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# THE SMOOTHNESS OF RIEMANNIAN SUBMERSIONS WITH NON-NEGATIVE SECTIONAL CURVATURE

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Let $M^n$ be a complete, non-compact and $C^\infty$-smooth Riemannian manifold with non-negative sectional curvature. Suppose that $\mathcal{S}$ is a soul of $M^n$ given by the fundamental theory of Cheeger and Gromoll, and suppose that $\Psi$: $M^n \to \mathcal{S}$ is a distance non-increasing retraction from the whole manifold to the soul (e.g. the retraction given by Sharafutdinov). Then we show that the retraction $\Psi$ above must give rise to a $C^\infty$-smooth Riemannian submersion from $M^n$ to the soul $\mathcal{S}$.

Moreover, we derive a new flat strip theorem associated with the Cheeger–Gromoll convex exhaustion for the manifold above.

*Keywords*: Non-negative sectional curvature; smooth submersion; Cheeger–Gromoll soul theory; Perelman's solution.

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In this article, we study the smoothness of Riemannian submersions for open manifolds with non-negative sectional curvature. Suppose that $M^n$ is a $C^\infty$-smooth, complete and non-compact Riemannian manifold with nonnegative sectional curvature. Cheeger–Gromoll [2] established a fundamental theory for such a manifold. Among other things, they showed that $M^n$ admits a totally convex exhaustion $\{\Omega_u\}_{u\geq 0}$ of $M^n$, where $\Omega_0 = \mathcal{S}$ is a totally geodesic and compact submanifold without boundary. Furthermore, $M^n$ is diffeomorphic to the normal vector bundle of the soul $\mathcal{S}$.

Sharafutdinov found that there exists a distance non-increasing retraction $\Psi$: $M^n \to \mathcal{S}$ from the open manifold $M^n$ of non-negative sectional curvature to its soul (cf. [8, 10]). Perelman [7] further showed that such a map $\Psi$ is indeed a $C^1$-smooth Riemannian submersion. Furthermore, $\Psi[\mathrm{Exp}_q(t\vec{v})] = q$ for any $q \in \mathcal{S}$ and $\vec{v} \perp T_q(\mathcal{S})$. Therefore, the fiber $F_q = \Psi^{-1}(q)$ is a $k$-dimensional submanifold, which is $C^\infty$-smooth *almost everywhere*, where $k = \dim(M^n) - \dim(\mathcal{S}) > 0$.

Guijarro [Gu] proved that the fiber $F_q$ is indeed a $C^2$-smooth submanifold for each $q \in S$. In this paper, we prove that the fibres are $C^\infty$-smooth.

**Theorem 1.** *Let $M^n$ be a complete, non-compact and $C^\infty$-smooth Riemannian manifold with non-negative sectional curvature. Suppose $\mathcal{S}$ is a soul of $M^n$. Then any distance non-increasing retraction $\Psi\colon M^n \to \mathcal{S}$ must give rise to a $C^\infty$-smooth Riemannian submersion.*

*Consequently, if $\mathbb{R}^k = \mathcal{N}_q(\mathcal{S}, M^n)$ is the normal space of the soul $\mathcal{S}$ in $M^n$ at $q$, then the fiber $F_q = \Psi^{-1}(q) = \mathrm{Exp}_q(\mathbb{R}^k)$ is a $k$-dimensional $C^\infty$-smooth submanifold of $M^n$, for any $q \in S$.*

Professor Wilking kindly informed us that he has recently obtained a similar result (cf. [9]). His method is completely independent of ours. Our proof of Theorem 1 uses a flat strip theorem associated with Cheeger–Gromoll exhaustion (cf. Theorem 4 below), an uniform estimate for cut-radii of convex subsets in [2] and a smooth extension theorem for ruled surfaces.

For each compact convex subset $\Omega \subset M^n$, we let $U_\epsilon(\Omega) = \{x \in M^n | d(x, \Omega) < \epsilon\}$. Its cut-radius is given by $\delta_\Omega = \sup\{\epsilon | \text{there is a unique nearest point projection } \mathcal{P}_\Omega\colon U_\epsilon(\Omega) \to \Omega\}$.

