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# Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature

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**Abstract.** Let $M^{2n}$ be a compact Riemannian manifold of *non-positive* sectional curvature. It is shown that if $M^{2n}$ is homeomorphic to a Kähler manifold, then its Euler number satisfies the inequality $(-1)^n\chi(M^{2n}) \geq 0$.

## Introduction

The results of this paper are related to a well-known problem, attributed to Hopf, to the effect that the Euler number $\chi(M^{2n})$ of a compact Riemannian manifold $M^{2n}$ of negative sectional curvature must satisfy the inequality $(-1)^n\chi(M^{2n}) > 0$. This conjecture is true in dimensions 2 and 4 [Ch] and it has been verified in the Kähler case for all $n$ by Gromov [G] and Stern [S] (the work in [S] also uses results of Greene and Wu; see [GW], p.183-215).

Gromov's arguments are rather general and establish the following result: "Let $M^{2n}$ be a compact Riemannian manifold of negative curvature. If $M^{2n}$ is homotopy equivalent to a compact Kähler manifold then $(-1)^n\chi(M^{2n}) > 0$"; see [G], Theorem 0.4.A and Example (a), p.265.

A companion conjecture asserts that, if the sectional curvature of a Riemannian manifold $M^{2n}$ is assumed to be only *non-positive,* then the Euler number must satisfy $(-1)^n\chi(M^{2n}) \geq 0$. Again, this second conjecture is known to be true in dimensions two and four [Ch]. The aim of this paper is to establish its validity for all $n$ in the Kähler case, thus complementing the above result of Gromov:

**Main Theorem.** *Let $M^{2n}$ be a compact Riemannian manifold of non-positive curvature. If $M^{2n}$ is homeomorphic to a Kähler manifold, then the Euler number of $M^{2n}$ satisfies the inequality* $(-1)^n\chi(M^{2n}) \geq 0$.

*Remark.* The proof of the Main Theorem works if the two manifolds are assumed to be only homotopy equivalent but, in view of Farrell and Jones [FJ], the man-

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ifolds are actually homeomorphic if $\dim M \geq 6$ (the case $\dim M = 4$ follows from [Ch]).

Dodziuk [Do] and Singer ([Yau], p. 672) have proposed to settle the Hopf conjecture using the Atiyah index theorem for coverings (see [At]). In this approach, one is required to prove a vanishing theorem for $L^2$ harmonic $k$-forms, $k \neq n$, on the universal covering of $M^{2n}$. The vanishing of these $L^2$ Betti numbers implies, by Atiyah's result, that $(-1)^n \chi(M^{2n}) \geq 0$. The strict inequality $(-1)^n \chi(M^{2n}) > 0$ follows provided one can establish the existence of non-trivial $L^2$ harmonic $n$-forms on the universal cover.

The program outlined above was carried out by Gromov [G] when the manifold in question is Kähler and is homotopy equivalent to a compact manifold with *strictly* negative sectional curvatures. The central idea in Gromov's approach is the notion of *d(bounded)* differential forms. As the terminology suggests, these are differential forms that are exterior derivatives of *bounded* forms, where boundedness is to be understood in the Riemannian sense. For instance, Gromov [G] points out that a bounded closed $k$-form, $k \geq 2$, on a complete simply-connected manifold whose sectional curvatures are bounded above by a negative constant is automatically $d$-bounded.

Following [G], a Kähler manifold is termed *Kähler-hyperbolic* if the Kähler form of its universal cover is $d$(bounded). Armed with the concept of Kähler-hyperbolicity, Gromov went on to prove the desired vanishing and existence theorems on the universal cover, thus establishing the inequality $(-1)^n \chi(M^{2n}) > 0$ when the Kähler manifold $M^{2n}$ is homotopy equivalent to a compact manifold with negative curvature.

Although Kähler hyperbolicity is a powerful idea in the case of strictly negative curvature, it is clearly inadequate for the problem "$K \leq 0 \Longrightarrow (-1)^n \chi(M^{2n}) \geq 0$". This can be seen by taking a compact Kähler manifold $M^{2n}$ satisfying $K \leq 0$ and $\chi(M^{2n}) = 0$ (e.g., a flat complex torus or, more generally, the product of such a torus and a Kähler manifold of negative curvature). The Kähler form on the universal cover of $M^{2n}$ is not $d$(bounded). Otherwise, by Gromov's work one would have the correct vanishing and existence theorems of $L^2$ harmonic forms and, by Atiyah's theorem, the Euler number of $M^{2n}$ would actually be non-zero.

