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CHEEGER ISOPERIMETRIC CONSTANTS OF GROMOV-HYPERBOLIC SPACES WITH QUASI-POLES

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Let X be a non-compact complete manifold (or a graph) which admits a quasi-pole and has bounded local geometry. Suppose that X is Gromov-hyperbolic and the diameters (for a fixed Gromov metric) of the connected components of $X(\infty)$ have a positive lower bound. Under these assumptions we show that X has positive Cheeger isoperimetric constant. Examples are also constructed to show that the Cheeger constant $h(X)$ may be zero if any of the above assumption on X is removed.

Applications of this isoperimetric estimate include the solvability of the Dirichlet problem at infinity for non-compact Gromov-hyperbolic manifolds X above. In addition, we show that the Martin boundary $\partial_\Delta X$ of such a space X is homeomorphic to the geometric boundary $X(\infty)$ of X at infinity.

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The main purpose of this paper is to estimate the Cheeger isoperimetric constant for non-compact Gromov-hyperbolic manifolds and graphs with quasi-poles. Gromov-hyperbolic spaces considered in this paper do *not* necessarily have compact quotients. In addition, it is known that some Gromov-hyperbolic spaces do *not* have positive Cheeger isoperimetric constants. For example, the real line $\mathbb{R}^1$ is Gromov-hyperbolic, but its Cheeger isoperimetric constant $h(\mathbb{R}^1)$ is zero. However, using some new techniques, we shall show that a large class of Gromov-spaces have positive Cheeger isoperimetric constants.

In fact, it will be shown that there are interesting relations between the geometry at infinity of a Gromov-hyperbolic space X and its Cheeger isoperimetric constant. It is clear that the geometric boundary at infinity of a real line consists of only two isolated points. Under the assumption that each connected components of $X(\infty)$ is not reduced to a single point and other mild assumptions, we show that $h(X) > 0$.

The isoperimetric estimate $h(X) > 0$ is closely related to the project of Ancona on the space of positive harmonic functions of Gromov-hyperbolic manifolds, (cf. [1–3]). Ancona actually considered the so-called coercive elliptic operators. A second order elliptic operator $\mathcal{L}$ is called coercive if the bottom, $\lambda_1(\mathcal{L})$, of the spectrum of $\mathcal{L}$ is positive. Ancona was able to obtain a series of results for Gromov-hyperbolic manifolds, *under additional assumptions that* $\lambda_1(\mathcal{L}) > 0$. When $\mathcal{L} = \Delta_X$ is the Laplace operator on X, a theorem of Cheeger implies that $\lambda_1(X) = \lambda_1(\Delta_X) \geq \frac{[h(X)]^2}{4}$. Our goal in the present work is to show that the Cheeger isoperimetric constant is indeed positive (and hence $\lambda_1(X) > 0$) for a large class of spaces that Ancona has considered. Therefore, the main results of this paper are needed to complete the geometric part of Ancona's project.

The recent work of Brin and Kifer [11] introduced the Brownian motion on piecewise smooth manifolds. The techniques of estimating isoperimetric constant in this paper can be applied to the piecewise spaces that Brin and Kifer recently considered as well.

More examples of non-compact Gromov-hyperbolic manifolds with arbitrarily prescribed topological structures at infinity will be constructed in Sec. 4 below. In following sections we will discuss isoperimetric inequalities and the Dirichlet problem at infinity for Gromov-hyperbolic spaces with quasi-poles.

1. Main Theorem and Applications

1.1. *The statement of main theorem*

Let us first recall one of equivalent definitions for manifolds and graphs that are hyperbolic in the sense of Gromov:

Definition 1.1. (cf. [25, 14] p. 328, [34]) Let X be a complete Riemannian manifold (or a graph). The space X is said to be Gromov-hyperbolic if there exists a constant $C = c(X)$ such that the following holds on X: For any pair of geodesic rays $\{\sigma_1, \sigma_2\}$ with the same initial point p and with $d(\sigma_1(t_0), \sigma_2(t_0)) > C$ for some t_0, any path from $\sigma_1(t_0 + r)$ to $\sigma_2(t_0 + r)$ lying outside $B_{(t_0+r)}(p)$ has length greater than $\frac{e^{r/C}}{C}$ for any $r > C$.

Suppose that M^n is a simply-connected Riemannian manifold of dimension n and that there exists a constant $\mu > 0$ such that every closed curve σ of length $L(\sigma)$ bounds a disk Σ of area $A(\Sigma) \leq \mu L(\sigma)$ in M. Then the manifold M^n is Gromov-hyperbolic (cf. [34]). The inequality $A(\Sigma) \leq \mu L(\sigma)$ is often referred to as the Gromov linear length-area isoperimetric inequality. It is a fact that a simply-connected Riemannian manifold M (with compact quotient) is Gromov-hyperbolic if and only if M satisfies a Gromov linear length-area isoperimetric inequality. We will discuss more examples of Gromov-hyperbolic spaces in Sec. 4 below.

If M^n is a complete non-compact Riemannian manifold, its Cheeger isoperimetric constant of M^n is defined to be $h(M^n) = \inf_\Omega\{\frac{\mathrm{vol}_{n-1}(\partial\Omega)}{\mathrm{vol}_n(\Omega)}\}$, where Ω ranges over all compact domains with rectifiable boundaries in M^n. Suppose that Γ is an infinite graph. The combinatorial Cheeger isoperimetric constant of Γ is defined to be

$$h(\Gamma) = \inf_A \frac{|\partial A|}{|A|},$$

where A ranges over all non-empty finite subsets of vertices in Γ, $\partial A = \{v \in \Gamma | d_\Gamma(v, A) = 1\}$ and $|A|$ denotes the cardinality of A.

For general manifolds and graphs, Gromov-hyperbolicity does *not* imply positivity of the Cheeger constant. For example, when $X = S^2 \times R^1$ or $X = [0, \infty)$, X is Gromov-hyperbolic but the Cheeger constant $h(X)$ of X is zero.

Let G be a Gromov-hyperbolic group. Suppose that G is not roughly quasi-isometric to $\mathbb{Z}$, the group of integers. Then $h(G) > 0$. However, Gromov-hyperbolic spaces considered in this paper do not necessarily admit co-compact group actions.

The existence of examples of Gromov-hyperbolic spaces with zero Cheeger constant led to a natural question:

Problem 1.1. Let X be a complete Gromov-hyperbolic manifold (or graph) with bounded local geometry. Under what additional conditions does X have positive Cheeger constant?

We say a Riemannian manifold M^n has *bounded local geometry* provided that about each $x \in M^n$ there is a geodesic ball $B_a(x)$ of radius a (independent of x) and a diffeomorphism $F\colon B_a(x) \to \mathbb{R}^n$ with

$$\frac{1}{c} d(z, y) \leq \|F(z) - F(y)\| \leq c d(z, y)$$

for all $z, y \in B_a(x)$, where c is independent of x. If M^n has positive injectivity radius and a lower bound on its Ricci curvature, then M^n has bounded local geometry, (cf. [5]). A graph Γ is said to have *bounded local geometry* if there exists a constant μ such that each vertex has at most μ neighbors in Γ.

Gromov-hyperbolicity is preserved by rough quasi-isometries, (cf. [34]). Recall that a (not necessarily continuous) map $f\colon X \to Y$ between metric spaces X and Y is called a "*rough quasi-isometry*" if (i) there exists a sufficient large $d_0 > 0$ such that the d_0-neighborhood of the image of f in Y coincides with Y itself and (ii) there are constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a} d(x_1, x_2) - b \leq d(f(x_1), f(x_2)) \leq a d(x_1, x_2) + b$$

for x_1, $x_2 \in X$.

Suppose that M^n is a complete manifold with bounded local geometry and that M^n is roughly quasi-isometric to Γ. Then by a theorem of Kanai [29], $h(M^n) > 0$ if and only if $h(\Gamma) > 0$.

Throughout this paper, all geodesics and geodesic segments are assumed to have unit speed and to be length minimizing. In order to study Problem 1.1, we introduce the following definition:

Definition 1.2. (compare [24] p. 5, Proposition 7.3 of [2, p. 20]) A complete manifold (or graph) X is said to have a quasi-pole in a compact subset $\Omega \subset X$ if there exists $C > 0$ such that each point of X lies in a C-neighborhood of some geodesic ray emanating from Ω.