For each $x \in M^n$, we let $\mathrm{Inj}_{M^n}(x)$ be the injectivity radius of $M^n$ at $x$. Similarly, let $\mathrm{Inj}_{M^n}(A) = \sup\{\mathrm{Inj}_{M^n}(x) | x \in A\}$.

A subset $\Omega$ of a complete Riemannian manifold $M^n$ is said to be *totally convex* if for any pair of points $\{p, q\} \subset \Omega$ and for any geodesic segment $\sigma$ joining $p$ and $q$, the geodesic segment $\sigma$ is contained in $\Omega$. There is a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of $M^n$ given in [2]. By comparing the inner angles of geodesic triangles, we have the following semi-global estimate for cut-radius.

**Lemma 2 ([1], [2, Lemma 2.4]).** *Let $A \subset \Omega_T$ be a connected, convex and compact subset in a Riemannian manifold $M^n$ with non-negative curvature, let $K_0 = \max\{K(x) | x \in \Omega_{T+1}\}$ be the upper bound of sectional curvature on $\Omega_{T+1}$, $\mathrm{Inj}_{M^n}(\Omega_T)$ and $\mathcal{S}$ be as above. Suppose that $\dim(\Omega_T) = n$. Then the subset $A$ has cut-radius bounded below by*

$$\delta_A \geq \delta_0(T) = \frac{1}{4} \min\left\{\mathrm{Inj}_{M^n}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\right\},$$

*where $\delta_0(T)$ is independent of choices of $A$ with $A \subset \Omega_T$.*

Let us briefly recall the Cheeger–Gromoll convex exhaustion. According to [2], there is a partition $a_0 = 0 < a_1 < \cdots < a_m < a_{m+1} = \infty$ of $[0, \infty)$ and an exhaustion $\{\Omega_u\}_{u \geq 0}$ of $M^n$ such that the following holds:

(1) $M^n = \cup_{u \geq 0} \Omega_u$. If $u > a_m$ then $\dim[\Omega_u] = n$. If $u \leq a_m$, then $\dim[\Omega_u] < n$.
(2) $\Omega_0 = \mathcal{S}$ is the soul of $M^n$, which is a totally geodesic $C^\infty$-smooth compact submanifold without boundary.
(3) If $u > 0$, $\Omega_u$ is a totally convex, compact subset of $M^n$ and hence $\Omega_u$ is a compact submanifold with a $C^\infty$-smooth relative interior. Furthermore,

$\dim(\Omega_u) = k_u > 0$ and $\Omega_u$ has a non-empty $(k_u - 1)$-dimensional relative boundary $\partial\Omega_u$;

(4) For any $u_0 \in [a_j, a_{j+1}]$ and $0 \leq t \leq u_0 - a_j$, the family $\{\Omega_{u_0-t}\}_{t\in[0,u_0-a_j]}$ is given by the *inward equidistant evolution*

$$\Omega_{u_0-t} = \{x \in \Omega_{u_0} | d(x, \partial\Omega_{u_0}) \geq t\}. \tag{1}$$

(5) If $u > a_m$ then $u - a_m = \max\{d(x, \partial\Omega_u) | x \in \Omega_u)\}$. If $0 \leq j \leq m-1$ then $a_{j+1} - a_j = \max\{d(x, \partial\Omega_{a_{j+1}}) | x \in \Omega_{a_{j+1}}\}$ and hence $\dim[\Omega_{a_j}] < \dim[\Omega_{a_{j+1}}]$ for $j \geq 0$.

Assume that $k = \dim[M^n] - \dim[\mathcal{S}] = \dim(F_q)$ for all $q \in \mathcal{S}$. Since $M^n = \cup_{T\geq 0}\Omega_T$, it is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a $k$-dimensional $C^\infty$-smooth interior, where $\delta_0(T)$ is given by Lemma 2 and $T > a_m$.

