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Finsler Geometry of Projectivized Vector Bundles

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Introduction

The purpose of this article is to reformulate the algebraic geometric concept of ampleness of a vector bundle E in differential geometric terms. As expected the condition involves the concept of a Finsler metric along the Fibers of a holomorphic vector bundle. By a Finsler metric (see section 4 for more details) on E we mean a non-negative function h on E with the following properties:

(1) *h is of class $\mathcal{C}^0$ on E;*

(2) *$h(z, \lambda v) = |\lambda| h(z, v)$ for all $\lambda \in \mathbf{C}$;*

(3) *$h(z, v) > 0$ on $E \setminus \{\text{zero} - \text{section}\}$;*

(4) *for z and v fixed the function $h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.*

For example the Kobayashi metric is a Finsler metric on the tangent bundle. Trivial examples are provided by the norm of a *hermitian* metric on E. Indeed the norm function satisfies, among others, the following additional properties:

(5) *h is of class $\mathcal{C}^\infty$ on $E \setminus \{\text{zero-section}\}$;*

(6) *h is strictly pseudoconvex on $E_x \setminus \{0\}$ for all $x \in M$.*

This last two properties are not shared by some of the naturally defined Finsler metrics, e.g., the Kobayashi metric. However, without these last 2 conditions differential geometric concepts become very complicated, if not impossible, to deal with. For this reason we shall only work with Finsler metrics satisfying properties (5) and (6). These shall be referred to as pseudoconvex (along the fibers) Finsler metrics smooth outside the zero section. With these conditions the mixed holomorphic bisectional curvature of E can be defined (see sections 2 and 4). The term mixed referred to the fact that we shall be taking one direction in the space direction and one fiber direction. If E is the tangent bundle and h a hermitian metric this coincides with the usual notion of holomorphic bisectional curvature. The main result is the following Theorem (see Theorem 3.2 and Theorem 4.5)

Theorem. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\odot^k E$ be the k-fold symmetric product and $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ be the "hyperplane bundle" over the projectivized bundle $\mathbf{P}(\odot^k E)$. Then the following statements are equivalent:*

(1) *E^* is ample;*

(2) *$\mathcal{L}_{\mathbf{P}(E)}$ is ample;*

(3) *$\odot^k E^*, k > 0$ is ample;*

(4) *$\mathcal{L}_{\mathbf{P}(\odot^k E)}, k > 0$ is ample;*

(5) *there exists a Finsler metric along the fibers of $\odot^k E$ with negative mixed holomorphic bisectional curvature;*

(6) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with negative mixed holomorphic bisectional curvature.*

Note that in condition (6) the metric is hermitian not merely Finsler and The positive integer m can be taken to be any integer so that $\mathcal{L}^m_{\mathbf{P}(E)}$ is *very* ample. The preceding Theorem can also be formulated in terms of the dual bundle :

Theorem. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M. Then the following statements are equivalent:*
(1) *E is ample;*
(2) *$\mathcal{L}_{\mathbf{P}(E^*)}$ is ample;*
(3) *$\odot^k E, k > 0$ is ample;*
(4) *$\mathcal{L}_{\mathbf{P}(\odot^k E^*)}, k > 0$ is ample;*
(5) *there exists a Finsler metric along the fibers of $\odot^k E$ with positive mixed holomorphic bisectional curvature;*
(6) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with positive mixed holomorphic bisectional curvature.*

In the special case of the tangent as well as the cotangent bundle we can say a little more (see Theorem 3.9 and Corollary 4.6):

Corollary. *Let $E = T^*M$ be the cotangent bundle of a compact complex n-dimensional manifold M then the following statements are equivalent:*
(1) *TM is ample;*
(2) *$\mathcal{L}_{\mathbf{P}(T^*M)}$ is ample;*
(3) *the anti-canonical bundle $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ is ample;*
(4) *$\odot^k TM, k > 0$ is ample;*
(5) *$\mathcal{L}_{\mathbf{P}(\odot^k T^*M)}, k > 0$ is ample;*
(6) *there exists a Finsler metric along the fibers of $\odot^k T^*M$ with positive holomorphic bisectional curvature;*
(7) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m TM$ with positive holomorphic bisectional curvature.*

Condition (3) is only true for the tangent bundle. The local calculations towards these results are valid even in the non-compact case. We take this opportunity to address the following question: suppose that M is a Kähler manifold and E is a holomorphic vector bundle over M; is E Kähler? The answer is clearly yes if M is compact. We are only able to get some partial results (see Theorem and Corollary 2.2 and Corollary 3.5). The general case remains open.

In the last section we include explicit formulas relating the bisectional curvatures of a vector bundle and the associated tensor bundles.

1. Riemannian metric on TTM

Before dealing with Kähler manifolds we review briefly the case of Riemannian manifolds (see Besse [B]). Let (M, g) be a Riemannian manifold of dimension n and $\pi : TM \to M$ the tangent bundle. It is quite obvious that the most natural way to put a Riemannian metric on TTM is to decompose the bundle in some natural way as a direct sum of a "vertical" and a "horizontal" sub-bundle each with a natural metric and the direct sum of these is a metric on TTM. The *vertical sub-bundle* is, by definition, the kernel of π_*:

$$\mathcal{V} = \ker \pi_* \subset TTM$$

where $\pi_* : TTM \to TM$ is the differential of π. In other words, it is the sub-bundle consisting of all vectors tangent to the fibers of $\pi : TM \to M$. There is a distinguished section, the position vector field P, of the vertical bundle. The most convenient way to describe this is via local coordinates. In terms of a local coordinate system $(U; x^1, ..., x^n)$, an element $v \in T_pM$ is of the form

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_p.$$

Obviously we may identify T_pM with $\mathbf{R}^n$ via the identification:

$$\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}|_p \mapsto (v^1, ..., v^n).$$

hence, on $\pi^{-1}(U) \cong U \times \mathbf{R}^n$ we have the natural coordinate system $(x^1, ..., x^n; v^1, ..., v^n)$. In terms of these, a local section $V \in \Gamma(\pi^{-1}(U), TTM)$ is of the form:

$$V = \sum_{i=1}^{n} a^i(x; v) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} b^\alpha(x; v) \frac{\partial}{\partial v^\alpha}.$$

We have used the same notation $\partial/\partial x^i$ which maybe considered as a vector field on U or on $\pi^{-1}(U)$. We shall, at times, write $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_\alpha = \partial/\partial v^\alpha$. It is clear that the vertical sections are spanned locally by $\partial/\partial v^\alpha, \alpha = 1, ..., n$:

$$\Gamma(\pi^{-1}(U), \mathcal{V}) = \{V \,|\, V = \sum_\alpha b^\alpha(x; v) \frac{\partial}{\partial v^\alpha}\}$$

is a sub-bundle of rank n. Alternatively,

$$\mathcal{V}|_{\pi^{-1}(U)} = \{V \,|\, \pi^* dx_i(V) = 0, i = 1, ..., n\} \subset TTM|_{\pi^{-1}(U)}.$$

The position vector field

$$P(x; v) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial v^\alpha}, \quad v = (v_1, ..., v_n)$$

is, by definition, vertical. The Riemannian metric g on M induces naturally an inner product on $\mathcal{V}$:

$$< V, W >_{\mathcal{V}} \stackrel{\text{def}}{=} \sum_{i,j=1}^{n} g_{ij}(\pi(v)) V^i W^j,$$

$V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial v^i}, W = \sum_{i=1}^{n} W^i \frac{\partial}{\partial v^i} \in \mathcal{V}_v$.

The horizontal sub-bundle is defined via the Riemannian connection ∇ associate to the metric g. The naturally induced connection on TTM is denoted by $\tilde{\nabla}$. The restriction $\tilde{\nabla}|_{\mathcal{V}}$ is a connection on $\mathcal{V}$. Let P be the position vector field defined above then the bundle map

$$\gamma : TTM \to \mathcal{V}, \;\; \gamma(X) := \tilde{\nabla}_X P$$

is a surjection. The kernel, denoted by $\mathcal{H}$, of γ is referred to as the *horizontal sub-bundle*. Thus we have an exact sequence of vector bundles:

$$0 \to \ker \gamma = \mathcal{H} \to TTM \xrightarrow{\gamma} \mathcal{V} \to 0$$

which implies that $\mathcal{V} \cong TTM/\mathcal{H}$. Moreover, the differential restricted to the horizontal sub-bundle $\pi_*|_{\mathcal{H}} : \mathcal{H} \to TM$ is an isomorphism. Using this isomorphism we define a metric on $\mathcal{H}$ by pulling back the Riemmian inner product on TM, i.e.,

$$< Z, W >_{\mathcal{H}} = < \pi_* Z, \pi_* W >_g, \;\; Z, W \in \mathcal{H}.$$

This together with the inner product on $\mathcal{V}$ defines an inner product $<, >_G$ on TTM; more precisely, for $Z_{\mathcal{H}} \in \mathcal{H}$ and $W_{\mathcal{V}} \in \mathcal{V}$ the inner product

$$< Z, W >_G = < W, Z >_G = 0$$

and for any $Z, W \in TTM$ there is a unique decomposition $Z = Z_{\mathcal{H}} + Z_{\mathcal{V}}$ and $W = Z_{\mathcal{H}} + W_{\mathcal{V}}$ into horizontal and vertical components then

$$< Z, W >_G = < Z_{\mathcal{H}}, W_{\mathcal{H}} >_{\mathcal{H}} + < W_{\mathcal{V}}, W_{\mathcal{V}} >_{\mathcal{V}} .$$

Beginning in the next section we shall extend the preceding construction to Kähler manifolds, in fact we shall deal with a more general situation, namely, the holomorphic tangent bundle of a hermitian holomorphic vector bundle over a Kähler manifold. As we shall see, the main difficulty there is that the canonical Riemannian metric constructed above is in general *not Kähler*.

2. The tangent bundle of a holomorphic vector bundle

In this section the construction of the preceding section shall be extended. Vertical and horizontal sub-bundles of the tangent bundle of a general holomorphic vector bundle E will be defined. *The main point is that the vertical bundle is a holomorphic sub-bundle of TE but the horizontal bundle is in general only a smooth but not a holomorphic sub-bundle.* Let (M, g) be a complex hermitian manifold of complex dimension n. Let $\pi : E \to M$ be a holomorphic vector bundle of rank r and the induced map $\pi_* : TE \to TM$. Let $e_1, ..., e_r$ be a local holomorphic frame for E over a local holomorphic coordinate system $(U; z = (z^1, ..., z^n))$. Elements of $E|_U$ are of the form $v = \sum_i v^i e_i$ and $\partial/\partial z^1, ..., \partial/\partial z^n; \partial/\partial v^1, ..., \partial/\partial v^r$ is a local basis for $TE|_{\pi^{-1}(U)}$. The *vertical sub-bundle* is, by definition, the kernel of π_*:

$$\mathcal{V} = \ker \pi_* \subset TE. \tag{2.1}$$

and is a holomorphic sub-bundle of rank r. It is clear that $\pi_*\partial/\partial v^i = 0$ hence

$$\mathcal{V}|_{\pi^{-1}(U)} = \{V \in TE|_{\pi^{-1}(U)} \,|\, V = \sum_{i=1}^{r} b^i(z;v)\frac{\partial}{\partial v^i}\}$$
$$= \{V \in TE|_{\pi^{-1}(U)} \,|\, dz_i(V) = 0, 1 \leq i \leq n\}.$$

The position vector field

$$P(z;v) = \sum_{i=1}^{r} v^i \frac{\partial}{\partial v^i}, \quad v = (v^1, ..., v^r) \tag{2.2}$$

is a *holomorphic* section of $\mathcal{V}$. Let h be a hermitian metric along the fibers of E:

$$< v, w >_h = \sum_{i,j=1}^{r} h_{i\bar{j}}(z) v^i \overline{w}^j, \quad v, w \in E_z. \tag{2.3}$$

This defines, tautologically, a hermitian metric along the fibers of $\mathcal{V}$:

$$< V, W >_{\mathcal{V}} = \sum_{i,j=1}^{r} h_{i\bar{j}}(z) V^i \overline{W}^j, \tag{2.4}$$

$V = \sum_{i=1}^{r} V^i \frac{\partial}{\partial v^i}, W = \sum_{i=1}^{r} W^i \frac{\partial}{\partial v^i} \in \mathcal{V}$. Denote by $\nabla^{\mathcal{V}}$ the hermitian connection on $\mathcal{V}$ with connection forms:

$$\theta_i^j(z,v) = \sum_{k=1}^{n} \gamma_{ik}^j(z) dz^k, \quad \gamma_{ik}^j(z) = \sum_{l=1}^{r} \frac{\partial h_{i\bar{l}}}{\partial z^k}(z) h^{\bar{l}j}(z) \tag{2.5}$$

$(1 \leq i, j \leq r)$ depending only on z. The curvature forms of the hermitian connection:

$$\Theta_i^j \stackrel{\text{def}}{=} d\theta_i^j - \sum_{k=1}^{r} \theta_i^k \wedge \theta_k^j = d\theta_i^j + \sum_{k=1}^{n} \theta_k^j \wedge \theta_i^k \tag{2.6}$$

are of bidegree $(1,1)$ so $\Theta_i^j = \overline{\partial}\theta_i^j$ which is equivalent to the condition that:

$$\partial\theta_i^j - \sum_{k=1}^{r} \theta_i^k \wedge \theta_k^j = \partial\theta_i^j + \sum_{k=1}^{r} \theta_k^j \wedge \theta_i^k = 0. \tag{2.7}$$

The components of the curvature forms are given by

$$\Theta_i^j = \overline{\partial}\theta_i^j = \sum_{k,l=1}^{n} K_{ik\bar{l}}^j dz^k \wedge d\bar{z}^l \tag{2.8}$$

with

$$K_{ik\bar{l}}^j = -\frac{\partial \gamma_{ik}^j}{\partial \bar{z}^l}$$
$$= -\sum_q \left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^k \partial \bar{z}^l} h^{\bar{q}j} - \frac{\partial h_{i\bar{q}}}{\partial z^k} \frac{\partial h^{\bar{q}j}}{\partial \bar{z}^l}\right)$$
$$= -\sum_q \frac{\partial^2 h_{i\bar{q}}}{\partial z^k \partial \bar{z}^l} h^{\bar{q}j} + \sum_{p,q,s} \frac{\partial h_{i\bar{q}}}{\partial z^k} \frac{\partial h_{p\bar{s}}}{\partial \bar{z}^l} h^{\bar{q}p} h^{\bar{s}j}.$$