In Proposition 7.3 of [2], Ancona assumes that for each point $x \in X$, there exists a length-minimizing geodesic ray σ starting at O such that $d(x, \sigma) \le C$ for some $C = c(X)$. Ancona's assumption implies that the space X has a quasi-pole at O by Definition 1.2. It should be pointed out that the main results of [2, 3] are independent of Proposition 7.3 of [2].

Given a compact subset $\Omega \subset X$, let $\mathcal{R}_\Omega$ denote the union of all length-minimizing geodesic rays from Ω. If a C-neighborhood of $\mathcal{R}_\Omega$ is equal to X for some $C > 0$, by definition X has a quasi-pole in Ω.

Recall that a Riemannian manifold M^n has a pole p if the exponential map $\exp_p \colon T_p(M^n) \to M^n$ is a diffeomorphism ([24]). Clearly, all manifolds which admit poles also admit quasi-poles. There are many other examples of manifolds which possess a quasi-pole. For instance, by a theorem of Cheeger and Gromoll (cf. [18]), we know that all complete non-compact manifolds with non-negative curvatures have compact souls. Hence, these manifolds also have quasi-poles.

If X is Gromov-hyperbolic, $\sigma(\infty)$ denotes the asymptotic class of σ and $X(\infty)$ denotes the space of all asymptotic classes of geodesic rays. If σ_1 and σ_2 are two geodesic rays with the same initial point x_0, the Gromov metric is defined to be

$$d_{x_0,\epsilon}(\sigma_1(\infty), \sigma_2(\infty)) = \liminf_{t \to \infty} e^{-\epsilon[t - \frac{1}{2} d(\sigma_1(t), \sigma_2(t))]} \,.$$

It is known that for sufficiently small ϵ, the function $d_{x_0,\epsilon}$ is a distance function on $X(\infty)$, cf. [23]. In this case, $d_{x_0,\epsilon}(\cdot\,,\cdot)$ is called a Gromov metric on $X(\infty)$. We now state our main theorem:

Main Theorem 1.1. *Let X be a non-compact complete manifold* (*or a graph*) *which admits a quasi-pole and has bounded local geometry. Suppose that X is Gromov-hyperbolic and the diameters of the connected components of $X(\infty)$ have a positive lower bound* (*with respect to a fixed Gromov metric*)*. Then X has positive Cheeger isoperimetric constant.*

Earlier results related to the main theorem can be found in [32, 36, 31] and [15]. The main theorem can also be used to estimate $\lambda_1(M^n)$, the bottom of the spectrum of the Laplacian on M^n. For a complete and non-compact Riemannian manifold M^n, the bottom of the spectrum of the Laplacian Δ is defined by

$$\lambda_1(M^n) := \inf_{f \neq 0} \frac{\int_{M^n} f \cdot \Delta f}{\int f^2} = \inf_{f \neq 0} \frac{\int_{M^n} \|\nabla f\|^2}{\int_{M^n} f^2} \,,$$

where f ranges over all smooth functions with compact support on M^n. If X is as in the main theorem, then $h(X) > 0$ which implies $\lambda_1(X) > 0$ by a theorem of Cheeger (cf. [17]). If M^n has bounded local geometry, then $h(X) > 0 \Leftrightarrow \lambda_1(X) > 0$, (cf. [13]).

The conclusion of the main theorem fails if one removes the assumption that the diameters of the connected components of $X(\infty)$ have a positive lower bound. For example, if X is a complete surface with constant curvature -1, finite area and three cusps, then $X(\infty)$ is of zero-dimensional since it consists of three points. Such a surface X has zero Cheeger constant. In addition, the surface X does *not* admit any bounded non-constant harmonic function. In particular, the Dirichlet problem at infinity for such a surface is not solvable.

We emphasize that the conclusion of the main theorem also fails for certain manifolds with no quasi-poles. For instance, let Γ_1 be a plane graph roughly quasi-isometric to the hyperbolic plane $\mathbb{H}^2$ of constant curvature -1. Choose an unbounded sequence $\{p_i\}_{i\geq 1}$ in Γ_1. For each i, we attach an interval $[0,i]$ (with integer vertices) to Γ_1 at p_i. The resulting graph $\Gamma = \Gamma_1 \cup_{\{p_i, i\geq 1\}} [0,i]$ has the property $\Gamma(\infty) = \mathbb{H}^2(\infty) = S^1$. The graph Γ does not have a quasi-pole, but it satisfies the other conditions of the main theorem. It is easy to check that the Cheeger constant of Γ is zero, because $h(\Gamma) \leq \lim_{i\to\infty} h([0,i]) = 0$.

1.2. *Two corollaries of the main theorem*

For any given continuous function ϕ defined on $M^n(\infty)$, one considers the Dirichlet problem at infinity:

$$\begin{cases} (\Delta u)(x) = 0\,, & \text{for } x \in M^n\,, \\ \lim\limits_{x\to\xi} u(x) = \phi(\xi)\,, & \text{for } \xi \in M^n(\infty)\,. \end{cases}$$

The definition of Laplacian operator acting on graphs will be reviewed in Sec. 2.

Corollary 1.1. *Let X be as in the main theorem. Then the Dirichlet problem at infinity is solvable on X. Consequently, X admits infinitely many linearly independent bounded non-constant harmonic functions.*

In order to study the space of normalized minimal positive harmonic functions on a manifold (or graph) X, one considers the so-called Martin boundary $\partial_\Delta X$ of X. The precise definition of Martin boundary can be found in [6]. Using the main theorem and the work of Ancona [1–3], we can extend a theorem of Anderson–Schoen [6] (and the work of Kifer [31]) to certain Gromov-hyperbolic manifolds:

Corollary 1.2. *Let X be as in the main theorem. Then the Martin boundary $\partial_\Delta X$ of X is homeomorphic to $X(\infty)$.*

Corollaries 1.1 and 1.2 follow immediately from the main theorem since for X Gromov-hyperbolic with $\lambda_1(X) > 0$ and bounded local geometry, they are special cases of known results, (cf. [2, 3]). In other words, because of the main theorem, Corollaries 1.1 and 1.2 become special cases of Ancona's theorems.

We also emphasize here that there are no assumptions made about sectional curvature in the main theorem and Corollaries 1.1 and 1.2, Proposition 4.1 and Corollary 1.2 also provide new examples of manifolds whose Martin boundaries have prescribed non-trivial topological types (other than spheres).

The study of positive harmonic functions on non-compact manifolds has been advanced by various authors including M. Anderson, Y. Kifer, W. Ballmann,

F. Ledrappier, D. Sullivan, A. Ancona, R. Schoen and S. T. Yau. In 1983, M. Anderson [4] and D. Sullivan [35] independently solved the Dirichlet problem at infinity for simply-connected manifolds with strictly negative sectional curvature $-b^2 \leq K \leq -b^2 < 0$. Earlier work in this direction was done by Kifer for manifolds with co-compact quotients. The Dirichlet problem at infinity was also solved for co-compact manifolds of rank one in the sense of Ballmann [7]. In 1985, Anderson and Schoen [6] showed that for any simply-connected manifold M^n with strictly negative sectional curvature $-b^2 \leq K \leq -b^2 < 0$ is homeomorphic to the sphere at infinity. They also provided a simple proof of the earlier result of Anderson and Sullivan. Anderson and Schoen's result was extended to simply-connected uniform visibility manifolds of non-positive curvature by Ancona (cf. [1]). Kifer [31] was able to further extend the result to simply-connected uniform visibility manifolds with no focal points. Kifer's results [31] were special cases of our main theorem and Corollaries 1.1 and 1.2. Every point in a manifold with no focal points is necessarily a pole. Most of the results above have very strong restriction on the underlying manifolds M^n, whose universal cover must be diffeomorphic to $\mathbb{R}^n$.

It should also be pointed out that Ancona's results [1–3] do *not* imply our isoperimetric estimate in the main theorem of this paper. Neither does Ancona's work [1–3] alone imply Corollaries 1.1 and 1.2 mentioned above. In fact, for the case of Laplace operator, in papers [1–3] all manifolds and graphs X under the consideration are *required* to have positive Cheeger isoperimetric constants (or $\lambda_1(X) > 0$). Our new contribution in this paper is to show that certain Gromov-hyperbolic manifolds (or graphs) have $\lambda_1(X) > 0$ so that Ancona's work can be applied. In addition, the method presented in this paper is more geometric, and hence completely different from Kifer's probabilistic approach.