For this purpose, we need to study the geometry of the equidistant hypersurfaces from $\partial\Omega_u$. Federer [3, p. 435] studied the smoothness of the *outward equidistant hypersurfaces* $\partial[U_\epsilon(\Omega)]$ for $0 < \epsilon < \delta_\Omega$. Following his approach, we consider the outward normal cone of $\Omega$ as follows:

$$\mathcal{N}^+(\Omega, M^m) = \{(p, \vec{v}) | p \in \Omega, d(\mathrm{Exp}_p(t\vec{v}), \Omega) = t|\vec{v}|, \text{ for } 0 \leq t|\vec{v}| < \delta_\Omega\}.$$

If $\{\Omega_u\}$ is the Cheeger–Gromoll convex exhaustion as above and $u > 0$, then the relative boundary $\partial\Omega_u$ is not necessarily smooth. When $u > 0$, we let $\mathrm{int}(\Omega_u)$ be the relative interior of the convex subset $\Omega_u$. We are going to study the corresponding decomposition of $\mathcal{N}^+(\Omega, M^m)$:

$$\mathcal{N}_p^+(\Omega_u, M^n) \subset \left[\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)\right], \tag{2}$$

where $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ is defined by

$$\begin{aligned}\mathcal{N}^+\big(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})\big) = \{&(p, \vec{v}) | p \in \Omega_u, d(\mathrm{Exp}_p(t\vec{v}), \Omega_u) = t|\vec{v}|, \\ &\text{for } 0 \leq t|\vec{v}| < \delta_{\Omega_u}, \mathrm{Exp}_p(t\vec{v}) \in \mathrm{int}(\Omega_{u+\epsilon})\}.\end{aligned}$$

Our next step is to choose $\epsilon$ sufficiently small so that (i) there is a nearest point projection $\mathcal{P}$: $\mathrm{int}(\Omega_{u+\epsilon}) \to \Omega_u$; and (ii) $\Omega_u = \{x \in \Omega_{u+\epsilon} | d(x, \partial\Omega_{u+\epsilon}) \geq \epsilon\}$ holds. We first find $j$ so that $a_j \leq u < a_{j+1}$ for some $0 \leq j \leq m$. Let $T = u + a_m + 1$ and $\delta_0(T)$ be given by Lemma 2. It follows from a result of Yim that there is a constant $C_T$ such that, for $0 \leq u_1 < u_2 \leq T$, we have

$$\max\{d(x, \Omega_{u_1}) | x \in \Omega_{u_2}\} \leq C_T(u_2 - u_1), \tag{3}$$

see [11, Theorem A.5(3)]. In what follows, we always choose

$$0 < \epsilon = \epsilon_u < \min\left\{[a_{j+1} - u], \frac{\delta_0(T)}{2C_T}\right\}, \tag{4}$$

where $u \in [a_j, a_{j+1})$, $T = u + a_m + 1$ and $\delta_0(T)$ is given by Lemma 2.

With such a choice of $\epsilon = \epsilon_u$ by (4), the geometry of $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ is determined by its minimal normal vectors which we now describe.

**Definition 3 (Minimal normal vector).** Let $\Omega_u$, $\Omega_{u+\epsilon}$ and $\mathcal{N}^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ be as above. Let $\sigma_{(p,\vec{v})}$: $[0,\epsilon] \to M^n$ be a geodesic given by $\sigma_{(p,\vec{v})}(t) = \mathrm{Exp}_p\left(t\frac{\vec{v}}{|\vec{v}|}\right)$, where $\vec{v} \neq 0$. If $\sigma_{(p,\vec{v})}$ is a length-minimizing geodesic from $p \in \Omega_u$ to $\partial\Omega_{u+\epsilon}$, then $\vec{v}$ is called a minimal normal vector in $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$.

It is known that any other normal vector $\vec{w} \in \mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ can be expressed as a linear combination of *minimal normal vectors* at $p$. Moreover, the convex hull of minimal normal vectors at $p$ is equal to $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ (cf. [10, Proposition 1.7]).

For each $p \in M^n$, we let $\mathcal{V}_p = T_p(F_{\Psi(p)})$ and $\mathcal{H}_p = [\mathcal{V}_p]^\perp$. A geodesic $\alpha$: $[a,b] \to M^n$ is said to be horizontal, if $\alpha'(t) \perp F_{\Psi(\alpha(t))}$ for all $t \in [a,b]$. We need the following flat strip theorem for the proof of Theorem 1.