The curvature depends only on the base variable $z = (z_1, ..., z_n)$ but not on the fiber variable variables $v = (v_1, ..., v_r)$. The connection $\nabla^{\mathcal{V}}$ defines a surjective bundle map:

$$\gamma : TE \to \mathcal{V}, \;\; \gamma(X) = \nabla^{\mathcal{V}}_X P \tag{2.9}$$

where P (see (2.2)) is the position vector field. In terms of local coordinates the map γ takes the following form:

$$\begin{aligned}\nabla^{\mathcal{V}}_X P &= \sum_{j=1}^{r}\{dv^j(X) + \sum_{i=1}^{r} v^i \theta_i^j(X)\}\frac{\partial}{\partial v^j}\\ &= \sum_{j=1}^{r}(b^j + \sum_{i=1}^{r}\sum_{k=1}^{n}\gamma_{ik}^j v^i a^k)\frac{\partial}{\partial v^j}\end{aligned}$$

for any vector field

$$X = \sum_{i=1}^{n} a^i(z; v)\frac{\partial}{\partial z^i} + \sum_{i=1}^{r} b^i(z; v)\frac{\partial}{\partial v^i}.$$

It is clear (for if $X \in \mathcal{V}$ then $a^i = 0$ for all i) that $\gamma|_{\mathcal{V}}$ is the *identity map* on $\mathcal{V}$. Notice that γ is smooth but, in general, *not* holomorphic (this is again clear because the definition of γ involves the connection). The kernel of γ, denoted $\mathcal{H}$, is referred to as the *horizontal sub-bundle* which is a smooth (but not holomorphic in general) sub-bundle of TE. However $\mathcal{H}$ is smoothly isomorphic to the quotient bundle $TE/\mathcal{V}$ which is holomorphic as $\mathcal{V}$ is a holomorphic sub-bundle of TE and

$$0 \to \mathcal{V} \to TE \to TE/\mathcal{H} = \mathcal{Q} \to 0$$

is an exact sequence of holomorphic vector bundles. On the other hand, we have an exact sequence

$$0 \to \ker\gamma = \mathcal{H} \to TE \xrightarrow{\gamma} \mathcal{V} \to 0$$

of *smooth* vector bundles and a *smooth* decomposition $TE = \mathcal{H} \oplus \mathcal{V}$. Thus the restriction of the map $\pi_* : TE \to TM$ to $\mathcal{H}$:

$$\pi_*|_{\mathcal{H}} : \mathcal{H} \xrightarrow{\cong} TM$$

is a smooth isomorphism. Using this isomorphism an inner product can be defined on $\mathcal{H}$ by pulling back the Kähler inner product on TM, i.e.,

$$< Z, W >_{\mathcal{H}} = < \pi_* Z, \pi_* W >_g, \;\; Z, W \in \mathcal{H}. \tag{2.10}$$

This together with the inner product, induced by the hermitian metric h of E, on $\mathcal{V}$ defines an inner product $<,>_G$ on TE. More precisely, if $Z \in \mathcal{H}$ and $W \in \mathcal{V}$ then $< Z, W >_G = < W, Z >_G = 0$ and for any $Z, W \in TTM$ we have unique decompositions $Z = Z_{\mathcal{H}} + Z_{\mathcal{V}}$ and $W = Z_{\mathcal{H}} + W_{\mathcal{V}}$ into horizontal and vertical components then

$$< Z, W >_G \stackrel{\text{def}}{=} < Z_{\mathcal{H}}, W_{\mathcal{H}} >_{\mathcal{H}} + < W_{\mathcal{V}}, W_{\mathcal{V}} >_{\mathcal{V}} \tag{2.11}$$

is a well-defined inner product on TE.

Given a vector field $V(z) = \sum_i a^i(z)\partial/\partial z^i$ on M the vector field

$$V^{\mathcal{H}}(z,v) = \sum_{i=1}^{n}\{a^i(z)\frac{\partial}{\partial z^i} - \sum_{j=1}^{r}\sum_{k=1}^{n}\gamma^i_{jk}v^j a^k\frac{\partial}{\partial v^i}\}$$

is horizontal and shall be referred to as the *horizontal lifting* of V. The horizontal lifts of the local basis $\{\partial_i = \partial/\partial z^i, i = 1, ..., n\}$ of TE:

$$\{\partial_i^{\mathcal{H}} = \frac{\partial}{\partial z^i} - \sum_{j,k=1}^{r}\gamma^k_{ji}v^j\frac{\partial}{\partial v^k}\,|\, i = 1, ..., n\}$$

is a basis of $\mathcal{H}$. By definition, we have:

$$< \partial_i^{\mathcal{H}}, \partial_j^{\mathcal{H}} >_G (z,v) =< \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} >_g (z) = g_{i\bar{j}}(z),$$
$$< \frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^i} >_G (z,v) =< \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} >_h (z) = h_{i\bar{j}}(z)$$

and $< \partial_i^{\mathcal{H}}, \frac{\partial}{\partial v^j} >_G = 0$ for all i and j. Let

$$\eta_i = \sum_{k=1}^{n} a_{ik}dz^k + \sum_{l=1}^{r} b_{il}dv^l,\ 1 \le i \le n,$$
$$\theta_j = \sum_{k=1}^{n} \alpha_{jk}dz^k + \sum_{l=1}^{r} \beta_{jk}dv^k,\ 1 \le j \le r$$

be the basis dual to $\{\partial_i^{\mathcal{H}}, \frac{\partial}{\partial v^j}\,|\, i = 1, ..., n; j = 1, ..., r\}$. The conditions that $\eta_i(\partial_i^{\mathcal{H}}) = 1, \eta_i(\partial_l^{\mathcal{H}}) = 0$ for $l \ne i$ and $\eta_i(\partial/\partial v^l) = 0$ for all i and l are equivalent to the following identities

$$\begin{cases} \eta_i(\partial_i^{\mathcal{H}}) = a_{ii} - \sum_{1\le j,k\le r}\gamma^k_{ji}v^j b_{ik} = 1, & 1 \le i \le n, \\ \eta_i(\partial_l^{\mathcal{H}}) = a_{il} - \sum_{1\le j,k\le r}\gamma^k_{jl}v^j b_{ik} = 0, & 1 \le l \ne i \le n, \\ \eta_i(\frac{\partial}{\partial v^l}) = b_{il} = 0, & 1 \le i \le n, 1 \le l \le r \end{cases}$$

which imply that $a_{il} = \delta_i^l$ for $1 \le i, l \le n$. Analogously, $\omega_i(\frac{\partial}{\partial v^i}) = 1, \omega_i(\frac{\partial}{\partial v^l}) = 0$ for $1 \le l \ne i \le r$ and $\omega_i(\partial_l^{\mathcal{H}}) = 0$ for all i and l resulting in the following identities:

$$\begin{cases} \theta_i(\frac{\partial}{\partial v^i}) = \beta_{ii} = 1, & 1 \le i \le r, \\ \theta_i(\frac{\partial}{\partial v^l}) = \beta_{il} = 0, & 1 \le l \ne i \le r, \\ \theta_i(\partial_l^{\mathcal{H}}) = \alpha_{il} - \sum_{j,k=1}^{r}\gamma^k_{jl}v^j\beta_{ik} = 0, & 1 \le i \le r, 1 \le l \le n \end{cases}$$

which imply that

$$\sum_{j=1}^{r}\gamma^i_{jl}v^j = \alpha_{il},\ 1 \le i \le r, 1 \le l \le n.$$

In other words, the dual basis is given by:

$$\text{(2.12)}\qquad \begin{cases} \eta^i = dz^i, & 1 \le i \le n, \\ \theta^i = \sum_{j=1}^{r}\sum_{k=1}^{n}\gamma^i_{jk}v^j dz^k + dv^i = dv^i + \sum_{j=1}^{r}\theta^i_k v^k, & 1 \le i \le r \end{cases}$$

and, with respect to this basis, the fundamental form of the metric G defined by the Kähler metric g on M and the fiber metric h on E takes the form:

$$\eta = \sqrt{-1}(\sum_{i,j=1}^{n} g_{i\bar{j}}(z)\, dz^i \wedge d\bar{z}^j + \sum_{i,j=1}^{r} h_{i\bar{j}}(z)\, \theta^i \wedge \overline{\theta}^j). \tag{2.13}$$

The next Theorem identifies the obstruction of η from being Kähler *assuming that* (M, g) *is Kähler*:

Theorem 2.1. *Let* (M, g) *be a complex Kähler manifold and* (E, h) *be an hermitian holomorphic vector bundle of rank* r *over* M*. Let* η *be the fundamental form of the hermitian metric* G *as defined above then*

$$\begin{aligned} d\eta &= \sqrt{-1}(\sum_{1\le i,j,k\le r} h_{i\bar{j}} v^k \Theta^i_k \wedge \overline{\theta}^j - \sum_{1\le i,j,k\le r} h_{i\bar{j}} \theta^i \wedge \bar{v}^k \overline{\Theta}^j_k) \\ &= \sqrt{-1}(\sum_{1\le i,j,k\le r} h_{i\bar{j}} \overline{\partial}\theta^i \wedge \overline{\theta}^j - \sum_{1\le i,j,k\le r} h_{i\bar{j}} \theta^i \wedge \partial\overline{\theta}^j) \end{aligned}$$

where $\theta^i = dv^i + \sum_k \theta^i_k v^k$, $(v^1, ..., v^r)$ *are the fiber coordinates and* θ^i_k, Θ^i_k *are resp. the hermitian connection and curvature forms of the metric* h.

Proof. The first term on the right hand side of (2.13) is closed as it is the Kähler form of the metric g. Thus exterior differentiation yields,

$$d\eta = \sqrt{-1}\sum_{i,j=1}^{r}(dh_{i\bar{j}} \wedge \theta^i \wedge \overline{\theta}^j + h_{i\bar{j}} d\theta^i \wedge \overline{\theta}^j - h_{i\bar{j}} \theta^i \wedge d\overline{\theta}^j).$$

We have (by the hermitian condition $\overline{h_{k\bar{i}}} = h_{i\bar{k}}$, the identities (2.5) and (2.12)),

$$\begin{aligned} dh_{i\bar{j}} &= \sum_k \theta^k_i g_{k\bar{j}} + \sum_k \overline{\theta^k_j g_{k\bar{i}}} = \sum_k \theta^k_i h_{k\bar{j}} + \sum_k \overline{\theta}^k_j h_{i\bar{k}}; \\ d\theta^i &= \sum_k dv^k \wedge \theta^i_k + \sum_k d\theta^i_k v^k, \\ d\overline{\theta}^j &= \sum_k d\bar{v}^k \wedge \overline{\theta^j_k} + \sum_k d\overline{\theta^i_k} \bar{v}^k. \\ \theta^i \wedge \overline{\theta}^j &= dv^i \wedge d\bar{v}^j + \sum_k \bar{v}^k dv^i \wedge \overline{\theta}^j_k + \sum_k v^k \theta^i_k \wedge d\bar{v}^j + \sum_{k,l} v^k \bar{v}^l \theta^i_k \wedge \overline{\theta}^j_l, \end{aligned}$$

hence,

$$\begin{aligned} dh_{i\bar{j}} \wedge \theta^i \wedge \overline{\theta}^j = &\sum_k h_{k\bar{j}} \theta^k_i \wedge dv^i \wedge d\bar{v}^j + \sum_k h_{i\bar{k}} \overline{\theta}^k_j \wedge dv^i \wedge d\bar{v}^j, \overline{\theta}^j_l \\ &- \sum_{k,l} h_{k\bar{j}} \bar{v}^l dv^i \wedge \theta^k_i \wedge \overline{\theta}^j_l - \sum_{k,l} h_{i\bar{k}} \bar{v}^l dv^i \wedge \overline{\theta}^k_j \wedge \\ &+ \sum_{k,l} h_{k\bar{j}} v^l \theta^k_i \wedge \theta^i_l \wedge d\bar{v}^j - \sum_{k,l} h_{i\bar{k}} v^l \theta^i_l \wedge \overline{\theta}^k_j \wedge d\bar{v}^j \end{aligned}$$

$$+\sum_{k,l,m} h_{k\bar{j}} v^l \bar{v}^m \theta_i^k \wedge \theta_l^i \wedge \overline{\theta}_m^j - \sum_{k,l} h_{i\bar{k}} v^l \bar{v}^m \theta_k^j \wedge \theta_l^i \wedge \overline{\theta}_m^j$$

$$\begin{aligned}
h_{i\bar{j}} d\theta^i \wedge \overline{\theta}^j &= \sum_k h_{i\bar{j}} dv^k \wedge \theta_k^i \wedge \overline{\theta}^j + \sum_k h_{i\bar{j}} v^k d\theta_k^i \wedge \overline{\theta}^j \\
&= -\sum_k h_{i\bar{j}} \theta_k^i \wedge dv^k \wedge d\bar{v}^j + \sum_{k,l} h_{i\bar{j}} \bar{v}^l dv^k \wedge \theta_k^i \wedge \overline{\theta}_l^j \\
&\quad + \sum_k h_{i\bar{j}} v^k d\theta_k^i \wedge d\bar{v}^j + \sum_{k,l} h_{i\bar{j}} v^k \bar{v}^l d\theta_k^i \wedge \overline{\theta}_l^j \\
-h_{i\bar{j}} \theta^i \wedge d\overline{\theta}^j &= -\sum_k h_{i\bar{j}} \omega^i \wedge d\bar{v}^k \wedge \overline{\theta}_k^j - \sum_k h_{i\bar{j}} \bar{v}^k \omega^i \wedge d\overline{\theta}_k^j \\
&= -\sum_k h_{i\bar{j}} \overline{\theta}_k^j \wedge dv^i \wedge d\bar{v}^k + \sum_{k,l} h_{i\bar{j}} v^l d\bar{v}^k \wedge \theta_l^i \wedge \overline{\theta}_k^j \\
&\quad - \sum_k h_{i\bar{j}} \bar{v}^k dv^i \wedge d\overline{\theta}_k^j - \sum_{k,l} h_{i\bar{j}} v^l \bar{v}^k \theta_l^i \wedge d\overline{\theta}_k^j.
\end{aligned}$$

Summing the above over i and j yields

$$\begin{aligned}
d\eta = \sqrt{-1}\Big(&-\sum_{i,j,k,l} h_{i\bar{k}} \bar{v}^l dv^i \wedge \overline{\theta}_j^k \wedge \overline{\theta}_l^j + \sum_{i,j,k,l} h_{k\bar{j}} v^l \theta_i^k \wedge \theta_l^i \wedge d\bar{v}^j \\
&+ \sum_{i,j,k,l,m} h_{k\bar{j}} v^l \bar{v}^m \theta_i^k \wedge \theta_l^i \wedge \overline{\theta}_m^j - \sum_{i,j,k,l,m} h_{i\bar{k}} v^l \bar{v}^m \theta_l^i \wedge \overline{\theta}_j^k \wedge \overline{\theta}_m^j \\
&+ \sum_{i,j,k,l} h_{i\bar{j}} v^k d\theta_k^i \wedge d\bar{v}^j + \sum_{i,j,k,l} h_{i\bar{j}} v^k \bar{v}^l d\theta_k^i \wedge \overline{\theta}_l^j \\
&- \sum_{i,j,k,l} h_{i\bar{j}} \bar{v}^k dv^i \wedge d\overline{\theta}_k^j - \sum_{i,j,k,l} h_{i\bar{j}} v^l \bar{v}^k \theta_l^i \wedge d\overline{\theta}_k^j\Big)
\end{aligned}$$

