1), (5.11.2) hold.

Note that from the construction of $\Gamma^{\#}(\mathcal{C})$, it is clear that $\hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$ is a finite normal covering of $\hat{W}_1(\mathcal{C})/[\Gamma(\mathcal{C})_{2d+1}]$ and that the free $\mathbb{R}^{k(\mathcal{C})}$-action on $\hat{W}_1(\mathcal{C})$ (which is a small deformation of the $\mathbb{R}^{k(\mathcal{C})}$-action on $\tilde{M}^n$) descends to a $T^{k(\mathcal{C})}$-fibration, $T^{k(\mathcal{C})} \to \hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C}) \to \underline{D}_{\hat{W}_1}$; see Subsect. 5.h. We will construct a $\Gamma^{\#}(\mathcal{C})$-invariant and $\mathbb{R}^{k(\mathcal{C})}$-invariant subset, $\hat{V}_1(\mathcal{C})$, which contains $\hat{W}_1(\mathcal{C})$ and which can be continuously deformed onto $K_1(\mathcal{C})$. Given this, it follows that the $T^{k(\mathcal{C})}$-fibration over $\hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$ induced by the $\mathbb{R}^{k(\mathcal{C})}$-action on $\tilde{M}^n$ is trivial, since its restriction to $K_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$

is trivial. Then from $\hat{W}_1(\mathcal{C}) \subset \hat{V}_1(\mathcal{C})$, we get the desired triviality of the $T^{k(\mathcal{C})}$-fibration on $\hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$.

Recall that $K_1(\mathcal{C})$ is a subset of the closed totally geodesic convex set $\mathrm{Min}(\mathcal{C}) \subset \tilde{M}^n$. Let $\hat{V}_1(\mathcal{C})$ denote the set of points, $\tilde{x}$, such that there is a unique minimal geodesic of length at most $\delta_{\underline{N}}$ from $\tilde{x}$ to $\mathrm{Min}(\mathcal{C})$, and in addition, this geodesic meets $\mathrm{Min}(\mathcal{C})$ at a point of $K_1(\mathcal{C})$. Then there is a deformation retraction, D_t, of $\hat{V}_1(\mathcal{C})$ onto $K_1(\mathcal{C})$. We now choose α so small that $\hat{W}(\mathcal{C}) \cap N(\mathcal{C}) \subseteq K_1(\mathcal{C})$. Consequently, $\hat{W}_1(\mathcal{C}) \subset \hat{V}_1(\mathcal{C})$. Thus, since $\hat{V}_1(\mathcal{C})$ is $\Gamma^{\#}(\mathcal{C})$-invariant, we have $\hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C}) \subset \hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$. Moreover, the free $\mathbb{R}^{k(\mathcal{C})}$-action on $\hat{V}_1(\mathcal{C})$ descends to a $T^{k(\mathcal{C})}$-fibration over $\hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$.

Clearly, the map, D_t, induces a deformation retraction, $\bar{D}_t$, of $\hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$ onto $K_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$. This implies that the $T^{k(\mathcal{C})}$-fibration over $\hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$ is trivial. Therefore, the $T^{k(\mathcal{C})}$-fibration over $\hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$, which is a small deformation of the restriction of the $T^{k(\mathcal{C})}$-fibration of $\hat{V}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$ over $\hat{W}_1(\mathcal{C})/\Gamma^{\#}(\mathcal{C})$, is also trivial.

This completes the proof of Theorem 0.4.

6. An example

In [CFG] and [CG2], an F-structure is constructed on any n-manifold with $|K| \leq 1$ and $inj_x \leq \epsilon(n)$, for all x. Even if the curvature of the compact manifold, M^n is nonpositive, this structure need not be injective; see Example 6.1. Thus the existence of an injective substructure (let alone a Cr-structure) does *not* follow from the general considerations of [CFG] and [CG2], which rely only on the boundedness of the curvature and are local in nature. Those of the present paper depend on the nonpositivity of the curvature and the compactness of the underlying manifold.

Example 6.1. Take a noncompact complete hyperbolic 3-manifold, N^3, of finite volume with one end, $T^2 \times [0, \infty)$. By Dehn surgery, one can cut off $T^2 \times [i, \infty)$ from N^3 and glue back a solid torus, $D^2 \times S^1$, along the torus boundary, such that the resulting compact 3-manifold, N_i^3, admits a metric with sectional curvature $-1 \leq K < 0$ and volume bounded above by $Vol(N^3)$; see [Gr3]. Moreover, the metric on N_i^3 has the following properties:

(6.2.1) The maximum value of the injectivity radius on $D^2 \times S^1$ converges to zero as $i \to \infty$ and at some points there are two locally independent short geodesic loops.

(6.2.2) The minimum set of any deck transformation has dimension 1.

Put $M_i = N_i^3 \times S^1_{i^{-1}}$ with the product metric, where $S^1_{i^{-1}}$ denotes the circle of radius i^{-1}, $i \gg 1$. Then M_i satisfies the assumptions of Theorem 0.6.

From (6.2.2), the splitting structure on $\tilde{M}_i$ is the metric product, $\tilde{N}_i^3 \times \mathbb{R}^1$, where $\tilde{N}_i^3$ is the universal covering of N_i^3.

Consider an F-structure, $\mathcal{F}$, on M_i given by the general construction in [CFG] and [CG2]. From (6.2.1), (6.2.2) and the fact that each orbit of $\mathcal{F}$ captures all short geodesic loops at a point, one can see that at a point of N_i^3 whose injectivity radius is $\geq \delta(3)$ (the Margulis constant) the orbit is S^1_{i-1} and at points in $(T^2 \times i) \times S^1_{i-1}$, the orbit is a 3-torus. Note that the three torus orbit is *not injective* since it is contained in $(D^2 \times S^1) \times S^1_{i-1}$.

Examination of the construction given in Sect. 5 reveals that the injective F-structure of Theorem 0.4 is defined by the global S^1-action given by rotation in the direction of the S^1-factor of M_i; compare Sect. 5.

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