Our contribution to this circle of ideas has been to single out a condition which is weaker than $d$-boundedness and can be applied to Kähler manifolds of *non-positive* curvature:

**Definition 1.** *A differential form $\alpha$ on a complete non-compact Riemannian manifold is called $d$(sublinear) if there exist a differential form $\beta$ and a number $c > 0$ such that $d\beta = \alpha$ and $|\beta(x)| \leq c(1 + \rho(x, x_0))$, where $\rho(x, x_0)$ stands for the Riemannian distance between $x$ and a base point $x_0$.*

The concept of $d$-sublinearity is both natural and flexible. This can be seen from the following results, both of which will be used to prove the Main Theorem in Sect. 3.

**Theorem 1.** *Let $M^n$ be a complete simply-connected manifold of non-positive sectional curvature and $\alpha$ a bounded closed $k$-form on $M^n$, $k \geq 1$. Then $\alpha$ is d(sublinear).*

**Theorem 2.** *Let $N^{2n}$ be a complete non-compact Kähler manifold of complex dimension $n$ whose Kähler form is d(sublinear). If $k \neq n$, then any $L^2$ harmonic $k$-form on $N^{2n}$ is identically zero.*

Other results on $L^2$ cohomology can be found in Anderson [An], Atiyah [At], Dodziuk [Do], Donnelly and Fefferman [DF], Donnelly and Xavier [DX], Elworthy and Rosenberg [ER] and Lott [L].

Extending Gromov's terminology, we propose the following:

**Definition 2.** *A Kähler manifold is Kähler-parabolic if the Kähler form on its universal cover is d(sublinear) but not d(bounded).*

Accordingly, our results imply that $(-1)^n \chi(M^{2n}) \geq 0$ if the compact manifold $M^{2n}$ is Kähler-parabolic, with strict inequality holding if $M^{2n}$ is Kähler-hyperbolic ([G]).

## 1 Proof of Theorem 1: Solutions of $d\beta = \alpha$ with sublinear growth

Throughout this section $(M^m, g)$ will be a complete simply-connected manifold of non-positive curvature. Let also $\alpha$ be a bounded smooth closed $k$-form on $M$. Since $M^m$ is diffeomorphic to $\mathbb{R}^m$ there exists a form $\beta$ such that $d\beta = \alpha$. The purpose of this section is to show that $\beta$ can be chosen to have sublinear growth, in the sense of the definition given in the introduction. The proof will follow from the Poincaré lemma by a comparison argument.

Fix $p \in M$ and denote by $exp_p : T_pM \to M$ the exponential map.

**Lemma 1.** *Consider the maps $\tau_t : M \to M$, given by $x \longmapsto exp_p(t\, exp_p^{-1}(x))$, where $0 \leq t \leq 1$. Then*

$$|(\tau_t)_* \xi| \leq t|\xi| \tag{1}$$

*for every tangent vector $\xi$.*

*Proof.* Let $\sigma : [0, 1] \to M^n$ be the geodesic segment joining $p$ to $x$, $\xi \in T_xM^n$ and $y = (exp_p)^{-1}(x) \in T_pM^n$. By a straightforward computation one has

$$\begin{aligned} (\tau_t)_*\xi &= (d\, exp_p)_{t(exp_p)^{-1}(x)}[td(exp_p^{-1})_{(x)}\xi] \\ &= (d\, exp_p)_{ty}\{t[d(exp_p)_y]^{-1}\xi\}. \end{aligned}$$

It is now manifest from the above formula that

$$J(t) := (\tau_t)_*\xi \tag{2}$$

is the Jacobi field along $\sigma$ satisfying $J(0) = 0,\ J(1) = \xi$. On the other hand, since the sectional curvatures are non-positive, the function $f(s) := |J(s)|$ is convex ([BGS], p.5). In particular, for $s \geq t$, one has

$$f(s) \geq f(t) + (s - t)\frac{f(t) - f(0)}{t - 0} = \frac{s}{t}f(t).$$

Setting $s = 1$ one has $f(t) \leq tf(1)$ and the result follows from (2). □

Recall that if $\alpha$ is a $k$-form and $Z$ is a vector field, then $(\alpha\lfloor_Z)$ is the $(k-1)$-form given by

$$(\alpha\lfloor_Z)(\xi_1, \cdots, \xi_{k-1}) = \alpha(Z, \xi_1, \cdots, \xi_{k-1}).$$

For the sake of completeness we give a proof of the following elementary result.