The right hand side above can be expressed in terms of the curvature (see (2.6) and (2.7)) and we arrive at the following expression for $d\eta$:

$$\begin{aligned}
d\eta = \sqrt{-1}\Big(&-\sum_{i,j,k,l} h_{i\bar{j}} \bar{v}^k dv^i \wedge \overline{\Theta}_k^j + \sum_{i,j,k,l} h_{i\bar{j}} v^k \Theta_k^i \wedge d\bar{v}^j \overline{\theta}_l^j \\
&+ \sum_{i,j,k,l} h_{i\bar{j}} v^k \bar{v}^l \Theta_k^i \wedge \overline{\theta}_l^j - \sum_{i,j,k,l} h_{i\bar{j}} v^l \bar{v}^k \theta_l^i \wedge \overline{\Theta}_k^j\Big)
\end{aligned}$$

$$= \sqrt{-1}\Big(\sum_{i,j,k,l} h_{i\bar{j}} v^k \Theta_k^i \wedge (d\bar{v}^j + \bar{v}^l \overline{\theta}_l^j) - \sum_{i,j,k,l} h_{i\bar{j}} (dv^i + v^l \theta_l^i) \wedge \bar{v}^k \overline{\Theta}_k^j\Big)$$

$$= \sqrt{-1}\Big(\sum_{i,j,k,l} h_{i\bar{j}} v^k \Theta_k^i \wedge \bar{\theta}^j - \sum_{i,j,k,l} h_{i\bar{j}} \theta^i \wedge \bar{v}^k \overline{\Theta}_k^j\Big).$$

Next we observe that

$$\partial\theta^i + \overline{\partial}\theta^i = d\theta^i = \sum_k dv^k \wedge \theta_k^i + \sum_k \partial\theta_k^i v^k + \sum_k \overline{\partial}\theta_k^i v^k$$

and comparing bi-degrees yield

$$\overline{\partial}\theta^i = \sum_k v^k \overline{\partial}\theta^i_k = \sum_k \Theta^i_k v^k \tag{2.14}$$

hence

$$d\eta = \sqrt{-1}(\sum_{i,j,k} h_{i\bar{j}} \overline{\partial}\theta^i \wedge \overline{\theta}^j - \sum_{i,j,k} h_{i\bar{j}} \theta^i \wedge \partial\overline{\theta}^j)$$

as claimed. QED

Corollary 2.2. *Let G be the metric in Theorem 2.1. Then the following conditions are equivalent*

(*i*) *G is Kähler*

(*ii*) *the one forms $\{\theta^i, i = 1, ..., n\}$ are holomorphic*

(*iii*) *the curvature of the vertical bundle $\mathcal{V}$ satisfies the conditions:*

$$\sum_{1\leq i,k\leq r} h_{i\bar{m}} v^k \Theta^i_k = 0, \ 1 \leq m \leq r;$$

(*iv*) *the curvature of the vertical bundle $\mathcal{V}$ is zero.*

Proof. It is clear that the holomorphicity of $\{\theta^i, i = 1, ..., n\}$ implies that $d\eta = 0$. For the converse, we see from the expression (see (2.8))

$$\Theta^i_k \wedge \overline{\theta}^j = \sum_{p,q=1}^n K^i_{kp\bar{q}} dz^p \wedge d\bar{z}^q \wedge (d\bar{v}^j + \sum_{s=1}^r \sum_{s=1}^n \overline{\gamma}^j_{r\bar{s}} \bar{v}^r d\bar{z}^s)$$

that, for any i, j, k, m

$$\iota_{\partial_{\bar{m}}} \Theta^i_k \wedge \overline{\theta}^j = \delta^j_m \sum_{p,q=1}^n K^i_{kp\bar{q}} dz^p \wedge d\bar{z}^q$$

where $\iota_{\partial_{\bar{m}}}$ denotes interior product with the vector field $\partial/\partial\bar{v}^m$. This shows that

$$\iota_{\partial_{\bar{m}}} d\eta = \sqrt{-1} \sum_{i,k,l} h_{i\bar{m}} v^k \Theta^i_k = \sqrt{-1} \sum_i h_{i\bar{m}} \overline{\partial}\theta^i$$

($\sum_k v^k \Theta^i_k = \overline{\partial}\theta^i, i = 1, ..., r$ by (2.14)) for all m and that

$$\sum_m h^{\bar{m}j} \iota_{\partial_{\bar{m}}} d\eta = \sqrt{-1} \sum_{i,m} h^{\bar{m}j} h_{i\bar{m}} \overline{\partial}\theta^i = \sqrt{-1} \sum_i \delta^j_i \overline{\partial}\theta^i = \sqrt{-1}\overline{\partial}\theta^j$$

for all j. From these it is clear that assertions (i), (ii) and (iii) are equivalent. From the identities $\sum_k v^k \Theta^i_k = \overline{\partial}\theta^i, i = 1, ..., r$ and the fact that the curvature forms $\{\Theta^i_k\}$ are independent of v, as $v = (v^1, ..., v^k)$ ranges over all $v \in E_z, z \in M$ we infer that $\Theta^i_k = 0$ for all i and k if and only if $\overline{\partial}\theta^i = 0$ for all i. QED

If $E = TM$ is the tangent bundle and $(E, h) = (TM, g)$ then

Corollary 2.3. *Let G be the metric in Theorem 2.1 with $(E, h) = (TM, g)$. Then the following conditions are equivalent*

(*i*) *G is Kähler*

(*ii*) *the one forms* $\{\theta^i, i = 1, ..., n\}$ *are holomorphic*
(*iii*) *the curvature of the vertical bundle* $\mathcal{V}$ *satisfies the conditions:*

$$\sum_{1\le i,k\le r} h_{i\bar{m}} v^k \Theta^i_k = 0,\ 1 \le m \le r;$$

(*iv*) *the curvature of the vertical bundle* $\mathcal{V}$ *is zero;*
(*v*) *the curvature of* (M, g) *is zero.*

Proof. The first 3 statements are the same as Corollary 2.2. Under the present assumptions that $h = g$ the curvature Θ^i_k of $\mathcal{V}$m defined by h, is the same as that of the curvature Ω^i_k of the metric g on M. QED

Corollary 2.2 shows that the natural metric G is generally not Kähler so we look for other means of producing Kähler metrics. We need a technical Lemma which is quite useful in local calculation:

Lemma 2.4. *Let* (E, h) *be a hermitian holomorphic vector bundle over a complex manifold* M*. Then at any point* $x_0 \in M$ *there exists a local holomorphic frame* $e_1, ..., e_r$ *over an open neighborhood of* x_0 *which is unitary and parallel at* x*, i.e.,* $h_{i\bar{j}}(x_0) =< e_i, e_j >_h (x_0) = \delta^j_i$ *and* $dh_{i\bar{j}}(x_0) = 0$ *for all* $1 \le i, j \le r$*. In particular, all connection forms relative to this local frame vanishes at* x_0 *and the curvature at* x_0 *is given by* $(\overline{\partial}\partial h)(x_0)$*.*

Proof. Since h is hermitian it is clear (by diagonalization and re-scaling) that there is a local holomorphic frame $e = (e_1, ..., e_r)$ over an open neighborhood U of x_0 such that $h_{i\bar{j}}(x_0) = \delta^j_i$ at the point x_0. Choose $(U, z = (z^1, ..., z^n)), n = \dim M$ to be a local coordinate neighborhood so that x_0 is the origin. Let $H(z) = I_r + A(z)$ where I_r is the $r \times r$ identity matrix and

$$A(z) = (\sum_{k=1}^{n} A^j_{ik} z^k)_{1\le i,j\le r},\ \ A^j_{ik} = -\frac{\partial h_{i\bar{j}}}{\partial z^k}(0).$$

Define a new frame $\tilde{e} = eH$. Denote by h_e the matrix $(< e_i, e_j >_h)_{1\le i,j\le r}$ and $h_{\tilde{e}}$ the matrix $(< \tilde{e}_i, \tilde{e}_j >_h)_{1\le i,j\le r}$ then

$$h_{\tilde{e}} = \overline{H}^t h_e H = (I_r + \overline{A})^t h_e (I_r + A).$$

Since $h_e = I_r$ and $A = 0_r$ (the $r \times r$ zero matrix) at 0 (i.e., the point x_0), we have:

$$h_{\tilde{e}} = I_r \text{ and } dh_{\tilde{e}} = d(\overline{A}^t) + dh_e + dA$$

at x_0. By construction $dA = -\partial h_e$ at x_0 and so, by the hermitian property of h, $d\overline{A}^t = -\overline{\partial} h_e$ at x_0. These imply that $dh_{\tilde{e}} = 0$ at x_0 and so $\tilde{e}$ is the required frame. The connection matrix $\theta_{\tilde{e}}$ relative to the frame $\tilde{e}$ is, by definition, $(\partial h_{\tilde{e}})h^{-1}_{\tilde{e}} = 0$ at x_0. This implies that, at the point x_0,

$$\Theta_{\tilde{e}} = d\theta_{\tilde{e}} - \theta_{\tilde{e}} \wedge \theta_{\tilde{e}} = d\theta_{\tilde{e}} = d((\partial h_{\tilde{e}})h^{-1}_{\tilde{e}}) = (\overline{\partial}\partial h_{\tilde{e}})h^{-1}_{\tilde{e}} - \partial h_{\tilde{e}} \wedge d\partial h^{-1}_{\tilde{e}}$$

hence $\Theta_{\tilde{e}} = \overline{\partial}\partial h_{\tilde{e}}$ as $h_{\tilde{e}}$ is the identity matrix and $\partial h_{h\tilde{e}} = 0$ at the point x_0. QED

The preceding Lemma extends the well-known fact that, for a Kähler metric, normal coordinates exist at any given point. For a hermitian but non-Kähler metric on the tangent bundle a normal holomorphic frame may not come from a local holomorphic coordinate.

Let (M, g) be a hermitian manifold and (E, h) a hermitian holomorphic vector bundle over M. Consider the global $(1,1)$-form on E:

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1}\partial\overline{\partial}||P||_{\mathcal{V}}^2 = \sqrt{-1}\partial\overline{\partial}\sum_{i,j=1}^{r} h_{i\bar{j}}(z)v^i\bar{v}^j \tag{2.15}$$

where P is the position vector field as defined in (2.2). A direct calculation shows that

$$\begin{aligned}
&\sqrt{-1}\partial\overline{\partial}||P||_G^2\\
&= \sqrt{-1}\sum_{i,j=1}^{r}\{h_{i\bar{j}}(z)dv^i \wedge d\bar{v}^j + v^i\bar{v}^j\sum_{k,l=1}^{n}\frac{\partial^2 h_{i\bar{j}}}{\partial z^k\partial\bar{z}^l}(z)dz^k \wedge d\bar{z}^l\}\\
&\quad + \sqrt{-1}\{\sum_{i,j=1}^{r}\{\sum_{k=1}^{n} v^i\frac{\partial h_{i\bar{j}}}{\partial z^k}(z)dz^k \wedge d\bar{v}^j + \sum_{l=1}^{n}\bar{v}^j\frac{\partial h_{i\bar{j}}}{\partial\bar{z}^l}(z)dv^i \wedge d\bar{z}^l\}.
\end{aligned}$$

By Lemma 2.4 we may choose a local frame of E which is normal at any given point z^* and, with respect to such a frame, we have, at the point z^*:

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1}\sum_{i=1}^{n}\{dv^i \wedge d\bar{v}^i + v^i\bar{v}^i\sum_{k,l=1}^{n}\frac{\partial^2 h_{i\bar{i}}}{\partial z^k\partial\bar{z}^l}dz^k \wedge d\bar{z}^l\}.$$

It is clear from this that $\sqrt{-1}\partial\overline{\partial}||P||_G^2$ is positive definite in the vertical direction. Moreover the curvature at z^* is given by (see (2.8))

$$\Theta_i^j = -\sum_{k,l=1}^{r}\frac{\partial^2 h_{i\bar{j}}}{\partial z^k\partial\bar{z}^l}dz^k \wedge d\bar{z}^l$$

thus the second sum in the expression above is a curvature term:

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}||P||_G^2 &= \sqrt{-1}\sum_{i=1}^{r}\{dv^i \wedge d\bar{v}^i - v^i\bar{v}^i\sum_{k,l=1}^{n}K_{ik\bar{l}}^i dz^k \wedge d\bar{z}^l\}\\
&= \sqrt{-1}\sum_{i,j=1}^{r}h_{i\bar{j}}dv^i \wedge d\bar{v}^j - \sqrt{-1} < K(\cdot,\cdot)P, P >_h
\end{aligned}$$

where the position vector field $P = \sum v^i\partial/\partial z^i$ is identified with the position vector $(v^1, ..., v^r)$ and K is the curvature operator which, in terms of the curvature matrix, is given as follows:

$$Kv = \sum_{i=1}^{r}\Theta_i^j v^i = \sum_{i=1}^{r}K_{ik\bar{l}}^j v^i,$$

and that

$$(2.16)\qquad < Kv, u >_h = \sum_{i,j,q=1}^{r} h_{j\bar{q}} \Theta_i^j v^i \overline{u}^q = \sum_{k,l=1}^{r} (\sum_{i,j,q=1}^{r} h_{j\bar{q}} K_{ik\bar{l}}^j v^i \bar{u}^q) \, dz^k \wedge d\bar{z}^l$$

for $v = \sum_{i=1}^r v^i e_i, u = \sum_{i=1}^r u^i e_i$ in E and for any tangent vectors X, Y of type $(1, 0)$ on M,

$$K(X, Y)v = \sum_{i=1}^{r} \Theta_i^j (X, \overline{Y}) v^i,$$

$$< K(X, Y)v, u >_h = \sum_{k,l=1}^{n} \sum_{i,j,q=1}^{r} h_{j\bar{q}} K_{ik\bar{l}}^j X^k \overline{Y}^l v^i \overline{u}^q.$$

If $||X||_g \neq 0$ and $||v||_h \neq 0$ the *mixed holomorphic bisectional curvature of* (E, h) is defined to be:

$$(2.17)\qquad k(X, v) = \frac{< K(X, X)v, v >_h}{||X||_g^2 ||v||_h^2}, \; X \in T_x M, v \in E_x.$$

Identifying the position vector field with the position vector we shall write $k(X, P)$ instead of $k(X, v)$ The preceding calculation shows that:

Theorem 2.5. *Let P be the position vector on E where (E, h) is a hermitian holomorphic vector bundle of rank r over a complex hermitian manifold (M, g). Let G be the metric along the fibers of TE defined by g and h then the $(1, 1)$-form $\sqrt{-1}\partial\overline{\partial}||P||_G^2$ is positive definite on $E \setminus \{zero - section\}$ if and only if the mixed holomorphic bisectional curvature $k(X, P)$ is strictly negative for all non-zero $X \in T_x M$.*

Proof. This is quite clear from the identity

$$\sqrt{-1}\partial\overline{\partial}||P||_G^2 = \sqrt{-1}(\sum_{i=1}^{r} dv^i \wedge d\bar{v}^i - \sum_{i=1}^{r} \sum_{k,l=1}^{n} v^i \bar{v}^i K_{ik\bar{l}}^i dz^k \wedge d\bar{z}^l)$$

as the first term on the right guaranteed that the $(1, 1)$-form is positive definite in the fiber directions while the second term is positive definite in the base directions if and only if the mixed holomorphic bisectional curvature $k(X, P)$ is strictly negative. QED

In the current situation $E = TM$ and $h = g$ hence we may write $v = Y \in T_x M$ and

$$k(X, Y) = \frac{< R(X, X)Y, Y >_g}{||X||_g^2 ||Y||_g^2}$$

if $X, Y \in T_x M$ are non-zero tangent vectors at x. We recognize from the definition that $k(X, X)$ (in which case we shall write simply $(k(X)$ instead of $k(X, X))$ is the usual holomorphic sectional curvature of g and $k(X, Y)$ is essentially the holomorphic bisectional curvature of g. More precisely, for X, Y linearly independent the holomorphic bisectional curvature of g is defined to be

$$b(X,Y) = \frac{< R(X,X)Y,Y >_g}{||X \wedge Y||_g^2}$$

(note that $b(X,X)$ does not make sense and is the main reason that we use $k(X,Y)$ instead of $b(X,Y)$) thus

$$b(X,Y) = \frac{||X||_g^2 ||Y||_g^2}{||X \wedge Y||_g^2} k(X,Y)$$

which shows that $b(X,Y)$ and $k(X,Y)$ have the same sign and we get from Theorem 2.5 that

Theorem 2.6. *Let P be the position vector field on TM where (M,g) is a complex hermitian manifold. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}||P||_g^2$ is positive definite on $TM \setminus \{zero - section\}$ if and only if the holomorphic bisectional curvature of g is strictly negative.*

In the next section the preceding Theorem shall be formulated on the projectivized bundle rather than on E. The reason for working on $\mathbf{P}(E)$ rather than E is that $\mathbf{P}(E)$ is compact if M is compact.

