A CLOSING LEMMA FOR FLAT STRIPS IN COMPACT SURFACES OF NON-POSITIVE CURVATURE

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Abstract

Let M^2 be a compact oriented Riemannian surface with non-positive curvature. Suppose that $F : \mathbb{R} \times [0, \varepsilon] \to M^2$ is an isometric immersion. We show that either M^2 is isometric to flat torus or the image of F is periodic, i.e., it is a periodic flat strip in M^2.

Introduction

The results of this paper are related to a well-known problem to the effect that the geodesic flow on the unit tangent bundle of a surface of non-positive curvature and genus ≥ 2 must be ergodic. This conjecture remains widely open. If a compact, non-positively curved Riemannian surface (M^2, g) has a flat strip, its geodesic flow is not necessarily hyperbolic in the sense of Anosov. Therefore, the ergodicity problem is related to the study the distribution of zero curvatures on the surface.

We shall show the following result

Main Theorem. *Let (M^2, g) be a compact surface without boundary. Suppose that the curvature is non-positive and $F : [0, \infty) \times [0, \varepsilon] \to M^2$ is an isometric immersion. Then either (M^2, g) is isometric to flat torus or the image of F is periodic, i.e., it is a periodic flat strip in M^2.*

We should point out that for any given flat torus (T^2, g), there exist examples of total geodesic isometric immersions F of flat strips such that F is not periodic. For instance, let (T^2, g_0) be a flat square torus which is isometric to $S^1 \times S^1$. If we choose a flat strip F with irrational slope, then the image of F is dense in T^2. However, such a strip is not periodic.

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The conclusion of our Main Theorem fails for surfaces of infinite area. For example, the infinite cylinder $M^2 = \mathbb{R} \times S^1$ admits non-periodic flat strips.

Using Main Theorem, one can further study the asymptotic behavior of geodesic rays. Let (M^2, g) be as in Main Theorem, and $(\widetilde{M}^2, \tilde{g})$ be its universal cover with the lifted metric. When the genus of M^2 is ≥ 2, the universal cover $(\widetilde{M}^2, \tilde{g})$ is hyperbolic in the sense of Gromov.

Definition 1. *(cf. [Gr], [Ca] p328, [Sh]) Let X be a complete riemannian manifold (or a graph). The space X is said Gromov-hyperbolic if the following dichotomy holds on X:*

(1) *either any pair of geodesic rays $\{\sigma_1, \sigma_2\}$ of unit speed with the same initial point p have bounded Hausdorff distance in X less than C;*
(2) *or the geodesic rays $\{\sigma_1, \sigma_2\}$ above diverge exponentially, which means if $d(\sigma_1(t_0), \sigma_2(t_0)) > C$ for some t_0, then any path from $\sigma_1(t_0 + r)$ to $\sigma_2(t_0 + r)$ lying outside $B_{(t_0+r)}(p)$ has length greater than $\frac{e^{r/C}}{C}$ for any $r > C$;*

where C is a constant independent of the choices of geodesic rays.

In fact, Gromov's hyperbolicity holds for any metric g on M^2 as long as M^2 has genus ≥ 2. In the presence of non-positive curvature, we have more refined result as follows.

Main Corollary. *Let (M^2, g) be as in Main Theorem, and $(\widetilde{M}^2, \tilde{g})$ be its universal cover with the lifted metric. Suppose that M^2 is not a torus. Then for any pair of geodesic rays $\{\sigma_1, \sigma_2\}$ of unit speed in $\widetilde{M}^2$, there are only three possibilities:*

(1) *The geodesic rays $\{\sigma_1, \sigma_2\}$ above diverge exponentially;*
(2) *There exists c such that $\lim_{t \to +\infty} d(\sigma_1(t - c), \sigma_2(t)) = 0$;*
(3) *The two geodesic rays $\{\sigma_1, \sigma_2\}$ are asymptotic to a periodic flat strip.*

§1 Proof of Main Theorem

A closed subset Ω of a Riemannian M^n is said to be locally convex if for each point $p \in \Omega$ there exists a constant $\epsilon(p) > 0$ such that the geodesic segment $\sigma_{p,q}$ form p to q is contained in Ω whenever its length $L(\sigma_{p,q}) < \epsilon(p)$ and $q \in \Omega$. For example, all geodesic balls in a Cartan-Hadamard manifold $\widetilde{M}^n$ are convex.