Our proof of the main theorem introduces several new ideas. One of them is the use of combinatorial isoperimetric inequalities for graphs embedded in manifolds to estimate the classical Cheeger isoperimetric constant for smooth Riemannian manifolds.

The rest of the paper is organized as follows. Section 2 provides an isoperimetric estimate for Gromov-hyperbolic manifolds with poles, where we use combinatorial isoperimetric inequalities on graphs embedded in Riemannian manifolds. We consider more general manifolds (or graphs) with quasi-poles in Sec. 3, where the main theorem will be proved. We will also discuss more examples of Gromov hyperbolic spaces in Sec. 4.

2. Isoperimetric Inequalities for Spaces which Possess a Pole

We first recall the notion of length and geodesics in metric spaces. Let (X, d) be a complete metric space. If $\eta\colon [a, b] \to X$ is a continuous curve, we let

$$L(\eta) = \limsup_{\max_i\{d(\eta(t_i),\eta(t_{i+1}))\}\to 0} \left\{ \sum_i d(\eta(t_i), \eta(t_{i+1})) | a = t_0 \leq t_1 \leq \cdots \leq t_{k+1} = b \right\}.$$

A curve $\eta\colon [a,b] \to X$ is called a geodesic segment if $d(\eta(s), \eta(t)) = |s-t|$ for all s, $t \in [a,b]$. If any pair of points p, $q \in X$ can be joined by a geodesic segment η with $d(p,q) = L(\eta)$, then X is called a *geodesic metric space.*

We also let Γ denote a connected graph with vertex set V and edge set E. For vertices x, $y \in V$, we write $x \sim y$ if x and y are adjacent and joined by an edge, which will be denoted by xy. Each element in the set

$$N(x) = \{y \in V | x \sim y\}$$

is called a neighbor of x. The number of neighbors of x is $|N(x)|$, called the valence of x. Here we consider *locally finite* graphs i.e. every vertex has a finite number of neighbors.

In this section, we prove our main theorem in the special case of a manifold with a pole. The general case will be discussed in the next section. Let S^{n-1} denote the unit $(n-1)$-sphere. Suppose that M^n has a pole at p. We consider the map $F_p\colon \bar{B_1}(0) \to M \cup M(\infty)$ such that $F_p(S^{n-1}) = M(\infty)$ by setting

$$F_p(v) = \begin{cases} \exp_p\left(\dfrac{v}{1-\|v\|}\right), & \text{if } \|v\| < 1 \\ \sigma_v(+\infty)\,, & \text{if } \|v\| = 1\,, \end{cases} \tag{2.1}$$

for σ_v a geodesic ray with $\sigma_v'(0) = v$.

Theorem 2.1. *Let X be a complete manifold (or a graph) as in the main theorem. Suppose that X is roughly quasi-isometric to a manifold M^n with a pole p, and that the above map F_p is a homeomorphism between $M^n(\infty)$ and S^{n-1} with $n \geq 2$. Then the Cheeger isoperimetric constant of X is positive.*

In order to prove Theorem 2.1 we would like to recall a result of Kanai. A graph Γ is said to be μ-uniform if each vertex p of Γ has at most μ neighbors, i.e.

$$\sup\{|N(p)|\,|p \in V\} \leq \mu$$

where $|A|$ denotes the cardinality of a subset $A \subset V$ and V is the vertex set of Γ. There is a natural metric structure on each graph. A sequence $\sigma = (p_0, \ldots, p_m)$ is called a path of length $m = L_\Gamma(\sigma)$ if each p_i is a neighbor of p_{i-1}. For any p, $q \in V$, we let

$$d_\Gamma(p,q) = \inf\{L_\Gamma(\sigma) | \sigma \text{ is a path joining } p \text{ and } q\}\,.$$

Given a subset $A \subset \Gamma$, we define its boundary ∂A by setting

$$\partial A = \{p \in \Gamma | d_\Gamma(p, A) \equiv 1\}$$

and $(\partial A)_E = \{e = xy | x \in A, y \in \partial A\}$.

The combinatorial Cheeger isoperimetric constant of Γ defined in Sec. 1.1. The combinatorial isoperimetric constant $h(\Gamma)$ of a δ-graph is related to the original Cheeger isoperimetric constant $h(M)$ of complete manifold M as follows.

Let us recall the definition of a δ-graph in a complete manifold. Suppose that M is a complete Riemannian manifold and d_M is the induced metric on M. A subset Γ of M is said to be δ-separated for $\delta > 0$, if $d_M(p,q) \geq \delta$ for all $p \neq q \in \Gamma$. A δ-separated subset is called maximal if it is maximal with respect to the order relation of inclusion. Let Γ be a maximal δ-separated subset of M. The neighboring relation N on Γ is given by $N = \{N(p) | p \in \Gamma\}$ and $N(p) = \{q \in \Gamma | 0 < d_M(p,q) \leq 2\delta\}$ for each $p \in \Gamma$. A maximal δ-separated subset of a complete Riemannian manifold M with the graph structure described above is called a δ-graph in M.

The following result about combinatorial isoperimetric constants is due to Kanai.

Proposition 2.1. ([29]) *Let M^n be a complete n-dimensional Riemannian manifold with bounded local geometry, and let Γ be a δ-graph in M^n. Then the combinatorial isoperimetric constant $h(\Gamma)$ of Γ is positive if and only if the Cheeger isoperimetric constant $h(M^n)$ of M^n is positive.*

In addition, for any other graph Γ_1 roughly quasi-isometric to the manifold M^n above, $h(\Gamma_1) > 0$ if and only if $h(M^n) > 0$.

In what follows, we will find sufficient conditions for manifolds (or graphs) to have positive Cheeger constants. Let us first make the following observation for manifolds and discuss its counterpart for graphs afterwards.

Proposition 2.2. *Let M^n be a complete Riemannian manifold. Suppose that there is a vector field ξ on M^n satisfying the following*:

(i) *the vector field ξ is uniformly bounded*: $\|\xi(y)\| \leq c_1$,
(ii) $\mathrm{div}(\xi)(y) > c_2 > 0$;

where c_1 and c_2 are independent of $y \in M^n$. Then $h(M^n) \geq \frac{c_2}{c_1} > 0$.

Proposition 2.2 is a direct consequence of Stokes Theorem, because for any compact domain $\Omega \subset M^n$ one has

$$c_2 \operatorname{vol}_n(\Omega) \leq \int_\Omega \mathrm{div}(\xi) d\operatorname{vol} = \int_{\partial\Omega} \langle \xi, \mathbf{n} \rangle dA \leq c_1 \operatorname{Area}(\partial\Omega)$$

where $\mathbf{n}$ is the outward unit normal vector of $\partial\Omega$.

We would now like to formulate a result analogous to Proposition 2.2 for graphs. Given a real-valued function f on the vertex set of Γ, let ∇f and Δf be its *gradient* and *Laplacian* respectively. The gradient of f assigns the following value to each ordered pair of vertices x, $y \in \Gamma$

$$\nabla_{xy} f = f(y) - f(x) \,.$$

The gradient ∇f can also be considered as a function on oriented edges: if $e = \mathbf{xy}$ then $\nabla_e f = \nabla_{xy} f$.

The *Laplace* operator Δ is defined on the space of functions f of the vertex set of Γ as follows (compare [19]):

$$\Delta f(x) = f(x) - \mathrm{Av}\, f(x) = \frac{1}{|N(x)|} \sum_{y \in N(x)} [f(x) - f(y)] = \frac{1}{|N(x)|} \sum_{y \in N(x)} \nabla_{yx} f\,,$$

where $\mathrm{Av}\, f(x) = \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y)$.

Proposition 2.3. *Let Γ be a μ-uniform graph. Suppose that there is a function f defined on the vertex set of Γ and satisfying the following*:

(i) *the gradient of f is uniformly bounded on the edge set of Γ; $|\nabla_e f| \le c_1$;*
(ii) $\Delta f(x) > c_2 > 0$,

for each vertex x and edge e, where c_1 and c_2 are independent of x and e. Then $h(\Gamma) \ge \frac{c_2}{\mu c_1} > 0$.

Using Proposition 2.3 one can show that any binary tree (or homogeneous k-tree with $k \ge 2$) Γ has $h(\Gamma) \ge \frac{k-1}{k} > 0$. The Proof of Proposition 2.1 uses the following discrete version of Green's formula (cf. [19], Sec. 6.3). If u and v are two functions on the vertex set V and one of them has finite support then

$$\sum_{x \in V} \Delta u(x) v(x) |N(x)| = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} u)(\nabla_{xy} v)\,. \tag{2.2}$$

The factor $\frac{1}{2}$ appears in the second term because each edge is counted twice.