3. THE TANGENT BUNDLE OF A PROJECTIVIZED VECTOR BUNDLE

Let (M,g) be a Kähler manifold with holomorphic tangent bundle $p_M : TM \to M = TM$ and let (E,h) be a hermitian holomorphic vector bundle of rank $r \geq 2$ over M with projection

$$p_E : E \to M. \tag{3.1}$$

Denote by $E_* = E \setminus \{\text{zero} - \text{section}\}$ then there is a natural $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ action on E_* and the quotient

$$[\,] : E_* \to \mathbf{P}(E) = E_*/\mathbf{C}^* \tag{3.2}$$

shall be referred to as the projectivized vector bundle. The natural projection map shall be denoted by

$$[p_E] : \mathbf{P}(E) \to M. \tag{3.3}$$

As the notations suggested, the following diagram commutes:

$$\begin{array}{ccc} E_* & = & E_* \\ [\,]\ \downarrow & & \downarrow p_E \\ \mathbf{P}(E) & \xrightarrow{[p_E]} & M \end{array}$$

The quotient map (3.2) induces a bundle map between the tangent bundles

$$[\,]_* : TE_* \to T\mathbf{P}(E).$$

The kernel of $[\,]_*$ is the trivial line bundle $< P >$ spanned by the position vector field P (defined in (2.2)) and we have a short exact sequence of holomorphic bundles

$$0 \to < P > \to TE_* \stackrel{[\,]_*}{\to} T\mathbf{P}(E) \to 0. \tag{3.4}$$

The pull-back $[p_E]^*E$ is a sub-bundle of $\mathbf{P}(E) \times E$ over $\mathbf{P}(E)$ and inherits projection maps $p_1 : [p_E]^*E \to \mathbf{P}(E)$ and $p_2 : [p_E]^*E \to E$ such that the following diagram is commutative:

$$\begin{array}{ccc} [p_E]^*E & \stackrel{p_2}{\longrightarrow} & E \\ \downarrow p_1 & & \downarrow p_E \\ \mathbf{P}(E) & \stackrel{[p_E]}{\longrightarrow} & M \end{array}$$

The tautological line bundle, denoted $\mathcal{L}^{-1}_{\mathbf{P}(E)}$, is a sub-bundle of $[p_E]^*E$ defined by:

$$\mathcal{L}^{-1}_{\mathbf{P}(E)} = \{((z,[v]),\lambda v) \in [p_E]^*E \,|\, (z,[v]) \in \mathbf{P}(E),\ \lambda \in \mathbf{C}\}. \tag{3.5}$$

The dual, denoted $\mathcal{L}_{\mathbf{P}(E)}$, shall be referred to as the "hyperplane bundle" over $\mathbf{P}(E)$. We shall often write, for simplicity, $\mathcal{L}$ instead of $\mathcal{L}_{\mathbf{P}(E)}$. The following definition is standard:

Definition 3.1. *Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold M. The dual bundle E^* is said to be ample (nef) if the line bundle $\mathcal{L}_{\mathbf{P}(E)}$ over $\mathbf{P}(E) = E_*/\mathbf{C}^*$ is ample (resp. nef).*

If the base manifold is compact then it is well-known that the existence of an ample vector bundle over M implies that M is projective.

Let P be the position vector field on TE then the function (see (2.15))

$$||P(z,v)||_h^2 = \sum_{i,j=1}^{r} h_{i\bar{j}}(z) v^i \bar{v}^j$$

is globally well-defined on E and is non-vanishing outside the zero section hence $\log ||P(z,v)||_h^2$ is well-defined on E_*. Moreover, since

$$\log ||P(z,\lambda v)||_h^2 = \log ||P(z,v)||_h^2 + \log |\lambda|^2$$

for all $\lambda \in \mathbf{C}^*$ the $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log ||P||_h^2$ descends to a well-defined $(1,1)$-form ϕ on $\mathbf{P}(E)$. Indeed , we may consider $||P||_h$ as a metric along the fibers of the tautological line bundle $\mathcal{L}^{-1}$ and

$$\frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial \log ||P||_h^2$$

descends to $-\phi = c_1(\mathcal{L}^{-1})$, the first Chern form of the line bundle $\mathcal{L}^{-1}$; equivalently,

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} \log ||P||_h^2$$

to $\phi = c_1(\mathcal{L})$ which is the first Chern form of the dual line bundle $\mathcal{L}$. Being a form on $\mathbf{P}(E)$ we have $\phi^{N-1} \equiv 0$ where $\dim \mathbf{P}(E) = \dim E - 1 = N - 1$ thus

$$(\partial\overline{\partial} \log ||P||_h^2)^{N-1} \equiv 0.$$

Indeed the position vector field is a zero eigen-vector of $\partial\overline{\partial} \log ||P||_h^2$, i.e.,

$$\iota_P \partial\overline{\partial} \log ||P||_h^2 = 0.$$

Theorem 3.2. *Let P be the position vector field on a holomorphic vector bundle (E, h) of rank $r \geq 2$ over a hermitian manifold (M, g) of dimension n. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial} \log ||P||_h^2$ descends to a well-defined form $\phi(= 2\pi c_1(\mathcal{L}_{\mathbf{P}(E)}))$ on $\mathbf{P}(E)$ moreover the following conditions are equivalent:*

(i) ϕ is positive definite (resp. positive semi-definite);

(ii) the mixed holomorphic bisectional curvature $k(X, P)$ of (E, h) is strictly negative (non-positive) for all non-zero $X \in TM$ and where P is the position vector field along the fibers of E.

Proof. Since

$$\partial\overline{\partial} \log ||P||_h^2 = \frac{\partial\overline{\partial}||P||_h^2}{||P||_h^2} - \frac{\partial||P||_h^2 \wedge \overline{\partial}||P||_h^2}{||P||_h^4} \tag{3.6}$$

we obtain in terms of a normal holomorphic frame at a point z^* (see Lemma 2.3),

$$\partial(\sum_{i,j=1}^{r} h_{i\bar{j}} v^i \bar{v}^j) = \sum_{i,j=1}^{r} \frac{\partial h_{i\bar{j}}}{\partial z_k} v^i \bar{v}^j dz_k + \sum_{i,j=1}^{r} h_{i\bar{j}} \bar{v}^j dv^i,$$

$$\overline{\partial}(\sum_{i,j=1}^{r} h_{i\bar{j}} v^i \bar{v}^j) = \sum_{i,j=1}^{r} \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_k} v^i \bar{v}^j d\bar{z}_k + \sum_{i,j=1}^{r} h_{i\bar{j}} v^i d\bar{v}^j$$

and from the computation of $\partial\overline{\partial}||P||_h^2$ in the last section (in the proof of Theorem 2.4),

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial} \log ||P||_h^2 = \sqrt{-1}&\frac{||P||_h^2 \sum_{i=1}^{r} dv^i \wedge d\bar{v}^i - \sum_{i,j=1}^{r} \bar{v}^j v^i dv^i \wedge d\bar{v}^j}{||P||_h^4} \\ &- \sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2}\end{aligned}$$

where K is the hermitian curvature of h. The first term on the right is $[\,]^*\omega_{FS}$ where ω_{FS} is the Fubini-Study metric of the fiber $\mathbf{P}(E_{z^*})$ and $[\,] : E_* \to \mathbf{P}(E)$ is the quotient map. Thus

$$\sqrt{-1}\partial\overline{\partial} \log ||P||_h^2 = [\,]^*\omega_{FS} - \sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2} \tag{3.7}$$

from which we infer easily that ϕ is positive definite in the fiber direction (the Fubini-Study metric is positive definite and, from (2.16), the second term is the pull-back of a form on M) and is positive definite in the base direction (by Theorem 2.4) if and only if the mixed bisectional curvature:

$$k(X,P) = \frac{< K(X,\overline{X})P, P >_h}{||X||_g^2 ||P||_h^2}$$

is strictly negative for non-zero $X \in TM$. This shows that (i) and (ii) are equivalent. QED

Since $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log ||P||_h^2$ is the first Chern form of $\mathcal{L}_{\mathbf{P}(E)}$ and the positivity of the Chern form implies the ampleness of $\mathcal{L}$ (which, by definition 2.1 this is equivalent to the ampleness of E^*). In other words, either of the condition implies that E^* is ample (resp. nef). A necessary and sufficient condition will be established in section 5. Actually the proof above says a little more, namely we get from (3.7) that $c_1(\mathcal{L}_{\mathbf{P}(E)})$ is non-negative if and only if the mixed holomorphic bisectional curvature $k(X,P)$ is bounded from above by zero.

Note that the hermitian curvature matrix is skew hermitian, consequently the mixed holomorphic bisectional curvature of a holomorphic hermitian bundle (E,h) is positive (resp. negative) if and only if the mixed holomorphic bisectional curvature of its dual (E^*,h^*) is negative (resp. positive). Theorem 3.2 applied to the dual (E^*,h^*) of (E,h) yields:

Theorem 3.3. *Let P^* be the position vector field on (E^*,h^*) the dual of a holomorphic vector bundle (E,h) of rank $r \geq 2$ over a complex hermitian manifold (M,g) of dimension n. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log ||P^*||_{h^*}^2$ descends to a well-defined form $\psi(= 2\pi c_1(\mathcal{L}_{\mathbf{P}(E^*)}))$ on $\mathbf{P}(E^*)$ moreover the following conditions are equivalent:*

(i) ψ is positive definite (resp. positive semi-definite) ;

(ii) the mixed holomorphic bisectional curvature of (E^*,h^*) is strictly negative (resp. non-positive);

(iii) the mixed holomorphic bisectional curvature of (E,h) is strictly positive (resp. non-negative);

Each of these conditions implies that E is ample (resp. nef).

Just as in section 3 the condition on the mixed bisectional curvature in Theorems 3.2 and 3.3 is reduced to the usual bisectional curvature if $(E,h) = (TM,g)$.

Fix hermitian metrics (M,g) and (E,h) and let $\mu_{g,h}$ be the supremum of the mixed holomorphic bisectional curvature $k(X,P)$; more precisely:

$$m_{g,h}(x) = \sup_{X\in T_xM, \sigma\in E_x} k(X,P), \quad \mu_{g,h} = \sup_{x\in M} m_{g,h}. \tag{3.8}$$

Obviously the function $m_{g,h}(x)$ is continuous and, if M is compact, $\mu_{g,h}$ is a finite constant. However $\mu_{g,h}$ may be infinite if M is non-compact, in which case it is necessary to work with $m_{g,h}(x)$. In any case, we have, by definition:

$$\sqrt{-1}\frac{< K(\cdot,\cdot)\sigma,\sigma >_h}{||\sigma||_h^2} \leq \mu_{g,h}\,\omega_g$$

where ω_g is the fundamental form associate to the metric g. This together with (3.7) implies that, for $\mu_{g,h}$ finite

$$c_1(\mathcal{L}_{\mathbf{P}(E)}) + c[p_E]^*\omega \geq \omega_{FS} + c[p_E]^*\omega - \mu_{g,h}[p_E]^*\omega_g.$$

($[p_E] : \mathbf{P}(E) \to M$ is the projection map) for any constant c and any $(1,1)$-form ω on M. If M is compact Kähler we may take ω to be a Kähler form on M then there exists a constant $c >> 0$ such that $c\omega - \mu_{g,h}\omega_g$ is positive definite (in particular, if ω_g is Kähler we mat take $\omega = \omega_g$ and c to be any number strictly larger than $\mu_{g,h}$). With this choice we see from the preceding inequality that $c_1(\mathcal{L}_{\mathbf{P}(E)}) + c[p_E]^*\omega$ is positive definite as ω_{FS} is positive semi-definite and is positive definite in the fiber directions while $c[p_E]^*\omega - \mu_{g,h}[p_E]^*\omega_g$ is positive semi-definite and is positive definite in the horizontal directions. If in addition ω is Kähler then $c_1(\mathcal{L}_{\mathbf{P}(E)}) + c[p_E]^*\omega$ is a Kähler metric on $\mathbf{P}(E)$. This is equivalent to the condition that $\sqrt{-1}\partial\overline{\partial}\log||P||_h^2 + cp_E^*\omega$ ($p_E : E \to M$ is the projection map) is positive semi-definite on $E \setminus \{\text{zero-section}\}$ and is positive definite in all directions transversal to the radial direction (i.e., the direction spanned by the position vector field P. This is equivalent to the condition that $\sqrt{-1}\partial\overline{\partial}||P||_h^2 + cp_E^*\omega$ is positive definite on E.