**Theorem 4.** *Let $\{\Omega_u\}$ be the Cheeger–Gromoll totally convex exhaustion of $M^n$ as above. Suppose that $\Psi$: $M^n \to \mathcal{S}$ be a distance non-increasing retraction and $F_q = \Psi^{-1}(q)$ be a fibre for some $q \in \mathcal{S}$. Then for $p \in F_q \cap \Omega_u$ and any $(p, \vec{v}) \in \mathcal{N}^+(\Omega_u, M^n)$, we have*

$$\Psi[\mathrm{Exp}_p([\mathbb{R}\{\vec{v}\}])] = \Psi(p) = q. \tag{5}$$

*Moreover, if* $\dim(\mathcal{S}) \geq 1, \vec{v} \in N_p^+(\Omega_u, M^n)$ *and if* $\vec{w} \in \mathcal{H}_p$ *has* $|\vec{w}| = 1 = |\vec{v}|$, *then the surface* $\Sigma^2_{\vec{v},\vec{w}} = \mathrm{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ *is totally geodesic immersed flat plane in $M^n$.*

A result similar to Theorem 4 was proved in [1] via a totally different method.

**Proof of Theorem 4.** Theorem 4 was proved by Perelman [7] for the case of $\Omega_0 = \mathcal{S}$. Applying Perelman's argument for the case of $p \notin \mathcal{S}$, Guijarro [4] found the following sufficient condition for (5).

$$\begin{array}{c}\vec{v} \in \mathcal{V}_p \textit{ stays vertical under parallel transport along} \\ \textit{any horizontal broken geodesic.}\end{array} \tag{6}$$

Guijarro showed that (5) follows from (6). Moreover, if (6) holds and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma^2_{\vec{v},\vec{w}} = \mathrm{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in $M^n$ (cf. [4, Theorem 3.1]).

In order to see that $\mathcal{N}^+(\Omega_u, M^n) \subset \mathcal{V}_p$ holds, we recall that any horizontal geodesic $\alpha$ is contained a tubular neighborhood of the soul $S$, by Perelman's theorem [7]. Hence, $\alpha$ is contained in a compact totally geodesic subset $\Omega_T$ for a sufficiently large $T$. It follows from [2, Theorem 5.1] that $\alpha \subset \partial\Omega_u$ for some $u$ (cf. [5]).

$$\begin{array}{l}\textit{Any horizontal geodesic } \alpha \textit{ with } \alpha(0) \in \partial\Omega_\lambda \textit{ must be entirely} \\ \quad \textit{contained in } \partial\Omega_\lambda. \textit{ Since } \Omega_u = \mathcal{S} \bigcup [\cup_{\lambda \leq u}(\partial\Omega_\lambda)], \textit{ we have} \\ \quad \mathcal{H}_p \subset T_p^-(\partial\Omega_\lambda) \subset T_p^-(\Omega_u), \\ \quad \textit{where } T_p^-(\partial\Omega_\lambda) \textit{ is the tangent cone of } \partial\Omega_\lambda \text{ at } p.\end{array} \tag{7}$$

Recall that $\mathrm{int}(\Omega)$ is the relative interior of the convex subset $\Omega$. If $p \in \mathrm{int}(\Omega_u)$ and if $\vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n)$, Guijarro [4, Corollary 3.2] showed that $\vec{v}$ satisfies (6), because $\mathrm{int}(\Omega_u)$ is totally geodesic and (7) holds.

It remains to consider the case when $p \in \partial\Omega_u$. Recall that by (2), we have

$$\mathcal{N}_p^+(\Omega_u, M^n) \subset \big[\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)\big].$$

For $\vec{v}$ in either $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ or $\mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$, we will show that such a $\vec{v}$ satisfies (6).

It follows from [2, Theorem 1.10] (or [10, Corollary 1.4]) that any *minimal normal vector* $\vec{v}$ of $\mathcal{N}_p^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ stays *minimal* under parallel transport along any geodesic in $\partial\Omega_u$. Since the convex hull of minimal normal vectors is equal to the outward normal cone (cf. [10, Proposition 1.7]), the bundle $\mathcal{N}^+(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon}))$ is invariant under parallel transport along any geodesic in $\partial\Omega_u$. This together with (7) implies that if $\vec{v} \in \mathcal{N}_p^+\big(\Omega_u, \mathrm{int}(\Omega_{u+\epsilon})\big)$ then $\vec{v}$ satisfies (6).