Proof of Proposition 2.3. For any given finite subset A of the vertex set V of Γ, we let χ_A be the characteristic function of A:

$$\chi_A(x) = \begin{cases} 1\,, & \text{if } x \in A \\ 0\,, & \text{if } x \notin A\,. \end{cases}$$

Since χ_A has compact support, using Green's formula (2.2) we get

$$\begin{aligned} c_2 |A| &\le \sum_{x \in A} \Delta f(x) \chi_A(x) |N(x)| = \frac{1}{2} \sum_{x,y \in A} (\nabla_{xy} f)(\nabla_{xy} \chi_A) \\ &\le c_1 \sum_{x \in \partial A} |N(x)| \le c_1 \mu |\partial A|\,. \end{aligned}$$

In particular,

$$h(\Gamma) = \inf_A \frac{|\partial A|}{|A|} \ge \frac{c_2}{\mu c_1}\,. \qquad \square$$

Using Proposition 2.3, we will prove the main theorem for the special case of manifolds with a pole. If o is a pole of M^n, by Gauss' Lemma one can show that the injectivity radius of M^n at o is infinite. Therefore, any geodesic ray $\sigma_v : t \to \exp_o(t\mathbf{v})$

emanating from o is a length-minimizing curve for $t \in [0, \infty)$. We now re-state the main theorem for the special case of manifolds with a pole.

Theorem 2.2. *Let M^n be a complete manifold with a pole and bounded local geometry. Suppose that M^n is Gromov-hyperbolic and that the map F_p of* (2.1) *gives rise to a homeomorphism between $M^n(\infty)$ and S^{n-1} with $n \geq 2$. Then M^n has positive Cheeger isoperimetric constant $h(M^n) > 0$.*

For the proof of Theorem 2.2, we need to use the exponential divergence property of geodesic rays in Gromov-hyperbolic manifolds:

Lemma 2.1. ([34], p. 36) *Let X be a complete, geodesic metric space. Suppose that X is Gromov-hyperbolic. Then there exists $\delta > 0$ such that the following holds: if σ_1 and σ_2 are two geodesic rays emanating from a point x with $d(\sigma_1(r), \sigma_2(r)) \geq \delta \geq 1$, and $\phi\colon [0, \ell] \to [X - \bar{B}_{r+t}(x)]$ is a path joining $\sigma_1(r+t)$ and $\sigma_2(r+t)$ lying outside of $B_{r+t}(x)$, then*

$$L(\phi) \geq 2^{\frac{t}{\delta}-2}$$

for any r, $t > 0$, where $L(\phi)$ denotes the length of ϕ.

The result above leads us to consider the distance function outside a geodesic ball. For any pair of points y, $z \in [X \setminus B_r(x)]$, we define

$$d^x_r(y, z) = \inf\{L(\sigma) | \sigma\colon [a, b] \to X \text{ is a piecewise smooth curve},$$

$$\sigma \subset [X \setminus B_r(x)], \sigma(a) = y, \sigma(b) = z\}.$$

The following is a refinement of Lemma 2.1.

Lemma 2.2. (Compare [14], p. 328) *Let X be a complete, geodesic metric space. Suppose that X is Gromov-hyperbolic. Then there exists $\delta > 0$ such that the following holds: If σ_1 and σ_2 are two geodesic rays emanating from x with unit speed and if $d^x_r(\sigma_1(r), \sigma_2(r)) \geq 2\delta$, then the inequality*

$$d^x_{r+t}(\sigma_1(r+t), \sigma_2(r+t)) \geq \frac{2}{3}\left(\frac{3}{2}\right)^{\frac{t}{\delta}} d^x_r(\sigma_1(r), \sigma_2(r)),$$

holds for any $t \geq \delta$, where δ is independent of x, σ_1 and σ_2.

Proof. By a result of [14], there exists $\delta > 0$ such that if $d^x_r(\sigma_1(r), \sigma_2(r)) \geq \delta$ then

$$d^x_{r+t}(\sigma_1(r+t), \sigma_2(r+t)) \geq 2d^x_r(\sigma_1(r), \sigma_2(r)) - \delta$$

for any $t \geq \delta$. Hence when $d^x_r(\sigma_1(r), \sigma_2(r)) \geq 2\delta$, we observe that

$$d^x_{r+t}(\sigma_1(r+t), \sigma_2(r+t)) \geq \frac{3}{2} d^x_r(\sigma_1(r), \sigma_2(r)) \geq 3\delta \geq 2\delta.$$

Similarly, if $t \geq k\delta$ then one has

$$d^x_{r+t}(\sigma_1(r+t), \sigma_2(r+t)) \geq \left(\frac{3}{2}\right)^k d^x_r(\sigma_1(r), \sigma_2(r))$$

for any positive integer $k \geq 1$. Let k be the largest integer less than or equal to $\frac{t}{\delta}$. It follows that

$$d^x_{r+t}(\sigma_1(r+t), \sigma_2(r+t)) \geq \left(\frac{3}{2}\right)^{\frac{t}{\delta}-1} d^x_r(\sigma_1(r), \sigma_2(r))$$

holds for any $t \geq \delta$. □

In what follows, we fix a base point x_0, assume that x_0 is a pole of M^n and let

$$d_r(y,z) = d^{x_0}_r(y,z)\,.$$

Using this extrinsic distance function d_r restricted to $S_r(x_0)$, we further let

$$\Sigma_\rho(y) = \{z \in S_r(x_0) | d_r(z,y) < \rho\}$$

for each $y \in S_r(x_0)$. There exist a family of projection maps:

$$\pi_t \colon M^n - \{x_0\} \to S_t(x_0)$$
$$x \to \exp_{x_0}\left(t\frac{\exp^{-1}_{x_0}(x)}{\| \exp^{-1}_{x_0}(x)\|}\right)\,.$$

Let us now return to the Proof of Theorem 2.2.

Proof of Theorem 2.2. By Proposition 2.1, it is sufficient to show that M^n is roughly quasi-isometric to a graph Γ with $h(\Gamma) > 0$.

Our construction of the graph Γ uses Lemma 2.2. Let x_0 be a pole of M^n. It follows that $[M^n \setminus B_r(x_0)]$ is diffeomorphic to $[\mathbb{R}^n \setminus B_r]$. By our assumption, $\dim(M^n) = n \geq 2$. Therefore, the subspace $[M^n \setminus B_r(x_0)]$ is path-connected for any $r > 0$. Thus, the distance function $d_r = d^{x_0}_r$ is non-trivial for any $r > 0$.

For each unit tangent vector $v \in T_{x_0}(M^n)$, we let $\sigma_v(t) = \exp_{x_0}(t\mathbf{v})$. Let δ be the constant given by Lemma 2.2. Since F_{x_0} is a homeomorphism between $M^n(\infty)$ and $S^{n-1} \subset T_{x_0}(M^n)$, no two distinct rays σ_{v_1} and σ_{v_2} from the same initial point x_0 are asymptotic. In particular, the diameter function $f_r = \sup\{d(x,y) | x, y \in S_r(x_0)\}$ is an unbounded function.

Let us now choose $r_1 > 0$ such that $\mathrm{Diam}(S_{r_1}(x_0)) \geq 10\delta$, where δ is given by Lemma 2.2. When $r \geq r_1 + 2\delta$, it follows from Lemma 2.2 that $\mathrm{Diam}(S_r(x_0)) \geq 10\delta$ as well. Recall that $\Sigma_\rho(y) = \{z \in S_r(x_0) | d_r(z,y) < \rho\}$ stands for a metric ball in $S_r(x_0)$ centered at $y \in S_r(x_0)$. We now consider a new n-dimensional domain Ω associated with $\Sigma = \Sigma_\rho(y)$ by setting

$$\Omega = \Omega_{4\rho}(\Sigma) = \bigcup_{r \leq t \leq r+4\rho} \pi_t(\Sigma)\,,$$

where $\pi_t(z) = \exp_{x_0}[\frac{t \exp^{-1}_{x_0} z}{\| \exp^{-1}_{x_0} z\|}]$ for any $z \in M^n - \{x_0\}$. The following assertions play important roles in the construction of the graph Γ:

Assertion 2.1. *Let $\sigma_v(t) = \exp_{x_0}(t\mathbf{v})$, $y = \sigma_v(r)\, p = \sigma_v(r + 2\rho)$ and $\Omega_{4\rho}(\Sigma_\rho(y))$ be as above. Suppose that $r \geq r_1 + 2\delta$ and $\rho \geq 2\delta$. Then $B_{\frac{\rho}{4}}(p) \subset \Omega_{4\rho}(\Sigma_\rho(y))$ for any $y \in S_r(x_0)$.*

Assertion 2.2. *Let r_1, $r \geq r_1 + 2\delta$ and $\rho \geq 2\delta$ be as in Assertion 2.1. Suppose that M^n has bounded local geometry and that $\{q_1, q_2, \ldots, q_m\}$ is a maximal 2ρ-separated set in the sphere $S_r(x_0)$ with respect to the metric d_r. Then there exists a constant $C = C(\rho)$ such that $k_i = \#\{q_j | d_r(q_j, q_i) \leq 4\rho\} \leq C$ for $i = 1, 2, \ldots, m$.*

We postpone the proofs of Assertions 2.1 and 2.2 to the end of this section. Let us return to the construction of the graph Γ with $h(\Gamma) > 0$. Choose $r_0 = \max\{r_1, 4\delta, \hat{t}_n(\delta)\}$ where

$$\hat{t}_n(\delta) = \log_{3/2}\{[8C(16\delta) + 1]2\delta\}\,.$$

Let us now divide M^n into a family of concentric spherical shells $\{\Omega_i\}_{i=0}^{\infty}$ where

$$\Omega_i = \{x \in M^n | ir_0 \leq d(x, x_0) \leq (i+1)r_0\}\,.$$

Clearly $M^n = \cup_{i=0}^{\infty}\Omega_i$. For $i \geq 1$, each spherical shell Ω_i has two boundary components: the inner sphere $S_{ir_0}(x_0)$ and the outer sphere $S_{(i+1)r_0}(x_0)$.

We shall further divide the spherical domain Ω_i into a disjoint union of subdomains by using the radial projection map. For this purpose, we choose a maximal discrete subset $\{p_{i,\alpha}\}_{\alpha=1}^{m_i}$ in S_{ir_0} with respect to the metric d_{ir_0}. More precisely, let $\mathcal{M}_i = \{A | A \text{ is a} 4\delta - \text{separated subset of } S_{ir_0}\}$. The subset $\{p_{i,\alpha}\}_{\alpha=1}^{m_i}$ is maximal in $\mathcal{M}_i$ with respect to the inclusion relation.

Setting

$$\Sigma_{i,\alpha} = \{y \in S_{ir_0}(x_0) | d_{ir_0}(y, p_{i,\alpha}) \leq \min_{\beta \neq \alpha}\{d_{ir_0}(y, p_{i,\beta})\}\}$$

we consider the domain

$$V_{i,\alpha} = \bigcup_{ir_0 \leq t \leq (i+1)r_0} \pi_t(\Sigma_{i,\alpha}) = \Omega_{r_0}(\Sigma_{i,\alpha})\,,$$

with base $\Sigma_{i,\alpha}$ and height r_0, where $\pi_t(q) = \exp_{x_0}[\frac{t \exp_{x_0}^{-1} q}{\| \exp_{x_0}^{-1} q\|}]$ for any $q \neq x_0$. When $i = 0$, we let $m_0 = 1$ and $V_{01} = B_{r_0}(x_0)$. Thus we obtain a partition $\{V_{i,\alpha}\}$ and

$$M^n = \bigcup_{i=0}^{+\infty} \Omega_i = \bigcup_{i=0}^{+\infty} \bigcup_{\alpha=1}^{m_i} V_{i,\alpha}\,.$$

By Assertion 2.1, for each (i,α), we can choose a point $v_{i,\alpha} \in V_{i,\alpha}$ such that $V_{i,\alpha}$ contains an open ball $B_\delta(v_{i,\alpha})$. Let the vertex set of Γ be $\mathcal{V} = \{v_{i,\alpha} | 1 \leq \alpha \leq m_i, i = 0, 1, 2, \ldots\}$. To define the set of edges $\mathcal{E}$ of Γ, we say that $v_{i,\alpha}$ and $v_{j,\beta}$ *neighbor each other* if $V_{i,\alpha}$ and $V_{j,\beta}$ are adjacent. Equivalently, $d_\Gamma(v_{i,\alpha}, v_{j,\beta}) \leq 1$ if and only if $\bar{V}_{i,\alpha} \cap \bar{V}_{j,\beta} \neq \emptyset$.

Note that $V_{i,\alpha}$ contains an open ball $B_\delta(v_{i,\alpha})$ for some $v_{i,\alpha} \in V_{i,\alpha}$. Furthermore, by our construction we get $\text{Diam}(V_{i,\alpha}) \leq d_0 = 2r_0 + 8\delta$ for all i, α. Recall that

$M^n = \cup \bar{V}_{i,\alpha}$ and M^n has bounded local geometry. Therefore, M^n is roughly quasi-isometric to Γ. Since M^n has bounded local geometry, Γ is a uniform μ-graph for some $\mu > 0$.

It remains to check $h(\Gamma) > 0$. For this purpose, we need to define a function f on the vertex set $\mathcal{V}$ as follows:

$$f(v_{i,\alpha}) = i$$

for all i, α. It is clear that $|\nabla_{xy} f| \le 1$ if x and y are adjacent. We now need to compute $(\Delta f)(v_{i,\alpha})$. For $i \ge 1$, by our construction and Assertion 2.2, we observe that

$$\begin{aligned}\mathcal{N}^-(v_{i,\alpha}) &= \#\{v_{i-1,\beta} | \bar{V}_{i,\alpha} \cap \bar{V}_{i-1,\beta} \neq \emptyset\} \\ &\le \#\{p_{i,\beta} | d_{ir_0}(p_{i,\beta}, p_{i,\alpha}) \le 16\delta\} \\ &\le C(16\delta) = C^* .\end{aligned}$$

On the other hand, by Assertion 2.1, Lemma 2.2 and our construction we obtain

$$\begin{aligned}\mathcal{N}^+(v_{i,\alpha}) &= \#\{v_{i+1,\beta} | \bar{V}_{i,\alpha} \cap \bar{V}_{i+1,\beta} \neq \emptyset\} \\ &\ge \frac{\mathrm{Diam}_{(i+1)r_0}[\pi_{(i+1)r_0}(\Sigma_{i,\alpha})]}{4\delta} - 1 \ge 2C(16\delta) = 2C^* ,\end{aligned}$$

where $\mathrm{Diam}_{(i+1)r_0}(A)$ denotes the diameter of a subset $A \subset [M^n \setminus B_{(i+1)r_0}]$ with respect to the distance function $d_{(i+1)r_0}$. Therefore, we have

$$\Delta f(v_{i,\alpha}) \ge \frac{\mathcal{N}^+(v_{i,\alpha}) - \mathcal{N}^-(v_{i,\alpha})}{\mu} \ge \frac{C^*}{\mu}$$

for all $i \ge 1$. Thus, $\Delta f(v_{i,\alpha}) \ge \min\{\frac{C^*}{\mu}, m_1\}$ for all i, α. It follows from Proposition 2.3 that $h(\Gamma) > 0$. This completes the proofs of Theorem 2.2, except for the verification of Assertions 2.1 and 2.2. □

Proof of Assertions 2.1 and 2.2. To prove Assertion 2.1 we argue by contradiction. Suppose that there is a point $z \in B_{\frac{\rho}{4}}(p) \cap [M^n \setminus \Omega_{4\rho}(\Sigma)]$. Then we would have

$$d(z, p) < \frac{\rho}{4} .$$

Let $s_1 = \frac{7\rho}{4}$. The contradiction will be obtained by showing that

$$d_{r+s_1}(\pi_{r+s_1}(z), \pi_{r+s_1}(p)) \le \rho$$

via triangle inequalities, and that $d_{r+s_1}(\pi_{r+s_1}(z), \pi_{r+s_1}(p)) \ge \frac{9}{4}\rho$ via Lemma 2.2. The details follow.

Recall that $p = \sigma(r+2\rho)$ with $d(p, x_0) = r+2\rho$. Let $\phi\colon [0,1] \to M^n$ be a length-minimizing geodesic segment from p to z. Because $s_1 = \frac{7\rho}{4}$, $d(p, x_0) = r+2\rho$ and $d(z,p) < \frac{\rho}{4}$, the geodesic segment ϕ must lie outside of $B(x_0, r+s_1)$. By the triangle inequalities for the distance functions, one can verify that $r+s_1 \le d(z, x_0) \le r+\frac{9\rho}{4}$.