The construction above, however, does not work if M is non-compact because $\mu_{g,h}$ may be infinite. In the non-compact case we have to deal with the function $m_{g,h}$. This can be done if M is Stein (i. e., there exists a strictly plurisubharmonic exhaustion function f on M) or, slightly more generally, by dropping the exhaustion condition (i.e., each of the level set of f is compact) on the strictly plurisubharmonic function f. On such M we may take the Kähler form to be the Levi form of f:

$$\omega_g = \sqrt{-1}\partial\overline{\partial}f.$$

By definition, at any point x

$$-\sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2} \leq -m_{g,h}\sqrt{-1}\partial\overline{\partial}f = -m_{g,h}\omega_g.$$

Let $\chi : \mathbf{R} \to \mathbf{R}$ be a positive convex increasing function then

$$\sqrt{-1}\partial\overline{\partial}(\chi \circ f) = \chi^{'}(f)\sqrt{-1}\partial\overline{\partial}f + \chi^{''}(f)\sqrt{-1}\partial f \wedge \overline{\partial}f \geq \chi^{'}(f)\sqrt{-1}\partial\overline{\partial}f$$

thus, by choosing χ such that

$$\chi^{'}(f(x)) > |m_{g,h}(x)| \tag{3.9}$$

(it is clear that such a function χ exists) then

$$-\sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2} \leq |m_{g,h}|\omega_g \leq \sqrt{-1}\partial\overline{\partial}(\chi \circ f)$$

and consequently,

$$\begin{aligned} & c_1(\mathcal{L}_{\mathbf{P}(E)}) + [p_E]^*\sqrt{-1}\partial\overline{\partial}(\chi \circ f) \\ & = \omega_{FS} + [p_E]^*\sqrt{-1}\partial\overline{\partial}(\chi \circ f) - \sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2} \end{aligned}$$

is positive definite on $\mathbf{P}(E)$ and

$$\sqrt{-1}\partial\overline{\partial}||P||^2 + p_E^*\sqrt{-1}\partial\overline{\partial}(\chi \circ f)$$

is positive definitive on E. We summarized the above in the following Theorem:

Corollary 3.5. *Let (M, g) be a Kähler manifold and $[p_E] : (E, h) \to M$ be a hermitian holomorphic vector bundle, of rank ≥ 2 over M. If M is compact then there exists a constant $c > 0$ such that $\sqrt{-1}\partial\overline{\partial}||P||_h^2 + cp_E^*\omega_g$ is a Kähler metric on the bundle space E. Here P is the position vector field on E and ω_g is the Kähler form associate to the metric g. If M is non-compact and admits a strictly plurisubharmonic function f then there exists an increasing convex function $\chi : \mathbf{R} \to \mathbf{R}$ such that $\sqrt{-1}\partial\overline{\partial}||P||_h^2$ $+p_E^*\sqrt{-1}\partial\overline{\partial}(\chi \circ f)$ is a Kähler metric on E.*

The case of a general non-compact Kähler manifold remains open.

Let $[p_E]_* : T\mathbf{P}(E) \to TM$ be the differential of the projection $[p_E] : \mathbf{P}(E) \to M$. The projectivized vertical sub-bundle $[\mathcal{V}_E] = \ker[p_E]_*$ consisting of tangent vectors of $\mathbf{P}(E)$ tangential to the fibers of $[p_E]$. Note that $\ker p_E = \mathcal{V}_E$ where $p_E : TE \to TM$ and $[\mathcal{V}_E] = [\ ]_*\mathcal{V}_E$ where $[\ ] : E \setminus \{\text{zero} - \text{section}\} \to \mathbf{P}(E)$ is the quotient map (see (3.2)). The kernel of the quotient map is spanned by the position vector field P hence $[\mathcal{V}_E] = \mathcal{V}_E / < P >_{\mathbf{C}}$. There is an exact sequence (which shall be referred to as the Euler sequence over $\mathbf{P}(E)$ see []):

$$0 \to \mathbf{C} \to [p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)} \xrightarrow{\rho} [\mathcal{V}_E] \to 0 \tag{3.10}$$

where $\mathcal{L}_{\mathbf{P}(E)}$ is the "hyperplane" bundle as defined in (3.5) and $\mathbf{C}$ is the trivial line bundle spanned by the tautological section

$$\tau(z, [v]) = \sum_{i=1}^{r} v^i \otimes e_i$$

with $v = \sum_i v^i e_i$ (if M is a single point then $E = \mathbf{C}^r$ and the preceding reduces to the classical Euler sequence for projective space is the exact sequence (see [])

$$0 \to \mathbf{C} \to \mathbf{C}^r \otimes \mathcal{L}_{\mathbf{P}^{r-1}} = \oplus^r \mathcal{L}_{\mathbf{P}^{r-1}} \to T\mathbf{P}^{r-1} \to 0$$

where $\mathcal{L}_{\mathbf{P}^{r-1}}$ is the hyperplane bundle on $\mathbf{P}^{r-1}$). The homomorphism ρ in (3.10) is given as follows. A local section σ of $[p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)}$ is of the form

$$\sigma(z, [v]) = \sum_{i=1}^{r} \sigma_i(z, [v]) \otimes e_i(z, [v])$$

where each σ_i is a local section of $\mathcal{L}_{\mathbf{P}(E)}$. The section σ determines a vector field on $[p_E]^* E$:

$$V_\sigma = \sum_{i=1}^{r} \sigma_i(z, [v^1, ..., v^r]) \otimes \frac{\partial}{\partial v^i}$$

which, by definition, is vertical ($v_1, , , , , v_n$ are fiber coordinates) and

$$\rho(\sigma) \stackrel{\text{def}}{=} [\ ]_* V_\sigma.$$

It is clear that the kernel of ρ is spanned by the tautological section (corresponding to the position vector field), i.e.,

$$\tau = \sum_{i=1}^{r} v^i \otimes e_i \Leftrightarrow V_\tau = P = \sum_{i=1}^{r} v^i \otimes \frac{\partial}{\partial v^i}$$

hence $[\ ]_* V_\tau = [\ ]_* P = 0$. Denote by $[\mathcal{V}_E^*]$ the dual of $\mathcal{V}_E$. The Euler sequence implies that

$$c_1([\mathcal{V}_E^*]) = -c_1([p_E]^* E \otimes \mathcal{L}_{\mathbf{P}(E)}) = -c_1([p_E]^* E) - rc_1(\mathcal{L}_{\mathbf{P}(E)}) \tag{3.11}$$

where we have also used the case $k = 1$ of the following identity for Chern classes of tensor product of a line bundle $\mathcal{F}$ and a rank r vector bundle $\mathcal{E}$:

$$c_k(\mathcal{E} \otimes \mathcal{F}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(\mathcal{E}) c_1^{k-i}(\mathcal{F}).$$

Note that in the classical Euler sequence this reduces to (as E is the trivial bundle) the well-known fact that $c_1(\mathcal{K}_{\mathbf{P}^{r-1}}) = -rc_1(\mathcal{L}_{\mathbf{P}^{r-1}})$. On the other hand, we have (by definition) an exact sequence:

$$0 \to [\mathcal{V}_E] \to T\mathbf{P}(E) \to TE/[\mathcal{V}_E] \to 0$$

where $T\mathbf{P}(E)/[\mathcal{V}_E]$ is $\mathcal{C}^\infty$-isomorphic to the horizontal sub-bundle $\mathcal{H}_E$ (which is holomorphically isomorphic to TM under the map $[p_E]_* : T\mathbf{P}(E) \to TM$). By duality we get a $\mathcal{C}^\infty$ exact sequence:

$$0 \to [p_E]^* T^* M \to T^* \mathbf{P}(E) \to [\mathcal{V}_E^*] \to 0$$

which implies that $c_1(T^*\mathbf{P}(E)) = c_1([p_E]^* T^* M) + c_1(\mathcal{V}_E^*)$, i.e.,

$$c_1(\mathcal{K}_{\mathbf{P}(E)}) = c_1([p_E]^* \mathcal{K}_M) + c_1([\mathcal{V}_E^*]) \tag{3.12}$$

where $\mathcal{K}_{\mathbf{P}(E)}$ and $\mathcal{K}_M$ are the canonical bundles of $\mathbf{P}(E)$ and M respectively. The preceding identities imply (cf. Griffiths [?], Kobayashi-Ochiai [?]):

Theorem 3.8. *For any holomorphic vector bundle E of rank $r \geq 2$ over a complex manifold M, we have*

$$\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M \otimes \det E^*) \otimes \mathcal{L}_{\mathbf{P}(E)}^{-r}$$

where $\mathcal{L}_{\mathbf{P}(E)}^{-r}$ is the dual of the r-fold tensor product of the "hyperplane bundle" $\mathcal{L}_{\mathbf{P}(E)}$; consequently, we have

$$c_1(\mathcal{K}_{\mathbf{P}(E)}) = [p_E]^* \{c_1(\mathcal{K}_M) + c_1(\det E^*)\} - rc_1(\mathcal{L}_{\mathbf{P}(E)}).$$

Proof. Recall that an exact sequence of vector bundles

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

induces an isomorphism: $\det E_1 \otimes \det E_3 \cong \det E_2$. We get from the dual Euler sequence (see (3.10))

$$\det \mathcal{V}_E^* \cong [p_E]^* \det E^* \otimes \mathcal{L}_{\mathbf{P}(E)}^{-r}.$$

On the other hand, by (3.11), it is clear that

$$\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M) \otimes \det \mathcal{V}_E^*$$

hence

$$\mathcal{K}_{\mathbf{P}(E)} \cong [p_E]^*(\mathcal{K}_M) \otimes [p_E]^* \det E^* \otimes \mathcal{L}_{\mathbf{P}(E)}^{-r}$$

as claimed. QED

Let $E = TM$ in the preceding Theorem then $\det E^* = \mathcal{K}_M$ and we get

$$\mathcal{K}_{\mathbf{P}(TM)} \cong [p_{TM}]^* \mathcal{K}_M^2 \otimes \mathcal{L}_{\mathbf{P}(TM)}^{-n}$$

where $n = \dim M$; equivalently, $\mathcal{K}_{\mathbf{P}(TM)}^{-1} \cong [p_{TM}]^* \mathcal{K}_M^{-2} \otimes \mathcal{L}_{\mathbf{P}(TM)}^{n}$. Let $E = T^*M$ in the preceding Theorem then $\det E^* = \mathcal{K}_M^{-1}$ and we get

$$\mathcal{K}_{\mathbf{P}(T^*M)} \cong \mathcal{L}_{\mathbf{P}(T^*M)}^{-n}$$

where $n = \dim M$; equivalently, $\mathcal{K}_{\mathbf{P}(T^*M)}^{-1} \cong \mathcal{L}_{\mathbf{P}(T^*M)}^{n}$.

Corollary 3.9. *Let M be a complex manifold then $\mathcal{K}_{\mathbf{P}(T^*M)}^{-1} \cong \mathcal{L}_{\mathbf{P}(T^*M)}^{n}, n =$ dimension of M, hence T^*M is ample if and only if the anti-canonical bundle $\mathcal{K}_{\mathbf{P}(T^*M)}^{-1}$ of $\mathbf{P}(T^*M)$ is ample.*

4. Finsler metrics

Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold M and let $\mathcal{L}^{-1} = \mathcal{L}_{\mathbf{P}(E)}^{-1}$ over $\mathbf{P}(E)$ be the "tautological" line bundle. By definition the bundle space of $\mathcal{L}^{-1}$ is the blowing up of the bundle space of E along the zero section. Thus there is a canonical isomorphism

$$\mathcal{L}^{-1} \setminus \{\text{zero} - \text{section}\} \cong E \setminus \{\text{zero} - \text{section}\}$$

compatible with the respective $\mathbf{C}^*$ structure associated to the respective bundle structures. Let H be a hermitian metric along the fibers of $\mathcal{L}^{-1}$ which, via the preceding isomorphism, determines uniquely a function

$$h : E \to \mathbf{R}_{\geq 0},\ h(z,v) = ||\beta^{-1}(z,v)||_H \tag{4.1}$$

(where $\beta : \mathcal{L}^{-1} \to E$ is the blowing up map) with the following properties:

(FM1) *h is of class $\mathcal{C}^0$ on E and is of class $\mathcal{C}^\infty$ on $E \setminus \{\text{zero} - \text{section}\}$;*

(FM2) *$h(z, \lambda v) = |\lambda| h(z,v)$ for all $\lambda \in \mathbf{C}$;*

(FM3) *$h(z,v) > 0$ on $E \setminus \{\text{zero} - \text{section}\}$;*

(FM4) *for z and v fixed the function $h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.*

A function on E satisfying properties (FM1), (FM2), (FM3) and (FM4) above shall be referred to as a *Finsler metric* of class $\mathcal{C}^\infty$. In the literature some authors do not include property (FM4) in the definition. With condition (FM4) a Finsler metric on E determines uniquely a hermitian metric along the fibers of $\mathcal{L}^{-1}$ via (5.1). *Note that we do not require that $h^2(z,v)$ be of class $\mathcal{C}^\infty$ along the zero-section; indeed, with this extra condition the Finsler metric is the norm of a hermitian metric along the fibers of E and we are in the situation of section 2.* A Finsler metric is said to be *strictly* *pseudoconvex along the fibers* if the following condition is satisfied:

(FM5) $h|_{E_z}$ *is a strictly pseudoconvex function on* $E_z \setminus \{0\}$ *for all* $z \in M$.