**Lemma 2.** *Let $\Psi$ be a closed k-form in $\mathbb{R}^m$. Then the $(k-1)$-form $\Phi$ defined by*

$$\Phi(x) = r\int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt \tag{3}$$

*satisfies $d\Phi = \Psi$; here $\frac{\partial}{\partial r} = \sum_{i=1}^m \frac{x_i}{r}\frac{\partial}{\partial x_i}$, $r = (\sum_{i=1}^m x_i^2)^{1/2}$ and $\tau_t(x) = tx$.*

*Proof.* By the standard proof of the Poincaré lemma ([ST], p.130), $\Phi$ can be taken to be

$$\Phi(x) = \sum_{i_1<\cdots<i_k}\sum_{j=1}^{k}(-1)^{j-1}x_{i_j}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_j}}\wedge\cdots\wedge dx_{i_k},$$

where $\Psi = \sum_{i_1<\cdots<i_k}\Psi_{i_1\cdots i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

In particular, one has

$$\begin{aligned}\Psi(x) &= \sum_{i_1<\cdots<i_k}\sum_{j=1}^{k}x_{i_j}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)(dx_{i_1}\wedge\cdots\wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial x_{i_j}}}\\ &= r\sum_{i_1<\cdots<i_k}\Big(\int_0^1 t^{k-1}\Psi_{i_1\cdots i_k}(tx)dt\Big)(dx_{i_1}\wedge\cdots\wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial r}}\\ &= r\int_0^1 t^{k-1}(\Psi\lfloor_{\frac{\partial}{\partial r}})(tx)dt\\ &= r\int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt,\end{aligned}$$

as desired. □

*Proof of Theorem 1:* Let $(x_1, ..., x_n)$ be Euclidean coordinates in $T_pM$ and consider the pull-back metric $h$ of the metric $g$ under $exp_p : T_pM \to M$. Observe that there are now two ways to interpret the map $\tau_t$. The first interpretation comes from Lemma 1 with $(M, g)$ being replaced by $(T_pM, h)$; alternatively, one can think of $\tau_t$ as the self-map of $T_pM$, $(x_1, ..., x_n) \longmapsto t(x_1, ..., x_n)$, that appears in the Poincaré lemma (Lemma 2). It is an easy and yet basic observation that these two ways of thinking about $\tau_t$ give rise to the same map.

We may also replace the form $\alpha$ that appears in the statement of Theorem 1 by a closed form $\Psi$ on $T_pM$ which is bounded in the induced metric $h$. Let $\Phi$ be given by Lemma 2 and observe that, by Lemma 1,

$$|(\tau_t)^*\varphi(x)|_h \leq Ct^{k-1}|\varphi(\tau_t(x))|_h, \quad k \geq 1, \tag{4}$$

holds for any $(k-1)$-form $\varphi$ on $T_pM$; here $|\cdot|_h$ is any one of the equivalent norms induced by $h$. Since $|\frac{\partial}{\partial r}| = 1$, it follows from (4) and Lemma 2 that

$$\begin{aligned}
|\Phi(x)|_h &\leq r\int_0^1 |[(\tau_t)^*(\Psi \llcorner_{\frac{\partial}{\partial r}})](x)|_h dt \\
&\leq Cr\int_0^1 t^{k-1}|\Psi \llcorner_{\frac{\partial}{\partial r}}(tx)|_h dt \\
&\leq C_1 r\int_0^1 t^{k-1}|\Psi(tx)|_h dt \\
&\leq C_1 r \sup_{0\leq t\leq 1} |\Psi(tx)|_h.
\end{aligned}$$

In particular,

$$|\Phi(x)|_h \leq C_1\rho_h(0, x) \sup|\Psi|_h \leq C_2\rho_h(0, x).$$

Hence $\Phi$ is $d$(sublinear) and the proof of Theorem 1 is complete. □

## 2 Proof of Theorem 2: Vanishing of $L^2$ harmonic forms

We begin the proof of Theorem 2 by recalling some basic facts in Hodge theory and Kähler geometry. If $M^m$ is an oriented complete Riemannian manifold, let $\delta$ be the adjoint operator of $d$ acting on the space of $L^2$ k-forms. Denote by $\Omega^k_{(2)}(M^m)$ and $\mathcal{H}^k_{(2)}(M^m)$ the spaces of $L^2$ $k$-forms and $L^2$ harmonic $k$-forms, respectively. By elliptic regularity and completeness of the manifold, a $k$-form in $\mathcal{H}^k_{(2)}(M^m)$ is smooth, closed and co-closed.