Lemma 2. *Let (M^2, g) be a compact surface without boundary. Suppose that the curvature of the surface is non-positive and $F : \mathbb{R} \times [0, \varepsilon] \to M^2$ is an isometric immersion. Then each path-connected component of $\Sigma = \overline{[M^2 - \overline{Image(F)}]}$ is locally convex.*

Proof: In fact, if A is a rectifiable set, its tangent cone is given by

$$T_p(A) = \{v | \lim_{t \to 0} \frac{d(Exp_p(tv), A)}{t} = 0\}.$$

Let $\Sigma = \overline{[M^2 - \overline{Image(F)}]}$. For each $p \in \partial\Sigma$, it is easy to see that any pair of vectors in the tangent cone $T_p(\Sigma)$ have angle $\leq \pi$. This is because for each $p \in \partial\Sigma$, $T_p(\overline{Image(F)})$ contains at least one half space. The angle of the cone $T_p(\overline{Image(F)})$ is at least π. Therefore, its complement $T_p(\Sigma)$ has angle at most π. □

The following lemma shows finite topological type of the closure of the image of immersed flat strip.

Lemma 3. *Let (M^2, g) be a compact surface without boundary. Suppose that the curvature of the surface is non-positive and $F : \mathbb{R} \times [0, \varepsilon] \to M^2$ is an isometric immersion. Then $\overline{Image(F)}$ is homeomorphic to a closed surface with at most finitely open disks removed.*

Proof: It follows from Lemma 2 that each path-connect component of $\Sigma = \overline{[M^2 - \overline{Image(F)}]}$ is locally convex. Let Ω_i and Ω_j are two distinct components. Since the width of $Image(F)$ is at least ϵ,

$$d(\Omega_i, \Omega_j) \geq \epsilon.$$

Choose exactly one point p_i from each Ω_i. Then the set $\{p_i\}$ is a discrete ϵ-separated subset of M^2. Because M^2 is compact, the set $\{p_i\}$ is a finite set. It follows that $\Sigma = \overline{[M^2 - \overline{Image(F)}]}$ has only finitely many connect components. □

In the case when $\overline{Image(F)}$ is not equal to the whole surface M^2, we consider the family of immersed flat half strips.

Definition 4. *Let $v_i = (p_i, \theta_i)$ be a unit tangent vector of a riemannian manifold $\widetilde{M}^n$ of non-positive curvature, and σ_{v_i} be the geodesic with $\sigma_{v_i}(0) = p_i$ and $\sigma'_{v_i}(0) = \theta_i$ for $i = 1, 2$. Suppose that δ is the injectivity radius of M^n and $d(p_1, p_2) < \delta$. Two unit vector $\{v_1, v_2\}$ are said to be parallel if $v_1 = \mathbb{P}_{p_1}^{p_2} v_2$, where $\mathbb{P}_{p_1}^{p_2}$ is the parallel transportation from p_1 to p_2 along the length-minimizing geodesic from p_1 to p_2.*

An immersed flat strip $G : \mathbb{R} \times [0, \epsilon] \to M^n$ is said to be maximum if its lifting strip $\widetilde{G} : \mathbb{R} \times [0, \epsilon] \to \widetilde{M}^n$ is maximun with respect to the relation of inclusion.

Proof of Main Theorem:

Suppose that M^2 is not flat. For any given flat half-strip $\widetilde{F} : [0, \infty) \times [0, \epsilon] \to \widetilde{M}^2$, there exists a flat strip of maximal with $\widetilde{G} : [0, \infty) \times [0, \epsilon_1] \to \widetilde{M}^2$ such that two geodesic rays $\widetilde{F}([0, \infty) \times \{0\})$ and $\widetilde{G}([0, \infty) \times \{0\})$ are parallel. Clearly, $\epsilon_1 \leq 4Diam(M^n, g)$.

Suppose that $\pi : \widetilde{M}^2 \to M^2$ is the universal covering map and $G = \pi \circ \widetilde{G}$. We shall show that G must be periodic by a contradiction method. Let $(t, u) \in [0, \infty) \times [0, \epsilon_1]$. We consider a sequence of unit vectors $\{v_i\}_{i=1}^{\infty}$, where $v_i = \frac{\partial G}{\partial t}(i, \epsilon_1)$. Since the unit tangent bundle SM^2 of M^2 is compact, there is a convergent sub-sequence $\{v_{i_j}\}_{j=1}^{\infty}$, which is convergent to a unit vector $v_0 = (p_0, \theta_0)$.