Note that we require that F be strictly pseudoconvex only in the fiber directions. Let $G = h^2$ then

$$G(z, \lambda v) = |\lambda|^2 G(z, v) \tag{4.2}$$

for all $\lambda \in \mathbf{C}$. Taking $\lambda = e^{\sqrt{-1}\theta}$ yields $G(z, e^{\sqrt{-1}\theta}v) = G(z, v)$ and we see that a level set $\{G = c\}$(c a constant) of G is invariant by the circle action. A set invariant under the circle action is said to be *circular*. We derive some basic formulas of the derivatives of G which are needed in later calculations. *It is understood that all differentiations are carried out off the zero section.* It is clear that the homogeneity property (5.2) remains valid for all partial derivatives of G in the base variables, i.e.,

$$\frac{\partial^{a+b} G}{\partial z_{\alpha_1} ... \partial z_{\alpha_a} \partial \bar{z}_{\beta_1} ... \bar{z}_{\beta_b}}(z, \lambda v) = |\lambda|^2 \frac{\partial^{a+b} G}{\partial z_{\alpha_1} ... \partial z_{\alpha_a} \partial \bar{z}_{\beta_1} ... \bar{z}_{\beta_b}}(z, v). \tag{4.3}$$

On the other hand, differentiating (4.2) with respect to the fiber variables $v = (v^1, ..., v^r)$ yields (for $\lambda \in \mathbf{C}^*, v \in E_z \setminus \{0\}$)

$$\frac{\partial G}{\partial v^i}(z, \lambda v) = \overline{\lambda} \frac{\partial G}{\partial v^i}(z, v), \ \frac{\partial G}{\partial \overline{v}^i}(z, \lambda v) = \lambda \frac{\partial G}{\partial \overline{v}^i}(z, v). \tag{4.4}$$

The identities (4.2), (4.3) and (4.4) imply that

$$\sum_{i=1}^r \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) dz^i = \sum_{i=1}^r \frac{1}{G(z, v)} \frac{\partial G}{\partial z^i}(z, v) dz^i,$$

$$\sum_{i=1}^r \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial v^i}(z, \lambda v) d(\lambda v^i) = \sum_{i=1}^r \frac{1}{G(z, v)} \frac{\partial G}{\partial v^i}(z, v) dv^i.$$

In other words, $\partial_z \log G, \partial_v \log G$ (resp. $\overline{\partial}_z \log G, \overline{\partial}_v \log G$) are invariant by the $\mathbf{C}^*$ action on E where ∂_z, ∂_v (resp. $\overline{\partial}_z, \overline{\partial}_v$) are the base component and fiber component of the operator ∂ (resp. $\overline{\partial}$) on E. More precisely, we have

$$\lambda^*(\partial_z \log G(z, v)) = \sum_{i=1}^r \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) dz^i = \partial_z \log G(z, v),$$

$$\lambda^*(\partial_z \log G(z, v)) = \sum_{i=1}^r \frac{1}{G(z, \lambda v)} \frac{\partial G}{\partial z^i}(z, \lambda v) d_(\lambda v^i) = \partial_v \log G(z, v)$$

and, as G is real-valued, a similar set of formulas for $\partial_{\bar{z}} \log G(z, v)$. This also implies that the level set $\{G(z, v) = \mathrm{c}\}$ is non-singular for any *positive* constant c. The identity (4.2) and (4.3) imply in particular that

$$\frac{1}{G(z, \lambda v)} \frac{\partial^2 G}{\partial z^i \partial \bar{z}^j}(z, \lambda v) = \frac{1}{G(z, v)} \frac{\partial^2 G}{\partial z^i \partial \bar{z}^j}(z, v) \tag{4.5}$$

and differentiating the first identity of (5.4) with respect to $\bar{v}^j$ yields:

$$\frac{\partial^2 G}{\partial v^i \partial \overline{v}^j}(z, \lambda v) = \frac{\partial^2 G}{\partial v^i \partial \overline{v}^j}(z, v), \tag{4.6}$$

in other words, both $G^{-1}\partial_z\overline{\partial}_z G$ and $G^{-1}\partial_v\overline{\partial}_v G$ are invariant by the the $\mathbf{C}^*$ action.. Since

$$\partial_z\overline{\partial}_z \log G = \frac{1}{G}\partial_z\overline{\partial}_z G - \partial_z \log G \wedge \overline{\partial}_z \log G,$$
$$\partial_v\overline{\partial}_v \log G = \frac{1}{G}\partial_v\overline{\partial}_v G - \partial_v \log G \wedge \overline{\partial}_v \log G$$

we conclude, from the preceding calculations that both $\partial_z\overline{\partial}_z \log G$ and $\partial_v\overline{\partial}_v \log G$ are invariant by the the $\mathbf{C}^*$ action. To deal with mixed derivatives we differentiate (r.4) with respect to z then

$$\frac{\partial^2 G}{\partial v^i \partial z^{\overline{\beta}}}(z,\lambda v) = \overline{\lambda}\frac{\partial^2 G}{\partial v^i \partial z^{\overline{\beta}}}(z,v), \ \frac{\partial^2 G}{\partial z^{\alpha} \partial \overline{v}^j}(z,\lambda v) = \lambda\frac{\partial^2 G}{\partial z^{\alpha} \partial \overline{v}^j}(z,v). \tag{4.7}$$

and these imply that $G^{-1}\partial_v\overline{\partial}_z G$ and $G^{-1}\partial_z\overline{\partial}_v G$ are both invariant by the $\mathbf{C}^*$ action:

$$\lambda^*(\partial_v\overline{\partial}_z G(z,v)) = \sum \frac{1}{G(z,\lambda v)}\frac{\partial^2 G}{\partial v^i \partial \bar{z}^j}(z,\lambda v) d(\lambda v^i) \wedge d\bar{z}^j = \partial_v\overline{\partial}_z G(z,v),$$
$$\lambda^*(\partial_v\overline{\partial}_z G(z,v)) = \sum \frac{1}{G(z,\lambda v)}\frac{\partial^2 G}{\partial z^i \partial \bar{v}^j}(z,\lambda v) d\bar{z}^j \wedge d(\lambda \bar{v}^j) = \partial_z\overline{\partial}_v G(z,v).$$

Since

$$\begin{aligned}\partial\overline{\partial}\log G &= \frac{1}{G}\partial\overline{\partial} G - \partial \log G \wedge \overline{\partial}\log G \\ &= \frac{1}{G}(\partial_z\overline{\partial}_z G + \partial_v\overline{\partial}_v G + \partial_v\overline{\partial}_z G + \partial_z\overline{\partial}_v G) \\ &\quad - (\partial_z \log G + \partial_v \log G) \wedge (\overline{\partial}_z \log G + \overline{\partial}_v \log G)\end{aligned}$$

we conclude that $\partial\overline{\partial}\log G$ is also invariant by the $\mathbf{C}^*$ action. These show that both $\partial\overline{\partial}\log G, \partial_z\overline{\partial}_z \log G$ and $\partial_v\overline{\partial}_v \log G$ descend to well-defined (1, 1)-forms on $\mathbf{P}(E)$. Moreover, if the Finsler metric is strictly pseudoconvex along the fibers then (5.3) implies that the restriction of the Levi-form $\partial_v\overline{\partial}_v \log G$, to the maximal complex tangent bundle of $\{G = c\} \cap E_z$, is strictly pseudoconvex for all $c > 0$. This is so because the maximal complex tangent bundle of $\{G = c\} \cap E_z$ is annihilated by $\partial_v \log G$. In other words we have shown that

Lemma 4.1. *Let F be a Finsler metric along the fibers of a holomorphic vector bundle, of rank ≥ 2, over a complex manifold. Then $\sqrt{-1}\partial\overline{\partial}\log G$ $(G = F^2)$ descends to a well-defined $(1,1)$-form ϕ on $\mathbf{P}(E)$. If G is strictly pseudoconvex then ϕ is positive definite when restrict to a fiber $\mathbf{P}(E_z)$.*

The final set of formulas are obtained by differentiating the identity (4.2) with respect to the variable λ and $\overline{\lambda}$ resulting in the identities

$$\sum_{i=1}^{n} v^i \frac{\partial G}{\partial v^i}(z,\lambda v) = \overline{\lambda} G(z,v); \sum_{i=1}^{n} \bar{v}^i \frac{\partial G}{\partial \bar{v}^i}(z, \lambda v = \lambda G(z,v); \tag{4.8}$$

in particular, we have

$$PG(z,\lambda v)=\sum_{i=1}^{n} v^i \frac{\partial G}{\partial v^i}(z,v)=G(z,v)=\sum_{i=1}^{n}\bar v^i \frac{\partial G}{\partial \bar v^i}(z,v)=\overline{P}G(z,\lambda v).$$

where $P=\sum_i v^i\partial/\partial v^i$ is the position vector field. Differentiating (5.8) with respect to $\overline{\lambda}$ and using (5.7) yields

$$\sum_{i,j=1}^{r} v^i\bar v^j \frac{\partial^2 G}{\partial v^i\partial \bar v^j}(z,\lambda v)=\sum_{i,j=1}^{r} v^i\bar v^j \frac{\partial^2 G}{\partial v^i\partial \bar v^j}(z,v)=G(z,v); \tag{4.9}$$

in other words $P\overline{P}G(z,\lambda v)=P\overline{P}G(z,v)=G(z,v)$. On the other hand, differentiating (5.7) with respect to λ yields

$$\sum_{i,j=1}^{r} v^i v^j \frac{\partial^2 G}{\partial v^i\partial v^j}(z,\lambda v)=\sum_{i=1}^{r} v^i v^j \frac{\partial^2 G}{\partial v^i\partial v^j}(z,v)=0. \tag{4.10}$$

Note that

$$\sum_{i=1} v^i\frac{\partial}{\partial v^i}\sum_{j=1}^{r} v^j\frac{\partial G}{\partial v^j}=\sum_{j=1}^{r} v^j\frac{\partial G}{\partial v^j}+\sum_{i,j=1}^{r} v^i v^j\frac{\partial^2 G}{\partial v^i\partial v^j}$$

hence, we see from (4.8) that (4.10) is equivalent to the condition that

$$P^2G(z,\lambda v)=\overline{\lambda}G(z,v). \tag{4.11}$$

Inductively we get

$$\sum_{i_1,\dots,i_k=1}^{r} v^{i_1}\dots v^{i_k}\frac{\partial^k G}{\partial v^{i_1}\dots\partial v^{i_k}}=0 \tag{4.12}$$

for $k\geq 2$ and

$$\sum_{i_1,\dots,i_k,j_1,\dots,j_l=1}^{r} v^{i_1}\dots v^{i_k}\bar v^{j_1}\dots\bar v^{j_l}\frac{\partial^{k+l} G}{\partial v^{i_1}\dots\partial v^{i_k}\partial\bar v^{j_1}\dots\partial\bar v^{j_l}}=0 \tag{4.13}$$

for $k\geq 2, l\geq 0$. In fact we see, by differentiating (4.6) with respect to λ (resp. $\overline{\lambda}$) that:

$$\sum_{k=1}^{r} v^k\frac{\partial^3 G}{\partial v^i\partial\bar v^j\partial v^k}=0=\sum_{k=1}^{r}\bar v^l\frac{\partial^3 G}{\partial v^i\partial\bar v^j\bar v^l}.$$

and hence

$$P^k\sum_{i,j=1}^{r}\frac{\partial^2 G}{\partial v^i\partial\bar v^j}=0 \tag{5.14}$$

for all $k\geq 1$ where $P=\sum v^i\partial/\partial v^i$ is the position vector field. Note that, by property (FM4) of a Finsler metric, formulas (4.7)-(4.14) are valid even at the zero section.

If h is a hermitian metric along the fibers of E:

$$< z,w >_h=\sum_{i,j} h_{i\bar j}(z)v^i\bar w^j$$

then its norm

$$F(z,v) = (\sum_{i,j} h_{i\bar{j}}(z) v^i \bar{v}^j)^{1/2}$$

is a Finsler metric strictly pseudoconvex along the fibers with the following additional properties:

(FM 6) $G = F^2$ *is smooth even at the zero-section,*

(FM 7) $G_{i\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^j}$ *is independent of* v *for all* i, j.

Indeed either of these properties characterizes the norm of a hermitian metric on E (see, for example []):

Lemma 4.2. *A Finsler metric* F *is the norm of a hemitian metric on* E *iff the function* $G = F^2$ *is smooth at the zero section iff the functions* $\{G_{i\bar{j}}, 1 \leq i, j \leq r\}$ *are independent of* v.

Given a Finsler metric F which is stricly pseudoconvex along the fibers we define a hermitian inner product on the vertical bundle $\mathcal{V} \subset TE$ by:

$$< V, W >_{\mathcal{V}} = \sum_{i,j=1}^{r} G_{i\bar{j}}(z,v) V^i \overline{W}^j, \quad G_{i\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^j} \tag{4.15}$$

for $V = \sum_i V^i \partial/\partial v^i, W = \sum_i W^i \partial/\partial v^i \in \mathcal{V}$. In section 2 the hermitian inner product along the fibers of the vertical bundle is defined by a hermitian metric on E and formula (2.4) is the same as (4.15) *except that the functions* G_{ij} *in (2.4) depend only on the base coordinates* $(z^1, ..., z^n)$ *but not on the fiber coordinate* $(v^1, ..., v^r)$. The hermitian inner product (4.15) defines uniquely a hermitian connection (known as the *Chern connection*) $\nabla^{\mathcal{V}}$ and the connection forms are given by (compare (2.5)):

$$\theta_i^k = \sum_{j=1}^{r} (\partial G_{i\bar{j}}) G^{\bar{j}k} = \sum_{\alpha=1}^{n} \Gamma_{i\alpha}^k dz^\alpha + \sum_{l=1}^{r} \gamma_{il}^k dv^l \tag{4.16}$$

where the horizontal and vertical Christoffel symbols are given respectively by:

$$\Gamma_{i\alpha}^k = \sum_{j=1}^{r} \frac{\partial G_{i\bar{j}}}{\partial z^\alpha} G^{\bar{j}k} \text{ and } \gamma_{il}^k = \sum_{j=1}^{r} \frac{\partial G_{i\bar{j}}}{\partial v^l} G^{\bar{j}k}.$$

If F comes from a hermitian metric then, by Lemma 4.2, the vertical Christoffel symbols γ_{il}^k vanish and (4.16) reduces to (2.5). The curvature forms of a hermitian connection are always of type (1, 1) hence

$$\Theta_i^k = d\theta_i^k - \sum_{j=1}^{r} \theta_i^l \wedge \theta_l^k = d\theta_i^k + \sum_{j=1}^{r} \theta_l^k \wedge \theta_i^l = \overline{\partial}\theta_i^k; \tag{4.17}$$

equivalently

$$\partial\theta_i^k - \sum_{j=1}^{r} \theta_i^l \wedge \theta_l^k = \partial\theta_i^k + \sum_{j=1}^{r} \theta_l^k \wedge \theta_i^l = 0.$$

These formulas are the same as (2.7) and (2.8) except that there are now horizontal, vertical and mixed components:

$$\Theta_i^k = \sum_{\alpha,\beta=1}^{n} K^k_{i\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta + \sum_{j,l=1}^{r} \kappa^k_{ij\bar l} dv^j \wedge d\bar v^l$$
$$+ \sum_{\alpha=1}^{n}\sum_{l=1}^{r} \mu^k_{i\alpha\bar l} dz^\alpha \wedge d\bar v^l + \sum_{j=1}^{r}\sum_{\beta=1}^{n} \nu^k_{ij\bar\beta} dv^j \wedge d\bar z^\beta.$$

The components are given as follows:

$$K^k_{i\alpha\bar\beta} = -\frac{\partial\Gamma^k_{i\alpha}}{\partial\bar z^\beta},\ \kappa^k_{ij\bar l} = -\frac{\partial\gamma^k_{ij}}{\partial\bar v^l},\ \mu^k_{i\alpha\bar l} = -\frac{\partial\Gamma^k_{i\alpha}}{\partial\bar v^l},\ \nu^k_{ij\bar\beta} = -\frac{\partial\gamma^k_{ij}}{\partial\bar z^\beta}. \tag{4.18}$$