Suppose that $M^{2n}$ is a complete Kähler manifold of complex dimension $n$ with Kähler form $w$. Since $w$ is bounded, the Lefschetz map $L : \Omega^k_{(2)}(M^{2n}) \to \Omega^{k+2}_{(2)}(M^{2n}), \alpha \to w \wedge \alpha$, is well-defined, it is injective if $k < n$ ([We], p182) and satisfies $L\mathcal{H}^k_{(2)}(M^m) \subset \mathcal{H}^{k+2}_{(2)}(M^m)$ ([We], p191). By duality, it is sufficient to consider the case $k < n$ in Theorem 2.

By hypothesis, there exists a 1-form $\beta$ with $d\beta = w$ and

$$|\beta(x)| \leq c(1 + \rho(x, x_0)),$$

where $c$ is an absolute constant.

In what follows we assume that the distance function $\rho(x, x_0)$ is smooth for $x \neq x_0$. The general case follows easily by an approximation argument.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 0, & \text{if } t \geq 1 \end{cases}$$

and consider the compactly supported function

$$f_j(x) = \eta(\rho(x, x_0) - j),$$

where $j$ is a positive integer.

Let $\alpha$ be a harmonic $k$-form in $L^2$ , $k < n$, and consider the form $\Phi = \beta \wedge \alpha$. Observing that $\delta(w \wedge \alpha) = 0$ since $w \wedge \alpha \in \mathcal{H}^{k+2}_{(2)}(M^m)$ and noticing that $f_j\Phi$ has compact support, one has

$$0 = < \delta(w \wedge \alpha), f_j\Phi >= < w \wedge \alpha, d(f_j\Phi) >,$$

where $<, >$ stands for the global $L^2$ inner product.

We further note that, since $d\beta = w$ and $d\alpha = 0$,

$$\begin{aligned} 0 = &< w \wedge \alpha, d(f_j\Phi) > \\ = &< w \wedge \alpha, f_j d\Phi > + < w \wedge \alpha, df_j \wedge \Phi > \\ = &< w \wedge \alpha, f_j w \wedge \alpha > + < w \wedge \alpha, df_j \wedge \Phi > \\ = &< w \wedge \alpha, f_j w \wedge \alpha > + < w \wedge \alpha, df_j \wedge \beta \wedge \alpha > . \end{aligned} \tag{5}$$

Since $0 \leq f_j \leq 1$ and $\lim_{j\to\infty} f_j(x)(w \wedge \alpha)(x) = (w \wedge \alpha)(x)$, it follows from the dominated convergence theorem that

$$\lim_{j\to\infty} < w \wedge \alpha, f_j w \wedge \alpha >= \|w \wedge \alpha\|^2. \tag{6}$$

Since $w$ is bounded, $\text{supp}(df_j) \subset B_{j+1} \diagdown B_j$ and $|\beta(x)| = O(\rho(x, x_0))$, one obtains

$$< w \wedge \alpha, df_j \wedge \beta \wedge \alpha > \leq (j + 1)C \int_{B_{j+1} \diagdown B_j} |\alpha(x)|^2 dx, \tag{7}$$

where $C$ is a constant independent of $j$.

We claim that there exists a subsequence $\{j_i\}_{i\geq 1}$ such that

$$\lim_{i\to\infty}(j_i+1)\int_{B_{j_i+1}\setminus B_{j_i}}|\alpha(x)|^2dx=0. \tag{8}$$