It follows that $d(z, \pi_{r+s_1}(z)) \le \frac{\rho}{2}$. Hence, one can find a broken geodesic joining Ψ from $\pi_{r+s_1}(p)$ to $\pi_{r+s_1}(z)$ of length less than or equal to ρ such that Ψ lies outside of $B(x_0, r+s_1)$. Thus, we get

$$d_{r+s_1}(\pi_{r+s_1}(z), \pi_{r+s_1}(p)) \le \rho\,. \tag{2.3}$$

Notice that $\frac{7}{4}\rho \ge 3\delta$. Using the assumption that $z \notin \Omega_{4\rho}(\Sigma)$, we obtain $d_r(\pi_r(z), \pi_r(p)) \ge \rho \ge 2\delta$. Applying Lemma 2.2 we derive

$$d_{r+s_1}(\pi_{r+s_1}(z), \pi_{r+s_1}(p)) \ge \frac{2}{3}\left(\frac{3}{2}\right)^3 \rho = \frac{9}{4}\rho\,,$$

which contradicts (2.3). Assertion 2.1 has now been verified.

With regards to Assertion 2.2, we first observe that if M^n has bounded local geometry then for any $\rho_2 > \rho_1 > 0$ there exists a constant $N = N(\rho_1, \rho_2) > 0$ such that any metric ball $B_{\rho_2}(y)$ contains at most N many disjoint balls of radius ρ_1.

Recall that $\{q_1, q_2, \ldots, q_m\}$ is a maximal 2ρ-separated set in the sphere $S_r(x_0)$ with respect to the metric d_r. Let

$$\Sigma_i = \{y \in S_r(x_0) | d_r(y, q_i) \le \min_{j \ne i}\{d_r(y, q_j)\}\}\,.$$

Clearly, $\mathrm{Diam}(\Sigma_i) \le 4\rho$. If $\Sigma_i \cap \Sigma_j \ne \emptyset$, then $d_r(q_i, y) \le 8\rho$ for any $y \in \Sigma_j$. Let us consider the set $\Omega_i = \cup_{r \le t \le r+4\rho} \pi_t(\Sigma_i)$. It follows that $\Omega_i \subset B_{16\rho}(q_i)$. On the other hand, it follows from Assertion 2.1 that Ω_i contains

$$k_i = \#\{j | d_r(q_j, q_i) \le 4\rho\}$$

disjoint balls of radius $\frac{\rho}{4}$. Since M^n has bounded local geometry, we have $k_i \le C(\rho) = N(16\rho, \frac{\rho}{4})$. Thus Assertion 2.2 has been verified. □

3. Isoperimetric Estimates for Spaces which Possess a Quasi-Pole

In this section, we shall derive isoperimetric estimates for a space with a quasi-pole and complete the proof of the main theorem. Our goal is to demonstrate the following:

Theorem 3.1. *If X is a space satisfying all conditions in the main theorem, then X is roughly quasi-isometric to a graph Γ with $h(\Gamma) > 0$.*

The main theorem then follows directly from Theorem 3.1 and Proposition 2.1. The Proof of Theorem 3.1 will use the techniques developed in the last section. We first consider a special class of Gromov-hyperbolic spaces for which the Proof of Theorem 2.2 can be applied.

Suppose that Σ is a compact metric space with diameter $D = D(\Sigma)$. We let $X = \Sigma \times [0, \infty)$ and define a new metric d_X on X as follows:

$$d_X((p,t),(q,s)) = 2\log\left(\frac{d_\Sigma(p,q) + \max\{e^{-t}, e^{-s}\}D}{e^{\frac{-(s+t)}{2}}D}\right).$$

Such a metric space (X, d_X) is called a hyperbolic quasi-cone of Σ, which we denote by $\mathrm{Con}^h(\Sigma)$.

For each point $p \in \Sigma$ we consider the ray $\sigma_p : t \to (p,t)$ for $t \in [0,\infty)$. A computation shows that $d(\sigma_p(t), \sigma_p(s)) = d((p,t),(p,s)) = |t-s|$. Hence, each σ_p is a length-minimizing geodesic ray. Furthermore, if $p_1 \neq p_2 \in \Sigma$, then

$$d(\sigma_{p_1}(t), \sigma_{p_2}(t)) = 2t + 2\log\left[\frac{d_\Sigma(p_1,p_2)}{D} + e^{-t}\right]$$

and hence $\lim_{t\to\infty} d(\sigma_{p_1}(t), \sigma_{p_2}(t)) = \infty$. Therefore, σ_{p_1} and σ_{p_2} are not asymptotic as long as $p_1 \neq p_2$.

Proposition 3.1. ([10]) *Let* Σ, $X = \Sigma \times [0,\infty)$ *and* $\mathrm{Con}^h(\Sigma) = (X, d_X)$ *be as above. Then*

(1) $\mathrm{Con}^h(\Sigma)$ *is a Gromov-hyperbolic space with* $[\mathrm{Con}^h(\Sigma)](\infty) = \Sigma$.
(2) *A Gromov-hyperbolic metric space* X *has a quasi-pole if and only if* X *is roughly quasi-isometric to a Gromov-hyperbolic metric on the hyperbolic cone* $\mathrm{Con}^h(X(\infty))$ *over* $X(\infty)$.

We now formulate the bounded local geometry condition for general metric spaces.

Definition 3.1. A metric space X has bounded growth (at some scale) if there are constants $r_2 > r_1 > 0$, and an integer N such that every open ball of radius r_2 in X can be covered by N open balls of radius r_1.

Any manifold (or graph) X with bounded local geometry has bounded growth (at some scale).

Theorem 3.2. *Let* Σ *be a compact metric space,* $X = \Sigma \times [0,\infty)$ *and* $\mathrm{Con}^h(\Sigma) = (X, d_X)$ *be as above. Suppose that* X *has bounded growth* (*at some scale*) *and that the diameters of the connected components of* Σ *have a positive lower bound. Then* X *is roughly quasi-isometric to a graph* Γ *with* $h(\Gamma) > 0$.

The proof of Theorem 3.1 is similar to that of Theorem 2.2. We will replace x_0 by the bounded set $\Sigma \times \{0\}$ in the proof. For this purpose, we introduce

$$B_r = \{x \in X | d(x, \Sigma \times \{0\}) \le r\} = \Sigma \times [0,r]$$

and

$$S_r = \{x \in X | d(x, \Sigma \times \{0\}) = r\} = \Sigma \times \{r\}\,.$$

Similarly, for any pair of points y, $z \in [X \setminus B_r]$ we define

$$d_r(y,z) = \inf\{L(\sigma) | \sigma\colon [a,b] \to [X \setminus B_r] \text{ is a continuous curve}, \\ \sigma(a) = y, \sigma(b) = z\}\,.$$

The following is a variant of Lemma 2.2:

Lemma 3.1. *Let* Σ, $X = \Sigma \times [0, \infty)$ *and* $\mathrm{Con}(\Sigma) = (X, d_x)$ *be as above. Then there exists* $\delta^* > 0$ *such that the following holds*: *If* $\sigma_1(t) = (p_1, t)$ *and* $\sigma_2(t) = (p_2, t)$ *are two geodesic rays emanating from* Σ *and if* $d_r^x(\sigma_1(r), \sigma_2(r)) \geq 2\delta^*$, *then the inequality*

$$d_{r+t}^x(\sigma_1(r+t), \sigma_2(r+t)) \geq \frac{2}{3}\left(\frac{3}{2}\right)^{\frac{t}{\delta^*}} d_r^x(\sigma_1(r), \sigma_2(r)),$$

holds for any $t \geq \delta^*$, *where* δ *is independent of* σ_1 *and* σ_2.

Using Lemma 3.1, we can carry out the Proof of Theorem 3.1.

Proof of Theorem 3.1. Using the extrinsic distance function d_r restricted to S_r, we let

$$\Sigma_\rho(y) = \{z \in S_r | d_r(z, y) < \rho\}$$

for each $y \in S_r$. We also consider the projection maps $\{\pi_t\}_{t \geq 0}$ where

$$\pi_t : X \to S_t$$

$$(p, s) \to (p, t).$$

Our construction of the graph Γ uses Lemma 3.1. Let us now choose $r_1 > 0$ such that $\mathrm{Diam}(S_{r_1}) \geq 10\delta^*$, where δ^* is given by Lemma 3.1. When $r \geq r_1 + 2\delta^*$, it follows from Lemma 3.1 that $\mathrm{Diam}(S_r) \geq 10\delta^*$ as well.