As in section 2 (compare (2.9)) the connection $\nabla^{\mathcal V}$ defines a surjection

$$\gamma : TE \to \mathcal{V},\ \ \gamma(X) = \nabla^{\mathcal V}_X P$$

for any $X \in TE$ and where $P = \sum_{i=1}^r v^i\partial/\partial v^i$ is the position vector field. This map is now more complicated, in terms of coordinates

$$\begin{aligned}\nabla^{\mathcal V}_X P &= \sum_{j=1}^{r}\{dv^j(X) + \sum_{i=1}^{r} v^i\theta^j_i(X)\}\frac{\partial}{\partial v^j}\\ &= \sum_{j=1}^{r}(b^j + \sum_{i=1}^{r}\sum_{\alpha=1}^{n}\Gamma^j_{i\alpha}v^ia^\alpha + \sum_{i,k=1}^{r}\gamma^j_{ik}v^ib^k)\frac{\partial}{\partial v^j}\\ &= \sum_{j=1}^{r}(b^j + \sum_{i=1}^{r}\sum_{\alpha=1}^{n}\Gamma^j_{i\alpha}v^ia^\alpha)\frac{\partial}{\partial v^j}\end{aligned} \tag{4.19}$$

for any vector field

$$X = \sum_{\alpha=1}^{n} a^\alpha(z;v)\frac{\partial}{\partial z^\alpha} + \sum_{i=1}^{r} b^i(z;v)\frac{\partial}{\partial v^i}.$$

The last identity of (4.19) follows from the definition (4.16) of γ^j_{ik} and identity (4.13). The kernel of γ is the horizontal sub-bundle $\mathcal H$. Given a vector field $V(z) = \sum_i a^i(z)\partial/\partial z^i$ on M the vector field

$$V^{\mathcal H}(z,v) = \sum_{i=1}^{n}\{a^i(z)\frac{\partial}{\partial z^i} - \sum_{i=1}^{r}\sum_{\alpha=1}^{n}\Gamma^i_{j\alpha}v^ia^\alpha)\frac{\partial}{\partial v^j}\}$$

is horizontal and shall be referred to as the *horizontal lifting* of V. The horizontal lifts of the local basis $\{\partial_\alpha = \partial/\partial z^\alpha, \alpha = 1,...,n\}$ of TE:

$$\{\partial^{\mathcal H}_\alpha = \frac{\partial}{\partial z^\alpha} - \sum_{j,k=1}^{r}\Gamma^k_{j\alpha}v^j\frac{\partial}{\partial v^k}\,|\,\alpha = 1,...,n\}$$

is a local basis of $\mathcal H$ and these together with $\{\partial^{\mathcal V}_i = \partial/\partial v^i, i = 1,...,r\}$ form a local basis for TE. Let g be a Finsler metric on M inducing a hermitian inner product

$< , >_{\mathcal{H}}$ along the fibers of $\mathcal{H}$ and we define an inner product $< >_{TE}$ along the fibers of TE by taking the direct sum:

$$< >_{TE} = < , >_{\mathcal{H}} + < , >_{\mathcal{V}} . \tag{4.20}$$

We shall alsp use the notation $< , >_g$ instead of $< , >_{\mathcal{H}}$ and $< , >_h$ instead of $< , >_{\mathcal{V}}$ indicating the fact that the inner product depends only on the Finsler metrics g and h respectively. Let P be the position vector field and, as is clear from the previous sections, the $(1,1)$-forms $\sqrt{-1}\partial\overline{\partial}||P||_h^2$ on E and $\sqrt{-1}\partial\overline{\partial}\log||P||_h^2$ on $E \setminus \{\text{zero-section}\}$. The expressions for these forms in terms of the metrics are formally the same but the computation is more complicated. We have

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}||P||_h^2 &= \sqrt{-1}\partial\overline{\partial}\sum_{i,j=1}^{r} G_{i\bar{j}}v^i\bar{v}^j \\
&= \sqrt{-1}\sum_{i,j=1}^{r}\{G_{i\bar{j}}dv^i\wedge d\bar{v}^j + v^i\bar{v}^j\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha\wedge d\bar{z}^\beta\} \\
&\quad + \sqrt{-1}\{\sum_{i,j=1}^{r}\{\sum_{\alpha=1}^{n} v^i\frac{\partial G_{i\bar{j}}}{\partial z^\alpha}dz^\alpha\wedge d\bar{v}^j + \sum_{l=1}^{n}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{z}^\beta}dv^i\wedge d\bar{z}^\beta\} \\
&\quad + \sqrt{-1}\{\sum_{i,j,k=1}^{r} v^i\frac{\partial G_{i\bar{j}}}{\partial v^k}dv^k\wedge d\bar{v}^j + \sum_{i,j,l=1}^{r}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{v}^l}dv^i\wedge d\bar{v}^l\} \\
&\quad + \sqrt{-1}\sum_{i,j,k,l=1}^{r} v^i\bar{v}^j\frac{\partial^2 G_{i\bar{j}}}{\partial v^k\partial\bar{v}^l}dv^k\wedge d\bar{v}^l.
\end{aligned}$$

By (4.9) the last 3 term on the right above vanish:

$$\sqrt{-1}\sum_{i,j,k=1}^{r} v^i\frac{\partial G_{i\bar{j}}}{\partial v^k}dv^k\wedge d\bar{v}^j = \sum_{i,j,k=1}^{r} v^i\frac{\partial G_{k\bar{j}}}{\partial v^i}dv^k\wedge d\bar{v}^j = 0,$$

$$\sqrt{-1}\sum_{i,j,l=1}^{r}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{v}^l}dv^i\wedge d\bar{v}^l = \sqrt{-1}\bar{v}^j\sum_{i,j,l=1}^{r}\frac{\partial G_{i\bar{l}}}{\partial\bar{v}^j}dv^i\wedge d\bar{v}^l = 0,$$

$$\sqrt{-1}\sum_{i,j,k,l=1}^{r} v^i\bar{v}^j\frac{\partial^2 G_{i\bar{j}}}{\partial v^k\partial\bar{v}^l}dv^k\wedge d\bar{v}^l = \sqrt{-1}\sum_{i,j,k,l=1}^{r} v^i\bar{v}^j\frac{\partial^2 G_{k\bar{l}}}{\partial v^i\partial\bar{v}^j}dv^k\wedge d\bar{v}^l = 0$$

thus we have:

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}||P||_h^2 &= \sqrt{-1}\sum_{i,j=1}^{r}\{G_{i\bar{j}}dv^i\wedge d\bar{v}^j + v^i\bar{v}^j\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha\partial\bar{z}^\beta}dz^\alpha\wedge d\bar{z}^\beta\} \\
&\quad + \sqrt{-1}\{\sum_{i,j=1}^{r}\{\sum_{\alpha=1}^{n} v^i\frac{\partial G_{i\bar{j}}}{\partial z^\alpha}dz^\alpha\wedge d\bar{v}^j + \sum_{l=1}^{n}\bar{v}^j\frac{\partial G_{i\bar{j}}}{\partial\bar{z}^\beta}dv^i\wedge d\bar{z}^\beta\}.
\end{aligned}$$

By Lemma 2.4 we may choose a local frame of E which is normal at any given point z^*, i.e., we may assume (as positivilty or negativity is independent of the choice of holomorphic frames) that $G_{i\bar{j}} = \delta_i^j, \partial G_{i\bar{j}}/\partial z^\alpha = 0$ and so

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\{\sum_{i=1}^r dv^i \wedge d\overline{v}^i + \sum_{i,j=1}^r v^i\overline{v}^j \sum_{\alpha,\beta=1}^n \frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha \wedge d\bar{z}^\beta\}$$

at the point z^*. Moreover, by (4.18), the second term on the right above is the base direction of the curvature hence,

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\sum_{i=1}^r \{dv^i \wedge d\overline{v}^i - \sum_{i,j=1}^r v^i\overline{v}^j \sum_{k,l=1}^n K_{i\bar{j}\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta\}.$$

For $\sigma = \sum_{i=1}^r v^i e_i, \tau = \sum_{i=1}^r \tau^i e_i$ in E_x (which maybe identified with the tangent vectors $\sum \sigma^i \partial/\partial v^i$ and $\sum \tau^j \partial/\partial v^j$) define the (1, 1)-form $< K(\cdot,\cdot)\sigma, \sigma >_h$:

$$< K(X,Y)\sigma, \tau >_h = \sum_{i,j,k=1}^r \sum_{\alpha,\beta=1}^n G_{j\bar{k}} K^j_{i\alpha\bar{\beta}} X^r \alpha \overline{Y}^s \sigma^i \overline{\tau}^k = \sum_{i,j=1}^r \sum_{\alpha,\beta=1}^n K_{i\bar{j}\alpha\bar{\beta}} X^r \overline{Y}^s \sigma^i \overline{\tau}^k$$

and tangent vectors X, Y of type $(1,0)$ on M and where $K^j_{i\alpha\bar{\beta}}$ is the base component of the curvature as defined in (4.18). The preceding computations show that

$$\sqrt{-1}\partial\overline{\partial}||P||_h^2 = \sqrt{-1}\sum_{i,j=1}^r G_{i\bar{j}} dv^i \wedge d\overline{v}^j - \sqrt{-1} < K(\cdot,\cdot)P, P >_h$$

The first term on the right above is an $(1,1)$-form in the fiber variables and is positive definite (in the fiber direction) by the assumption that the Finsler metric F is strictly pseudoconvex along the fibers. The second term is an $(1,1)$-form in the base variables, hence $\sqrt{-1}\partial\overline{\partial}||P||_h^2$ is positive definite if and only if the second term is also positive definite (in the base direction) on $E \setminus \{\text{zero} - \text{section}\}$. We define the (base component) of the holomorphic bisectional curvature by

$$k(X,P) = \frac{< K(X,X)P, P >}{||X||_g^2 ||P||_h^2}.$$

We have:

Theorem 4.3. *Let (E,h) be a Finsler holomorphic vector bundle over a complex Finsler manifold (M,g). Assume that h is strictly pseudoconvex along the fibers and let K be the curvature of the Chern connection associated to the Finsler metric h. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}||P||_{\mathcal{V}}^2$ is positive definite on $E \setminus \{zero - section\}$ if and only if the base component of the mixed holomorphic bisectional curvature is strictly negative in the direction of X and P and on $E \setminus \{zero - section\}$:*

$$< K(X,X)P, P > = \sum_{i,j=1}^r \sum_{k,l=1}^n K_{i\bar{j}\alpha\bar{\beta}} v^i \overline{v}^j X^\alpha \overline{X}^\beta < 0.$$

for all nonzero tangent vector X of type $(1,0)$ on M.

The expression for $\partial\overline{\partial}\log ||P||_h^2$ can now be carried ouy just as in section 3:

$$\partial\overline{\partial}\log ||P||_h^2 = \frac{\partial\overline{\partial}||P||_h^2}{||P||_h^2} - \frac{\partial||P||_h^2 \wedge \overline{\partial}||P||_h^2}{||P||_h^4}.$$

In terms of a normal holomorphic frame at a point z^*, we get

$$\partial(\sum_{i,j=1}^{r} G_{i\bar{j}} v^i \bar{v}^j) = \sum_{i=1}^{r} \bar{v}^i dv^i,$$
$$\overline{\partial}(\sum_{i,j=1}^{r} G_{i\bar{j}} v^i \bar{v}^j) = \sum_{j=1}^{r} v^j d\bar{v}^j$$

and from the computation in Theorem 5.3,

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2 &= \sqrt{-1}\frac{||P||_h^2 \sum_{i=1}^{r} dv^i \wedge d\bar{v}^i - \sum_{i,j=1}^{r} \bar{v}^j v^i dv^i \wedge d\bar{v}^j}{||P||_h^4} \\ &\quad - \sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_{\mathcal{V}}}{||P||_h^2} \\ &= [\;]^*\omega_{FS} - \sqrt{-1}\frac{< K(\cdot,\cdot)P, P >_h}{||P||_h^2}\end{aligned}$$

where K is defined as in (4.17), ω_{FS} is the Fubini-Study metric of the fiber $\mathbf{P}(E_{z^*})$ and $[\;] : E_* = E \setminus \{\text{zero} - \text{section}\} \to \mathbf{P}(E)$ is the quotient map. Thus ϕ is positive definite in the fiber direction and (see Theorem 3.2)

Theorem 4.4. *Let P be the position vector field on a holomorphic vector bundle E of rank $r \geq 2$ over a complex manifold M of dimension n with a Finsler metric F which is strictly pseudoconvex along the fibers of E. Then the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2$ descends to a well-defined form $\phi(= c_1(\mathcal{L}_{\mathbf{P}(E)}))$ on $\mathbf{P}(E)$; moreover ϕ is positive definite if and only if the base component of the mixed holomorphic bisectional curvature $< K(X,X)P, P >_h$ is strictly negative in the direction of P on $E \setminus \{\text{zero} - \text{section}\}$ and non-zero tangent vector $X \in TM$.*

As pointed out at the beginning of this section, a Finsler metric on E is identified with a hermitian metric along the fibers of the "tautological" line bundle $\mathcal{L}^{-1}$ over $\mathbf{P}(E)$. Abusing the notation we shall denote by h these two metrics. The (1,1)-form $\sqrt{-1}\partial\overline{\partial}\log ||P||_h^2$ descends to the Chern form of $\mathcal{L}$ with the dual metric h^*. Thus the existence of a Finsler metric such that $c_1(\mathcal{L}_{\mathbf{P}(E)}, h^*)$ is positive definite is equivalent to the condition that the line bundle $\mathcal{L}$ is ample. This is equivalent to the condition that $\mathcal{L}^m$ is very ample for some positive integer m. Let $\{\sigma_0, ..., \sigma_N\}$ be a basis of global holomorphic sections of $\mathcal{L}^m$ then

$$\Phi = [\sigma_0, ..., \sigma_N] : \mathbf{P}(E) \to \mathbf{P}^N$$

is a holomorphic embedding and that $\mathcal{L}^m$ is the pull-back of the hyperplane section bundle $\mathcal{O}_{\mathbf{P}^N}(1)$ with the dual canonical metric h_0^*, i.e., $c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*)$ is the

Fubini-Study form on $\mathbf{P}^N$. Moreover the Chern form $mc_1(\mathcal{L}, h) = c_1(\mathcal{L}^m, h^m)$ is cohomologous to $c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*)$. The canonical metric h_0 on the tautological line bundle $\mathcal{O}_{\mathbf{P}^N}(-1)$ is by definition:

$$h_0([w_0, ..., w_N]) = (\sum_{i=0}^{N} |w_i|^2)^{1/2}, \; [w_0, ..., w_N] \in \mathbf{P}^N(\mathbf{C})$$

where $[w_0, ..., w_N]$ are the homogeneous coordinates on $\mathbf{P}^N$. Thus

$$h_\Phi = (\Phi^* h_0)^m = (\sum_{i=0}^{N} \sigma_i \otimes \overline{\sigma}_i)^{1/2} = (\sum_{i=0}^{N} \sigma_i \otimes \overline{\sigma}_i)^{\frac{1}{2m}} \tag{4.21}$$

is a well-defined hermitian metric on $\mathcal{L}^{-1}$ with $c_1(\mathcal{L}, h_\Phi^*) = \frac{1}{m}\Phi^* c_1(\mathcal{O}_{\mathbf{P}^N}(1), h_0^*) > 0$. The corresponding Finsler metric on E can be similarly expressed via Grothendieck's Theorem:

Theorem. (Grothendieck) *Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold M. Let $p = [p_E] : \mathbf{P}(E) \to M$ be the projection map then for any coherent sheaf $\mathcal{S}$ over X:*

$$p_*(\mathcal{L}^m_{\mathbf{P}(E)} \otimes p^*\mathcal{S}) \cong \odot^m E^* \otimes \mathcal{S}$$

for all $m \geq 0$ and

$$R^i p^*(\mathcal{L}^m_{\mathbf{P}(E)} \otimes p^*\mathcal{S}) = 0$$

for all $m \geq 0$ and $i > 0$ where $R^i p_*$ denotes the i-th direct image. Consequently, the corresponding cohomology groups are also isomorphic, i.e.

$$H^i(\mathbf{P}(E), \mathcal{L}^m_{\mathbf{P}(E)} \otimes p^*\mathcal{S})) \cong H^i(X, \odot^m E^* \otimes \mathcal{S})$$

for all integers $i \geq 0$ and $m \geq 0$.