If not, there would exist a positive constant $a$ such that

$$(j+1)\int_{B_{j+1}\setminus B_j}|\alpha(x)|^2dx\geq a>0,\ \ j\geq 1.$$

This inequality implies

$$\begin{aligned}\int_{M^{2n}}|\alpha(x)|^2dx&=\sum_{j=0}^{\infty}\int_{B_{j+1}\setminus B_j}|\alpha(x)|^2dx\\&\geq a\sum_{j=0}^{+\infty}\frac{1}{j+1}=+\infty,\end{aligned}$$

a contradiction to the assumption $\int_{M^{2n}}|\alpha(x)|^2dx<\infty$. Hence, there exists a subsequence $\{j_i\}_{i\geq 1}$ for which (8) holds. Using (7) and (8), one obtains

$$\lim_{i\to\infty}<w\wedge\alpha,df_{j_i}\wedge\beta\wedge\alpha>=0. \tag{9}$$

It now follows from (5), (6) and (9) that $w\wedge\alpha=0$. Since $L$ is injective ($k<n$), $\alpha=0$ as desired. □

## 3 Proof of the Main Theorem

We begin with the following:

**Lemma 3.** *Let $F:M_1\to M_2$ be a smooth homotopy equivalence between two compact Riemannian manifolds, $\pi:\widetilde{M}_i\to M_i$ the universal covering maps for $i=1,2$. Then, for any closed differential form $\alpha$ on $M_2$, $\pi^*(\alpha)$ is d(sublinear) on $\widetilde{M}_2$ if the form $(F\circ\pi)^*(\alpha)$ is d(sublinear) on $\widetilde{M}_1$.*

*Proof.* Assume that $\alpha$ is a closed form on $M_2$, and that $(F\circ\pi)^*(\alpha)$ is $d$(sublinear) on $\widetilde{M}_1$. Our goal is to show that $\pi^*(\alpha)$ is $d$(sublinear) on $\widetilde{M}_2$.

Since $F$ is a homotopy equivalence, there exists a smooth map $G:M_2\to M_1$ such that both $F\circ G$ and $G\circ F$ are homotopic to the identity maps.

Clearly, the maps $F$ and $G$ can be lifted to the universal covering spaces. Let then $\widetilde{F}:\widetilde{M}_1\to\widetilde{M}_2$ and $\widetilde{G}:\widetilde{M}_2\to\widetilde{M}_1$ be the lifted maps, so that the following diagram commutes:

$$\begin{tikzcd}\widetilde{M}_1 \arrow[r, "\widetilde{F}"] \arrow[d, "\pi"'] & \widetilde{M}_2 \arrow[r, "\widetilde{G}"] \arrow[d, "\pi"'] & \widetilde{M}_1 \arrow[d, "\pi"'] \\ M_1 \arrow[r, "F"] & M_2 \arrow[r, "G"] & M_1\end{tikzcd}$$

Since $M_1$ is compact, $\alpha$ is bounded and $\pi$ is a local isometry, $(F \circ \pi)^*(\alpha)$ is a bounded form on $\widetilde{M}_1$. By the assumption and the commutativity of the diagram, there exists $\beta$ with sublinear growth such that

$$d\beta = (F \circ \pi)^*(\alpha) = (\pi \circ \widetilde{F})^*(\alpha) = \widetilde{F}^*(\pi^*(\alpha)).$$

In particular,

$$d(\widetilde{G}^*(\beta)) = \widetilde{G}^* \circ \widetilde{F}^*(\pi^*(\alpha)).$$

On the other hand, since $F \circ G$ is homotopic to the identity map, the closed forms $\alpha$ and $(F \circ G)^*(\alpha)$ are in the same de Rham cohomology class. Hence there exists a form $\Theta$ on $M^{2n}$ such that $\alpha = (F \circ G)^*(\alpha) + d\Theta$. It follows that

$$\begin{aligned}
\pi^*(\alpha) &= d(\pi^*(\Theta)) + (\pi^* G^* F^*)(\alpha) \\
&= d(\pi^*(\Theta)) + (\widetilde{G}^* \pi^* F^*)(\alpha) \\
&= d(\pi^*(\Theta)) + (\widetilde{G}^* \widetilde{F}^* \pi^*)(\alpha) \\
&= d(\pi^*(\Theta)) + d(\widetilde{G}^*(\beta)) \\
&= d[\pi^*(\Theta) + \widetilde{G}^*(\beta)].
\end{aligned}$$

Since $\Theta$ is bounded, $\beta$ is sublinear and $\widetilde{G}_*$ is bounded, it follows that $\pi^*(\alpha)$ is $d$(sublinear), as desired. □

*Proof of Main Theorem:* Suppose that $F : M_1^{2n} \to M_2^{2n}$ is a homotopy equivalence from a compact Riemannian manifold $M^{2n}$ of non-positive sectional curvature to a compact Kähler manifold $M_2^{2n}$. By an approximation argument if necessary, one may assume that $F$ is a smooth map.