Recall that $\Sigma_\rho(y) = \{z \in S_r(x_0) | d_r(z, y) < \rho\}$ stands for a metric ball in $S_r(x_0)$ centered at $y \in S_r$. We now consider a new domain Ω associated with $\Sigma = \Sigma_\rho(y)$ by setting

$$\Omega := \Omega_{4\rho}(\Sigma) = \bigcup_{r \leq t \leq r + 4\rho} \pi_t(\Sigma).$$

Using Lemma 3.1 one can prove the following assertions in the same way as in Sec. 2.

Assertion 3.1. *Let* $y = (p, r)$, $\sigma(t) = (p, t)$ *and* $\Omega_{4\rho}(\Sigma_\rho(y))$ *be as above. Suppose that* $r \geq r_1 + 2\delta^*$ *and* $\rho \geq 2\delta^*$. *Then* $B_{\frac{\rho}{4}}(p) \subset \Omega_{4\rho}(\Sigma_\rho(y))$ *for any* $y \in S_r$.

Assertion 3.2. *Let* r_1, $r \geq r_1 + 2\delta$ *and* $\rho \geq 2\delta$ *be as in Assertion* 2.1. *Suppose that* X *has bounded growth* (*at some scale*) *and that* $\{q_1, q_2, \ldots, q_m\}$ *is a maximal* 2ρ-*separated set in the sphere* S_r *with respect to the metric* d_r. *Then there exists a constant* $C = C(\rho)$ *such that* $k_i = \#\{q_j | d_r(q_j, q_i) \leq 4\rho\} \leq C$ *for* $i = 1, 2, \ldots, m$.

Using Assertions 3.1 and 3.2 above, we can construct a graph $\Gamma \subset X$ with $h(\Gamma) > 0$ as in the Proof of Theorem 2.2 such that X is roughly quasi-isometric to Γ. We omit the straightforward details here. $\square$

Theorem 3.1 now becomes a direct consequence of Theorem 3.2 and Proposition 3.1. As we pointed out earlier, the main theorem follows from Theorem 3.1 and Proposition 2.1.

4. Examples of Gromov-Hyperbolic Spaces

We now show that any non-compact manifold of finite topological type carries a complete Gromov-hyperbolic metric.

Proposition 4.1. (Compare [10] Sec. 7, [25] Sec. 1.8) *Let Y^n be a compact manifold with non-empty boundary ∂Y and let $M^n = \mathrm{int}(Y)$ be the interior of Y. Then M^n admits a complete Riemannian metric g^* such that (M^n, g^*) is Gromov-hyperbolic, it has a quasi-pole, and furthermore $M^n(\infty) = \partial Y$.*

Proof. We begin with a Riemannian metric d_Y on Y. Therefore, its boundary ∂Y has the induced metric $d_{\partial Y}$. Let $\{\Sigma_1, \ldots, \Sigma_m\}$ be the connected components of ∂Y.

For each path-connected component $\Sigma = \Sigma_i$ of ∂Y, there is a Gromov-hyperbolic metric d on $\Sigma \times [0, \infty)$:

$$d((p,t),(q,s)) = 2\log\left(\frac{d_\Sigma(p,q) + \max\{e^{-t}, e^{-s}\}D}{e^{\frac{-(s+t)}{2}}D}\right). \qquad (4.1)$$

The metric d is not necessarily a smooth Riemannian metric. We can approximate the metric d by a Riemannian metric g^* in such a way that d and g^* are roughly quasi-isometric. The construction of g^* goes as follows: There exist constants $a > 1$ and b such that d_Σ and $d|_{\Sigma\times\{0\}}$ are (a,b)-roughly quasi-isometric. We choose $g_0 = d_\Sigma$. For each integer $i \geq 1$, we choose a Riemannian metric g_i on $\Sigma \times \{i\}$ such that g_i is (a,b)-roughly quasi-isometric to the metric $d|_{\Sigma\times\{i\}}$. When $t \in [i, i+1]$, we let

$$g_t = (i+1-t)g_i + (t-i)g_{i+1}\,.$$

Finally, we consider a warped-product metric g^* on $\Sigma \times [0, \infty)$:

$$g^* = g_t + dt^2$$

Clearly, the metric g^* is roughly quasi-isometric to d on $\Sigma \times [0, \infty)$. We now glue the space $(\Sigma \times [0,\infty), g^*)$ back to Y along Σ. The resulting manifold $M = \mathrm{int}(Y) \cup (\bigcup_{1\leq i\leq m} \Sigma_i \times [0,\infty))$ is diffeomorphic to $\mathrm{int}(Y)$.

Our manifold M admits a complete Riemannian metric g^* which is Gromov-hyperbolic. Furthermore, $M(\infty)$ is homeomorphic to ∂Y. Clearly, M has a quasi-pole $p \in \mathrm{int}(Y)$. The second assertion follows from Proposition 3.1. □

We have pointed in Sec. 1 that for certain Gromov-hyperbolic manifolds, the Martin boundary is equal to the Gromov geometric boundary. It is natural to ask "*under which conditions on $M(\infty)$ endowed with Gromov topology τ^G is $M(\infty)$ homeomorphic to the unit sphere*?" We will derive an affirmative answer for the case when M is a simply-connected uniform visibility manifold with no conjugate points.

A complete Riemannian manifold M^n is said to have no conjugate points if any pair of points $\{p, q\} \subset M^n$ can be joined by a unique geodesic segment. If M^n has no conjugate points, all points in M^n are poles.

In the early 1970's, P. Eberlein and B. O'Neill (cf. [22]) introduced so-called visibility manifolds. In what follows, we let M be a simply-connected manifold with no conjugate points. If p, q are distinct points of M we denote by σ_{pq} the unique geodesic joining them parameterized so that $\sigma_{pq}(0) = p$ and $\sigma_{pq}(a) = q$, where $a = d(p, q)$. If $p \in M$ and $q \in M$ is distinct from p, we define $v(p, q) = \sigma'_{pq}(0)$. If $p \in M$ and x_1, x_2 are points of M distinct from p, we then define $\sphericalangle_p(x_1, x_2) = \sphericalangle(v(p, x_1), v(p, x_2))$, the angle at p subtended by x_1 and x_2.

Definition 4.1. (Visibility Manifolds, [21] p. 153) Let M^n be a complete, simply-connected Riemannian manifold with no conjugate points. M^n is said to satisfy the visibility axiom if for each point $p \in M^n$ and $\varepsilon > 0$ there exists a constant $R = R(p, \varepsilon) > 0$ such that if $\sigma : [a, b] \to M^n$ is a geodesic segment satisfying the condition $d(p, \sigma) \geq R$ then $\sphericalangle_p(\sigma(a), \sigma(b)) \leq \varepsilon$. If the constant $R = R(p, \varepsilon)$ above can be chosen to be independent of p, M^n is called a uniform visibility manifold.

If a Riemannian manifold M^n with no conjugate points has a compact quotient M^n/Γ with the induced metric, then the visibility and uniform visibility axioms are equivalent. When M^n/Γ is a compact Riemannian manifold with non-positive curvature, the universal covering space M^n, endowed with lifted metric, is a uniform visibility manifold if and only if M^n contains no totally geodesic isometric embedding of the plane $\mathbb{R}^2$, (cf. [21]). In the presence of non-positive curvature, Gromov-hyperbolicity is equivalent to the uniform visibility axiom:

Lemma 4.1. *Let M^n be a simply-connected and complete Riemannian manifold with non-positive sectional curvature. Then M^n is a uniform visibility manifold if and only if M^n is Gromov-hyperbolic.*

Proof. Proposition 1.14 of [16] was stated for surfaces of non-positive curvature. However, its proof is also applicable to simply-connected manifolds M^n of $\dim(M^n) \geq 3$ as well. Hence, we omit the argument here. □

When M^n does not have non-positive curvature, we make the following observation.

Lemma 4.2. *Let M^n be a simply-connected manifold with no conjugate points. Let σ_1 and σ_2 be two unit speed geodesics with $\sigma_1(0) = \sigma_2(0) = p$ and $\sigma'_1(0) \neq \sigma'_2(0)$ in $T_p(M^n)$. Suppose that M^n is a visibility manifold. Then*

$$2t - 2c \leq d(\sigma_1(t), \sigma_2(t)) \leq 2t\,,$$

for $t \geq 0$, where $c = R(p, \frac{\theta}{2})$ and $\theta = \sphericalangle_p(\sigma'_1(0), \sigma'_2(0))$.