For $\vec{v} \in \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$, the assertion (6) follows from [4, Corollary 3.2]. In fact, since $\mathrm{int}(\Omega_{u+\epsilon})$ is totally convex and totally geodesic, both $T(\mathrm{int}(\Omega_{u+\epsilon}))$ and $N^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$ are invariant under parallel transport along any geodesic in $\mathrm{int}(\Omega_{u+\epsilon})$. This together with (7) implies that (6) holds for any vector $\vec{v} \in \mathcal{N}_p^+(\mathrm{int}(\Omega_{u+\epsilon}), M^n)$.

Therefore, (6) holds for any $\vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n)$. This completes the proof of Theorem 4. □

In order to see that Theorem 4 implies Theorem 1, we need to establish a bootstrap argument for the smoothness of ruled surfaces. A $C^1$-smooth one-parameter family of a straight lines in $\mathbb{R}^3$ gives rise to a ruled surface. Suppose that $\{\beta(s), \vec{v}(s)\}$ are $C^1$-smooth vector valued functions with $[\beta'(s) + t\vec{v}'(s)] \wedge \vec{v}(s) \neq 0$ for all $(s,t) \in (a,b) \times (c,d)$. Then we have a corresponding $C^1$-smooth immersed ruled surface.

$$\begin{aligned} F\colon (a,b) \times (c,d) &\to \mathbb{R}^3, \\ (s,t) &\to \beta(s) + t\vec{v}(s). \end{aligned}$$

Our bootstrap argument is motivated by the following observation.

**Lemma 5 (The smooth extension for ruled surfaces in $\mathbb{R}^3$).** *Let $F\big((a,b) \times (c,d)\big) = \Sigma^2$ be an embedded ruled surface in $\mathbb{R}^3$ and let $F\colon (a,b) \times (c,d) \to \mathbb{R}^3$ be a $C^{1,1}$-smooth embedding map be as above. Suppose that a subset $\hat{\Sigma}^2_\epsilon = F\big((a,b) \times (\epsilon_1, \epsilon_2)\big)$ is a $C^\infty$-smooth embedded surface of $\mathbb{R}^3$, where $(\epsilon_1, \epsilon_2) \subset (c,d)$. Then the whole ruled surface $\Sigma^2$ is a $C^\infty$-smooth surface of $\mathbb{R}^3$.*

**Proof.** By our assumption, $F$ is an embedding map, and hence the surface $\hat{\Sigma}^2_\epsilon = F\big((a,b) \times (\epsilon_1, \epsilon_2)\big)$ is foliated by straight lines. Because the surface $\hat{\Sigma}^2_\epsilon$ and each orbit (each straight line) are $C^\infty$, the quotient space $Q = [\hat{\Sigma}^2/{\sim}]$ is a $C^\infty$-smooth 1-dimensional space as well, where $\sim$ is the equivalent relation induced by the orbits (the ruling straight lines). Thus, we have a fibration $(\epsilon_1, \epsilon_2) \to \hat{\Sigma}^2_\epsilon \to Q$. We may assume that the quotient space $Q$ is diffeomorphic to an open interval $(0,1)$. Let $\pi\colon \hat{\Sigma}^2_\epsilon \to Q$ be the quotient map. Because the fibration is topologically trivial, we can find two *disjoint* $C^\infty$-smooth cross-sections

$$\begin{aligned} h_i\colon Q &\to \hat{\Sigma}^2_\epsilon, \\ u &\to h_i(u), \end{aligned}$$

for $i = 0, 1$, where $\pi(h_i(u)) = 0$. (Since the fibre is 1-dimensional line, we may assume that the graph of the cross-section $h_1$ lies above that of $h_0$.) Because $h_0(Q)$ and $h_1(Q)$ are disjoint, we obtain a new $C^\infty$-smooth parametrization of the ruled surface

$$\begin{aligned} G\colon Q \times \mathbb{R} &\to \mathbb{R}^3, \\ (u, \lambda) &\to h_0(u) + \lambda \frac{[h_1(u) - h_0(u)]}{\|h_1(u) - h_0(u)\|}. \end{aligned}$$