Let δ be the injectivity radius of (M^2, g). For sufficiently large j, we may assume $d(v_{i_j}, v_0) < \frac{\delta}{2}$. Choose a pre-image $\widetilde{p}_0 \in \pi^{-1}(p_0) \subset M^2$ such that $\widetilde{p}_0$ is the nearest point to $\widetilde{G}([0,\infty) \times [0, \epsilon_1])$ in $\pi^{-1}(p_0)$. Notice that $\pi|_{B_\delta(\widetilde{p}_0)} : B_\delta(\widetilde{p}_0) \to B_\delta(p_0)$ is an isometry. This map $\pi|_{B_\delta(\widetilde{p}_0)}$ induces an isometry between the corresponding tangent bundles, say $\Psi : SB_\delta(\widetilde{p}_0) \to SB_\delta(p_0)$.

Let $w_0 = (\widetilde{p}_0, \theta_0)$, $w_j = \Psi^{-1}(v_{i_j}) = (\widetilde{p}_j, \theta_j)$ and $F_j : [i_j, \infty) \times [0, \epsilon_1] \to \widetilde{M}^2$ be the lifted flat strip tangent to θ_j at p_j. Because the limit of flat half strips remain a flat half-strip, we may assume that $\{F_j\}$ converges to $\widetilde{G}_0 : [0,\infty) \times [0, \epsilon_1] \to \widetilde{M}^2$ tangent to θ_0 at $\widetilde{p}_0$.

Assertion 5. Let $F_j, \widetilde{G}_0$ be flat half-strips as above. If M^2 is not flat, then

(1) Either $\pi \circ F_j$ is periodic. Consequently, $G = \pi \circ \widetilde{G}_0$ is periodic;
(2) Or there is a sequence of positive number $\{L_j\} \to \infty$ as $j \to \infty$ such that $\widetilde{G}_0|_{[c_j, c_j+L_j]\times[0,\epsilon_1]\}}$ can be extended to a large isometric embedding

$$\widetilde{G}_0 : [c_j, c_j + L_j] \times [0, \frac{9\epsilon_1}{8}] \to \widetilde{M}^2$$

for some $c_j \geq 0$.

Assuming that Assertion 5 is true at a moment, we prove the Main Theorem. When $\pi \circ F_j$ is periodic, it is easy to check that G is periodic as well. Suppose now that Assertion 5 (2) holds, by a limiting argument we see that a subsequence of $\pi \circ \widetilde{G}_0 : [c_j, c_j + L_j] \times [0, \frac{9\epsilon_1}{8}] \to M^2$ is convergent to an immersed flat half-strip $G_0^* : [0,\infty) \times [0, \frac{9\epsilon_1}{8}] \to M^2$. Let us now choose a connected component, say F_0, of $\pi^{-1}(G_0^*)$ such that F_0 is parallel to $\widetilde{G}$. Notice that F_0 has the width $\frac{9\epsilon_1}{8} > \epsilon_1$, which contradicts to the assumption that $\widetilde{G}$ has maximal width among all flat half-strips parallel to $\widetilde{G}$. The main theorem now follows, excerpt for the proof Assertion 5.

The proof of Assertion 5 uses the following elementary fact:

Fact 6. Let F_j and $\widetilde{G}_0$ be two transversal flat strips of the same width ϵ_1 be as above. Suppose that two flat strips intersect at q_j with angle α, where q_j are in the intersection of ∂F_j and $\partial \widetilde{G}_0$. Then there exists a rectangle $R_j = [0, L_j] \times (0, \frac{\epsilon_1}{8}]$ contained in the closure of $F_j - \widetilde{G}_0$ such that (1) one side of R_j, $[0, L_j] \times \{0\}$ is contained in the ray $\widetilde{G}_0([0,\infty) \times \{\epsilon_1\}$ and (2) $L_j \geq \frac{\epsilon_1}{16 \sin \alpha}$.

To see that Fact 6 is true, we observe that $F_j \cup \widetilde{G}_0$ is simply-connected. Hence, $F_j \cup \widetilde{G}_0$ can be isometrically embedded into $\mathbb{R}^2$.

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