Proof. If M is a visibility manifold, we can choose a constant $c = R(p, \frac{\theta}{2})$ given by the visibility axiom. Let γ_t be the geodesic segment joining $\sigma_1(t)$ and $\sigma_2(t)$. Since $\sphericalangle_p(\sigma_1(t), \sigma_2(t)) = \theta > \frac{\theta}{2}$, there exists a point $q \in \gamma_t$ such that

$$d(p,q) = d(p,\gamma_t) \leq R\left(p, \frac{\theta}{2}\right) = c\,. \tag{4.2}$$

It is clear that

$$L(\gamma_t) = d(\sigma_1(t), q) + d(q, \sigma_2(t)) \tag{4.3}$$

where $L(\gamma_t)$ denotes the length of the geodesic segment γ_t. It follows from (4.2) and (4.3) that

$$\begin{aligned} d(\sigma_1(t), \sigma_2(t)) &= L(\gamma_t) \\ &\geq [d(p, \sigma_1(t)) - d(p,q)] + [d(p, \sigma_2(t)) - d(p,q)] \geq 2(t-c)\,. \end{aligned}$$

It is clear $d(\sigma_1(t), \sigma_2(t)) \leq d(p, \sigma_1(t)) + d(p, \sigma_2(t)) = 2t$. □

If M is a visibility manifold, Lemma 4.2 implies that for any given point $p \in M$ and a point $\xi \in M(\infty)$ there is a unique unit speed geodesic $\sigma : \mathbb{R} \to M$ with $\sigma(0) = p$ and $\sigma(+\infty) = \xi$. In this case, for each point $p \in M$, let $\overline{B_1(0)}$ be the closed unit ball in T_pM and define a bijection $F_p : \overline{B_1(0)} \to M \cup M(\infty)$ such that $F_p(S^{n-1}) = M(\infty)$ by setting

$$F_p(v) = \begin{cases} \exp_p\left(\dfrac{v}{1 - \|v\|}\right), & \text{if } \|v\| < 1 \\ \sigma_v(+\infty)\,, & \text{if } \|v\| = 1\,, \end{cases} \tag{4.4}$$

for σ_v a geodesic ray with $\sigma_v'(0) = v$.

When M is a visibility manifold, for each $p \in M$ and each point $\xi \in M(\infty)$ there is by Lemma 4.2 a unique geodesic $\sigma_{p\xi}$ that belongs to ξ and satisfies $\sigma_{p\xi}(0) = p$. Hence, $M(\infty)$ can be identified with the unit tangent sphere $S^{n-1} \subset T_p(M)$ at the point p.

For $p \in M$, $\xi \in M(\infty)$, $\varepsilon > 0$ and $R > 0$, we let $U_{R,\varepsilon}(p, \xi) = \{y \in M \cup M(\infty) | y \neq p, \sphericalangle_p(y, \xi) < \varepsilon \text{ and } d(p,y) > R\}$ be a truncated cone. For a fixed $p \in M$, we let τ_p^E be the topology generated by open sets of M and the truncated cones with given vertex p. Thus, $(M \cup M(\infty), \tau_p^E)$ is homeomorphic to $\overline{B_1(0)}$ via the map F_p defined by (4.4).

On the other hand, for a Gromov-hyperbolic space M, Gromov also introduced the topological structure τ_o^G at infinity, (see [34]). Gromov's compactification is related to the following function:

$$(x,y)_o = \frac{1}{2}[d(x,0) + d(y,0) - d(x,y)]\,,$$

where o is a given base point in M. The Gromov's topology on $M \cup M(\infty)$ is generated by open subsets in M and open subsets $\mathcal{C}_{\xi,k} = \{z \in M \cup M(\infty) | (\xi, z)_o > k\}$ for all $\xi \in M(\infty)$ and $k > 0$.

Theorem 4.1. *Let M^n be a simply-connected Gromov-hyperbolic visibility manifold. Then there exists a natural homeomorphism*

$$\Phi\colon (M \cup S^{n-1}(\infty), \tau_p^E) \to (M \cup M(\infty), \tau_o^G)$$

from the Eberlein compactification of M to the Gromov compactification, for any $p \in M^n$. In particular, the Gromov geometric boundary $M(\infty)$ endowed with the topology τ_o^G of M is homeomorphic to the unit sphere S^{n-1}.

Proof. We will use the same notations as in [34]. Let $\sigma_{x,y}$ denote the geodesic from x to y and $d(o, \sigma_{x,y}) = \inf\{d(o,z) | z \in \sigma_{x,y}\}$. Then by the triangle inequality, it is easy to check that

$$(x, y)_o \le d(o, \sigma_{x,y})\,.$$

If $x \in M$, we let $\Phi(x) = x$. If $\sigma : \mathbb{R}^+ \to M$ is a geodesic ray with unit speed, then Gromov observed that $\{\sigma(i)\}_{i=1}^{+\infty}$ converges to a point ξ in $M(\infty)$ in the sense of Gromov, i.e.

$$\lim_{i,j \to +\infty} (\sigma(i), \sigma(j))_o = +\infty$$

for any base point o in M.

Fix a base point $o \in M^n$, let $S^{n-1} \subset T_o(M^n)$ be the unit tangent sphere of M^n at o. It follows from Lemma 4.2 that the space of asymptotic geodesic rays can be identified with S^{n-1}. It follows from Propositions 2.1 and 2.2 of Chap. 2 in [20] that the map

$$\Phi\colon S^{n-1} \longrightarrow \Sigma_\infty(M^n)/\sim \sigma(+\infty) \longrightarrow [\{\sigma(i)\}_{i=1}^{+\infty}]\,,$$

is bijective, where σ is a geodesic ray with initial point $\sigma(0) = o$.

In what follows, we are going to show that the inverse Φ^{-1} of Φ is a continuous map, where

$$\Phi^{-1}\colon (M \cup \Sigma_\infty(M^n)/\sim, \tau_o^G) \to (M \cup S^{n-1}, \tau_o^E) \tag{4.5}$$

and M is a visibility manifold. If $x \in M$, Φ^{-1} is clearly continuous at x. We only have to show that Φ^{-1} is continuous at points of $M(\infty)$.

Since the map Φ is bijective, any point $\xi \in M(\infty)$ can be uniquely expressed as the endpoint of a geodesic ray $\sigma\colon \mathbb{R} \to M$ such that $\sigma(0) = o$ and $\xi = [\{\sigma(i)\}_{i=1}^{+\infty}]$. By definition, $\Phi^{-1}(\xi) = \sigma(+\infty)$ is the asymptotic class of σ. For any truncated cone

$$U_{r,\theta} = \{y \in M \cup M(\infty) | \sphericalangle_o(y, \sigma'(0)) < \theta, d(y, o) > r\}$$

we choose an integer $k \geq \max\{r+1, R_o(\frac{\theta}{2})+1\}$, where $R_o(\frac{\theta}{2}) = R(o, \frac{\theta}{2})$ is given by the visibility axiom. We claim that the open subset $\mathcal{C}_{\xi,k} = \{z \in M \cup M(\infty) | (\xi, z)_o > k\}$ is contained in $\Phi(U_{r,\theta})$. It follows that

$$d(o,z) \geq (\xi, z)_o > k > r \, . \tag{4.6}$$

Furthermore, if $\sigma_{z\xi} : \mathbb{R}^+ \to M$ is a geodesic ray joining z and ξ, then

$$d(o, \sigma_{z\xi}) = \inf_{x \in \sigma_{z\xi}} d(x, o) \geq (\xi, z)_o \geq k \geq R_o\left(\frac{\theta}{2}\right) \, . \tag{4.7}$$

Hence, by (4.7) and the visibility axiom we have that

$$\sphericalangle_o(z, \xi) \geq \frac{\theta}{2} < \theta \, .$$

The argument above shows that

$$\mathcal{C}_{\xi,k} \subset \Phi(U_{r,\theta}) \tag{4.8}$$

holds for any $k \geq \max\{r+1, R_o(\frac{\theta}{2})+1\}$. Therefore, we have shown that Φ^{-1} is continuous.

Recall that the topological space $M \cup M(\infty)$ is compact. Theorem 4.1 now follows from the fact that a bijective and continuous map between two compact spaces is a homeomorphism. □

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