Clearly, $G$ is a $C^\infty$-smooth map with $\Sigma^2 \subset G(Q \times \mathbb{R})$. Because $F$ is an embedding map, on the subset $G^{-1}(\Sigma^2)$, one can check that $G$ remains to be injective and with non-vanishing Jacobi $G_u \wedge G_\lambda \neq 0$. Hence, $G|_{G^{-1}(\Sigma^2)}$ is an embedding as well. Thus, $\Sigma^2$ is a $C^\infty$-smooth embedded surface. □

The proof of Lemma 5 can be applied to the proof of Theorem 1 as follows. Let $\Omega_u$ be a totally convex subset as above. By a theorem of Federer, the hypersurface $\partial[U_\epsilon(\Omega_u)]$ is $C^{1,1}$-smooth if the positive number $\epsilon$ is less than the cut-radius of $\Omega_u$, (see [3, Theorem 4.8(9), p. 435]). Assume that $T > u$ and $d = \delta_T - \epsilon > 0$. Let $\vec{v}(x)$ be the outward unit normal vector of $\partial[U_\epsilon(\Omega)]$ at $x$. There is an embedding:

$$\begin{aligned} F\colon \partial[U_\epsilon(\Omega)] \times (c, d) &\to M^n, \\ (x, t) &\to \mathrm{Exp}_x[t\vec{v}(x)]. \end{aligned} \tag{8}$$

where $c = -\epsilon$.

**Proposition 6 (The smooth extension for the ruled sub-manifold).** *For each $y \in \partial[U_\epsilon(\Omega)]$, we let $B^{k-1}_r(y) \subset \partial[U_\epsilon(\Omega)]$ be a small $(k-1)$-dimensional ball around $y$ which is $C^1$-diffeomorphic to $B^{k-1}(0) \subset \mathbb{R}^{k-1}$ and let $\Omega_u, \delta, \delta_T, c, d$ and $F$ be as above. Suppose that $F$ is an $C^{1,1}$-smooth embedding and that a smaller subset $\hat{\Sigma}^k_\epsilon = F\big(B^{k-1}_r(y) \times (\epsilon_1, \epsilon_2)\big)$ is a $C^\infty$-smooth embedded $k$-submanifold of $M^n$, where $(\epsilon_1, \epsilon_2) \subset (c, d)$. Then the whole ruled submanifold $\Sigma^k = F\big(B^{k-1}_r(y) \times (c, d)\big)$ is a $C^\infty$-smooth submanifold of $M^n$.*

**Proof.** The proof of Proposition 6 is the same as above with minor modifications. By our assumption, $\hat{\Sigma}^k_\epsilon$ is foliated by $C^\infty$-smooth open geodesic segments. The quotient space $Q = [\hat{\Sigma}^k_\epsilon/\sim]$ is a $C^\infty$-smooth $(k-1)$-dimensional open manifold. Because the fibration $(\epsilon_1, \epsilon_2) \to \hat{\Sigma}^k_\epsilon \to Q$ is trivial, we can choose two two *disjoint* cross sections $h_0\colon Q \to \hat{\Sigma}^k_\epsilon$ for $i = 0, 1$. If $\pi\colon \hat{\Sigma}^k_\epsilon \to Q$ is the quotient map, then $\pi \circ h_i(u) = u$ for all $u \in Q$. Since the two cross-sections are disjoint, we may assume that $r(h_1(u)) > r(h_0(u))$ for all $u \in Q$, where $r(y) = d\big(y, \partial[U_\delta(\Omega)]\big)$. For each $u \in Q$, we consider the unit vector

$$\vec{\eta}(u) = \frac{\mathrm{Exp}^{-1}_{h_0(u)}[h_1(u)]}{\|\mathrm{Exp}^{-1}_{h_0(u)}[h_1(u)]\|},$$

at the point $h_0(u)$. Similarly, we consider a new $C^\infty$-smooth parametrization

$$G\colon Q \times \mathbb{R} \to M^n,$$
$$(u, \lambda) \to \mathrm{Exp}_{h_0(u)}[\lambda \vec{\eta}(u)].$$

Clearly, we have $\Sigma^k = F\big(B^{k-1}_r(y) \times (c, d)\big) \subset G\big(Q \times \mathbb{R}\big)$. This completes the proof. □

With Lemma 2, Theorem 4 and Proposition 6, we are ready to prove Theorem 1.