Let $\pi : \widetilde{M}_1^{2n} \to M_1^{2n}$ be the universal covering map and $w$ the Kähler form on $M_2^{2n}$. Since $(F \circ \pi)^*(w)$ is a bounded closed form on $\widetilde{M}_1^{2n}$, it follows from Theorem 1 that $(F \circ \pi)^*(w)$ is $d$-sublinear. By Lemma 3, the lifted Kähler form $\widetilde{w}$ on $\widetilde{M}_2^{2n}$ is $d$-sublinear as well. It follows from Theorem 2 that the $L^2$ cohomology of $\widetilde{M}_2^{2n}$ is concentrated in the middle dimension. The Atiyah index theorem for covers [At] then gives $(-1)^n \chi(M_2^{2n}) \geq 0$. Since $\chi(M_1^{2n}) = \chi(M_2^{2n})$, the conclusion follows. □

*Added in proof:* This paper was widely circulated in preprint form in the Fall of 1997 (cf. [Be], p 138). The delay in its publication is due to circumstances beyond the control of the authors. After our preprint was submitted we learned of the related independent work [JZ].

## References

[An] Anderson, M., $L^2$-harmonic forms on complete Riemannian manifolds. Springer Lect. Notes in Math. **1339** (1989), 1–19

[At] Atiyah, M., Elliptic operators, discrete groups and Von Neumann algebra. Astérisque, **32–33** (1976), 43–47

[BGS] Ballmann, W., Gromov, M., Schroeder, V., Manifolds of non-positive curvature. Birkhäuser, Boston (1985)

[Be] Berger, M., Riemannian geometry during the second half of the twentieth century. University Lect. Series, 17. Amer. Math. Soc., Providence, RI, 2000. Reprint of the 1998 original (Jahresber. Deutsch. Math.-Verein. 100 (1998), no. 2, 45–208)

[Ch] Chern, S. S., On curvature and characteristic classes of a Riemannian manifold. Abh. Math. Sem. Univ. Hamburg, **20** (1955), 117–126

[Do] Dodziuk, J., $L^2$ harmonic forms on complete manifolds. In: Yau, S. T. (ed.) Seminar on Differential Geometry, Princeton University Press, Princeton. Ann. Math Studies, **102** (1982), 291–302

[DF] Donnelly, H. Fefferman, C., $L^2$-cohomology and the index theorem for the Bergman metric. Ann. Math. **118** (1983), 593–618

[DX] Donnelly, H., Xavier, F., On the differential form spectrum of negatively curved Riemannian manifolds. Amer J. Math., **106** (1984), 169–185

[ER] Elworthy, K.D., Rosenberg, S., The Witten Laplacian on negatively curved simply-connected manifolds. Tokyo J. Math. **16** (1995), 513–524

[FJ] Farrell, F.T., Jones, L.E., Topological rigidity for compact nonpositively curved manifolds. Proc. of Symposia in Pure Math., Amer. Math. Soc. **54**, Part III (1993), 229–274

[G] Gromov, M., Kähler hyperbolicity and $L^2$-Hodge theory. J. Differential Geom. **33** (1991), 263–292

[GW] Greene, R., Wu, H., Function theory on manifolds which posses a pole. Springer Lect. Notes in Math. **699** (1979)

[H] Hopf, H., Differential Geometrie und topogische Gestalt. Jahresber. Deutsch Math. Verein **41** (1932), 209–229

[JZ] Jost, J., Zuo, K., Vanishing theorems for $L^2$-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. Comm. Anal. Geom. **8** (2000), 1–30

[L] Lott, J., The zero-in-the-spectrum question. L'Enseignement Math. **42** (1996), 341–376

[ST] Singer, I., Thorpe, J., Lecture Notes on Elementary Topology and Geometry. Springer, New York, 1976

[St] Stern, M., $L^2$-cohomology of negatively curved Kähler manifolds. Preprint, Instit. Advanced Study, Princeton, 1989

[We] Wells, R. L., Jr., Differential analysis on complex manifolds. Springer, New York, 1979

[Yau] Yau, S.T., Problem section. In: Yau, S. T. (ed.) Seminar on Differential Geometry. Princeton University Press. Ann. Math Studies, Princeton, **102** (1982), 669–706