**Proof of Theorem 1.** Let $\{\Omega_u\}$ be a Cheeger–Gromoll convex exhaustion described as above. It is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a $k$-dimensional $C^\infty$-smooth interior for any given $T > a_m$ and $q \in \mathcal{S}$, where $\delta_0(T)$ is given by Lemma 2.

Fix $T > a_m$ with $\dim[\Omega_T] = n$. Let $C_T$ be given by (3). Choose a partition $0 = u_0 < u_1 < \cdots < u_N = T$ of $[0, T]$ such that $u_j - u_{j-1} < \frac{2C_T}{\delta_0(T)}$ for $j = 1, \ldots, N$, where $N = N_T$ is a number depending on $T$.

We will prove the following assertion by induction on $j = 0, 1, \ldots, N$.

**Assertion *j*.** *The sub-level set $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q]$ has the $k$-dimensional $C^\infty$-smooth interior, where $q \in \mathcal{S}$ and $k = \dim[F_q]$.*

It follows from Perelman's theorem or Theorem 4 that $\mathrm{Exp}_q[\mathcal{N}^+_q(\mathcal{S}, M^n)] \subset F_q$. Since the soul $\mathcal{S}$ has the cut radius $\geq \delta_0(T)$ and $\mathcal{S}$ is $C^\infty$-smooth, Assertion 0 holds.

Let $\epsilon_1 = \frac{\delta_0(T)}{16}$ and $\epsilon_2 = \frac{\delta_0(T)}{8}$. We consider

$$A(\Omega_{u_j}, r_1, r_2) = \{z \in F_q \,|\, 0 < r_1 < d(z, \Omega_{u_j}) < r_2\}.$$

It is clear that $A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset U_{\delta_0(T)}(\mathcal{S})$. It follows from Assertion 0 that the subset $\hat{\Sigma}^k_\epsilon = A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset F_q \cap U_{\delta_0(T)}(\mathcal{S})$ is $C^\infty$-smooth $k$-dimensional open sub-manifold. By Theorem 4, we let $\Sigma^k_1 = A\left(\Omega_{u_1}, \frac{\delta_0(T)}{16}, \delta_0(T)\right)$ be the ruled $k$-dimensional submanifold. Recall that $\Sigma^k_1$ is $C^{1,1}$-smooth by a theorem of Federer. Since $\hat{\Sigma}^k_\epsilon$ is $C^\infty$-smooth by Assertion 0, it follows from Proposition 6 (the smooth extension theorem for the ruled submanifold) that $\Sigma^k_1$ is a $C^\infty$-smooth $k$-dimensional submanifold of $M^n$. Observe that the subset $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q]$ is contained in the union $\{[U_{\delta_0(T)}(\mathcal{S}) \cap F_q] \cup \Sigma^k_1\}$. Since $\Sigma^k_1$ is a $C^\infty$-smooth, Assertion 1 follows from Assertion 0.

Similarly, using Theorem 4 and Proposition 6 we can verify that if Assertion $(j-1)$ is true then Assertion $j$ holds as well for $j \geq 2$. In fact, by induction we see that $A(\Omega_{u_j}, \epsilon_1, \epsilon_2) \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q]$ is $C^\infty$-smooth. It follows from

Theorem 4 and Proposition 6 that the ruled submanifold $\Sigma_j^k = A(\Omega_{u_j}, \frac{\delta_0(T)}{16}, \delta_0(T))$ must be of $C^\infty$-smooth as well. Since $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q] \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q] \cup \Sigma_j^k$, Assertion $j$ follows. Theorem 1 follows from Assertion $N_T$ for any arbitrarily large $T$. □

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