For negative powers we have:

Theorem. *Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold M. Then for any coherent sheaf $\mathcal{S}$ over X:*

$$R^i p^*(\mathcal{L}^{-m}_{\mathbf{P}(E)} \otimes p^*\mathcal{S}) = 0$$

for all $m > 0$ and $i \neq r - 1$ and

$$R^{r-1} p^*(\mathcal{L}^{-m}_{\mathbf{P}(E)} \otimes p^*\mathcal{S}) \cong \odot^{m-r} E \otimes \mathcal{S}$$

for all $m \geq 0$ (by convention $\odot^k E = 0$ if $k > 0$). Consequently,

$$H^i(\mathbf{P}(E), \mathcal{L}^{-m}_{\mathbf{P}(E)} \otimes p^*\mathcal{S})) \cong H^{i-r+1}(X, \odot^{m-r} E \otimes \mathcal{S})$$

for all integer $m \geq 0$.

Denote by $\gamma : H^0(\mathbf{P}(E), \mathcal{L}^m) \cong H^0(M, \odot^m E^*)$. Under this isomorphism the basis $\sigma_0, ..., \sigma_N$ of $H^0(\mathbf{P}(E), \mathcal{L}^m)$ is identified with a basis $\omega_0, ..., \omega_N$ of $H^0(M, \odot^m E^*)$, i.e., $\gamma^*\omega_i = \sigma_i$ where $\odot^m E^*$ is the symmetric product. The Finsler metric on E corresponding to h_Φ will be denoted, by abuse of notation, also by h_Φ and is given by

$$h_\Phi(a) = (< a^{\otimes m}, a^{\otimes m} >_m)^{1/2m} = (\sum_{i=0}^{N} |\omega_i(a^{\otimes m})|^2)^{1/2m}, \;\; a \in E. \tag{4.22}$$

Observe that the basis of sections $\{\omega_i\}$ of $H^0(M, \odot^m E^*)$ actually defines a *hermitian inner product* on $\odot^m E$:

$$< A, B >_m \stackrel{\text{def}}{=} \sum_{i=0}^{N} \omega_i(A)\, \overline{\omega}_i(\overline{B}), \;\; A, B \in \odot^m E. \tag{5.23}$$

Moreover, the norm $||\;||_m$ of the inner product is the Finsler metric h_Φ on E:

$$||v||_{h_\Phi} = (\sum_{i=0}^{N} \omega_i(\otimes^m v)\, \overline{\omega}_i(\overline{\otimes^m v}))^{1/2m} = || \otimes^m v||_m^{1/2m}, \; v \in E. \tag{5.24}$$

This is clear as the homogeneity condition, $||\lambda v||_{h_\Phi} = ||\lambda^m \otimes^m v||_m^{1/2m} = ||\lambda v||_{h_\Phi} = |\lambda|||v||_{h_\Phi}$, is clearly satisfied. We summarize these in the following Theorem:

Theorem 4.5. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold M and for any positive integer k let $\mathcal{L}_{\mathbf{P}(\odot^k E)}$ be the "hyperplane bundle" over $\mathbf{P}(\odot^k E)$. Then the following statements are equivalent:*

(1) *E^* is ample;*

(2) *$\mathcal{L}_{\mathbf{P}(E)}$ is ample;*

(3) *there exists a positive integer m such that $\mathcal{L}^m_{\mathbf{P}(E)}$ is very ample;*

(4) *there exists a hermitian metric along the fibers of $\mathcal{L}_{\mathbf{P}(E)}$ with positive definite Chern form;*

(5) *there exists a Finsler metric along the fibers of E with negative mixed holomorphic bisectional curvature in the direction of the position vector field on E and any non-zero tangent vector $X \in TM$;*

(6) *there exists a Finsler metric along the fibers of $\odot^k E$ with negative mixed holomorphic bisectional curvature in the direction of the position vector field on $\odot^k E$ and any non-zero tangent vector $X \in TM$;*

(7) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m E$ with negative mixed holomorphic bisectional curvature in the direction of the position vector field on $\odot^m E$ and any non-zero tangent vector $X \in TM$;*

(8) *$\odot^k E^*, k > 0$ is ample;*

(9) *$\mathcal{L}_{\mathbf{P}(\odot^k E)}, k > 0$ is ample.*

Proof. By definitions (1), (2), (3) and (4) are equivalent. Since $c_1(\mathcal{L}^k_{\mathbf{P}(E)}) = kc_1(\mathcal{L}_{\mathbf{P}(E)})$ (4) and (8) are equivalent. By definitions (8) and (9) are also equivalent. The equivalence of (4) and (5) is a consequence of Theorem 5.4. The discussion preceding the Theorem shows that (3) is equivalent to (7). The equivalence of (5), (7) and (8) follows from the definitions. QED

The Theorem applies of course to the tangent as well as the cotangent bundle and, in the later case we get from Theorem 3.3 and Corollary 3.9 that:

Corollary 4.6. *Let $E = T^*M$ be the cotangent bundle of a compact complex n-dimensional manifold M then the following statements are equivalent:*

(1) *TM is ample;*

(2) *$\mathcal{L}_{\mathbf{P}(T^*M)}$ is ample;*

(3) *the anti-canonical bundle $\mathcal{K}^{-1}_{\mathbf{P}(T^*M)}$ is ample;*

(4) *there exists a positive integer m such that $\mathcal{L}^m_{\mathbf{P}(T^*M)}$ is very ample;*

(5) *there exists a hermitian metric along the fibers of $\mathcal{L}_{\mathbf{P}(T^*M)}$ with positive definite Chern form;*

(6) *there exists a Finsler metric on M with positive holomorphic bisectional curvature;*

(7) *there exists a Finsler metric along the fibers of $\odot^k T^*M$ with positive mixed holomorphic bisectional curvature in the direction of the position vector field on $\odot^k T^*M$ and any non-zero tangent vector $X \in TM$;*

(8) *there exists a positive integer m and a Hermitian metric along the fibers of $\odot^m TM$ with positive mixed holomorphic bisectional curvature in the direction of the position vector field on $\odot^m TM$ and any non-zero tangent vector $X \in TM$;*

(9) *$\odot^k TM, k > 0$ is ample;*

(10) *$\mathcal{L}_{\mathbf{P}(\odot^k T^*M)}, k > 0$ is ample.*

If (M, g) is a Kähler metric then Theorem 3.1 extends also to (E, h) where h is only a Finsler metric. The calculation is formally similar but more complicated we include the calculations below for the sake of completeness. A calculation as in section 2 (and section 3) shows that

$$\begin{cases} \eta^\alpha = dz^i, & 1 \leq \alpha \leq n, \\ \theta^i = dv^i + \sum_{j=1}^r \sum_{\alpha=1}^n \Gamma^i_{j\alpha} v^j dz^\alpha, & 1 \leq i \leq r \end{cases} \tag{4.25}$$

is a dual basis (compare (2.12)). In terms of the dual frame, the fundamental form of the inner product on TE is given by (compare (2.13))

$$\eta = \eta_F = \eta_{TE} = \sqrt{-1}\Big(\sum_{\alpha,\beta=1}^n g_{\alpha\bar\beta}(z)\, dz^i \wedge d\bar z^j + \sum_{i,j=1}^r G_{i\bar j}(z, v)\, \theta^i \wedge \overline{\theta}^j \Big) \tag{4.26}$$

where the first term on the right is the Kähler form on M. The obstruction of η from being Kähler is given analogously (in fact formally the same; cf. Theorem 2.1) by :

Theorem 4.7. *Let (M, g) be a complex Kähler manifold and E be a holomorphic vector bundle of rank r over M with Finsler metric h which is strictly pseudoconvex along the fibers. Let $\eta = \eta_F$ be the fundamental form of the hermitian inner product $< \,, >_{TE}$ as defined in* (5.20) *then*

$$\begin{aligned} d\eta &= \sqrt{-1}\Big(\sum_{1\leq i,j,k\leq r} G_{i\bar j} v^k \Theta^i_k \wedge \overline{\theta}^j - \sum_{1\leq i,j,k\leq r} G_{i\bar j} \theta^i \wedge \bar v^k \overline{\Theta}^j_k \Big) \\ &= \sqrt{-1}\Big(\sum_{1\leq i,j\leq r} G_{i\bar j} \bar\partial \theta^i \wedge \overline{\theta}^j - \sum_{1\leq i,j\leq r} G_{i\bar j} \theta^i \wedge \partial \overline{\theta}^j \Big) \end{aligned}$$

where $\theta^i = dv^i + \sum_k \theta^i_k v^k$.

Proof. The notations were set up so that the proof is formally the same as that of Theorem 2.1. The first term of (5.26) is closed hence

$$d\eta = \sum_{i,j=1}^{r} (dG_{i\bar{j}} \wedge \theta^i \wedge \overline{\theta}^j + G_{i\bar{j}} d\theta^i \wedge \overline{\theta}^j - G_{i\bar{j}} \theta^i \wedge d\overline{\theta}^j).$$

We have,

$$\begin{aligned} d\theta^i &= \sum_k dv^k \wedge \theta^i_k + \sum_k d\theta^i_k v^k, \\ d\overline{\theta}^j &= \sum_k d\bar{v}^k \wedge \overline{\theta^j_k} + \sum_k d\overline{\theta^i_k} \bar{v}^k. \\ \theta^i \wedge \overline{\theta}^j &= dv^i \wedge d\bar{v}^j + \sum_k \bar{v}^k dv^i \wedge \overline{\theta}^j_k + \sum_k v^k \theta^i_k \wedge d\bar{v}^j + \sum_{k,l} v^k \bar{v}^l \theta^i_k \wedge \overline{\theta}^j_l, \end{aligned}$$

and

$$dG_{i\bar{j}} = \sum_k \theta^k_i g_{k\bar{j}} + \sum_k \overline{\theta^k_j g_{k\bar{i}}} = \sum_k \theta^k_i G_{k\bar{j}} + \sum_k \overline{\theta}^k_j G_{i\bar{k}};$$

hence,

$$\begin{aligned} dG_{i\bar{j}} \wedge \theta^i \wedge \overline{\theta}^j &= \sum_k G_{k\bar{j}} \theta^k_i \wedge dv^i \wedge d\bar{v}^j + \sum_k G_{i\bar{k}} \overline{\theta}^k_j \wedge dv^i \wedge d\bar{v}^j, \\ &\quad - \sum_{k,l} G_{k\bar{j}} \bar{v}^l dv^i \wedge \theta^k_i \wedge \overline{\theta}^j_l - \sum_{k,l} G_{i\bar{k}} \bar{v}^l dv^i \wedge \overline{\theta}^k_j \wedge \overline{\theta}^j_l \\ &\quad + \sum_{k,l} G_{k\bar{j}} v^l \theta^k_i \wedge \theta^i_l \wedge d\bar{v}^j - \sum_{k,l} G_{i\bar{k}} v^l \theta^i_l \wedge \overline{\theta}^k_j \wedge d\bar{v}^j \\ &\quad + \sum_{k,l,m} G_{k\bar{j}} v^l \bar{v}^m \theta^k_i \wedge \theta^i_l \wedge \overline{\theta}^j_m - \sum_{k,l} G_{i\bar{k}} v^l \bar{v}^m \theta^j_k \wedge \theta^i_l \wedge \overline{\theta}^j_m \\ G_{i\bar{j}} d\theta^i \wedge \overline{\theta}^j &= - \sum_k G_{i\bar{j}} \theta^i_k \wedge dv^k \wedge d\bar{v}^j + \sum_{k,l} G_{i\bar{j}} \bar{v}^l dv^k \wedge \theta^i_k \wedge \overline{\theta}^j_l \\ &\quad + \sum_k G_{i\bar{j}} v^k d\theta^i_k \wedge d\bar{v}^j + \sum_{k,l} G_{i\bar{j}} v^k \bar{v}^l d\theta^i_k \wedge \overline{\theta}^j_l \\ -G_{i\bar{j}} \theta^i \wedge d\overline{\theta}^j &= - \sum_k G_{i\bar{j}} \overline{\theta}^j_k \wedge dv^i \wedge d\bar{v}^k + \sum_{k,l} G_{i\bar{j}} v^l d\bar{v}^k \wedge \theta^i_l \wedge \overline{\theta}^j_k \\ &\quad - \sum_k G_{i\bar{j}} \bar{v}^k dv^i \wedge d\overline{\theta}^j_k - \sum_{k,l} G_{i\bar{j}} v^l \bar{v}^k \theta^i_l \wedge d\overline{\theta}^j_k. \end{aligned}$$

Summing the above over i and j yields

$$\begin{aligned} d\eta = &- \sum_{i,j,k,l} G_{i\bar{k}} \bar{v}^l dv^i \wedge \overline{\theta}^k_j \wedge \overline{\theta}^j_l + \sum_{i,j,k,l} G_{k\bar{j}} v^l \theta^k_i \wedge \theta^i_l \wedge d\bar{v}^j \\ &+ \sum_{i,j,k,l,m} G_{k\bar{j}} v^l \bar{v}^m \theta^k_i \wedge \theta^i_l \wedge \overline{\theta}^j_m - \sum_{i,j,k,l,m} G_{i\bar{k}} v^l \bar{v}^m \theta^i_l \wedge \overline{\theta}^k_j \wedge \overline{\theta}^j_m \end{aligned}$$

$$+\sum_{i,j,k,l} G_{i\bar{j}} v^k d\theta^i_k \wedge d\bar{v}^j + \sum_{i,j,k,l} G_{i\bar{j}} v^k \bar{v}^l d\theta^i_k \wedge \overline{\theta}^j_l$$
$$-\sum_{i,j,k,l} G_{i\bar{j}} \bar{v}^k dv^i \wedge d\overline{\theta}^j_k - \sum_{i,j,k,l} G_{i\bar{j}} v^l \bar{v}^k \theta^i_l \wedge d\overline{\theta}^j_k$$

The right hand side above can be expressed in terms of the curvature (see (5.17) and (5.18)) and we arrive at:

$$\begin{aligned} d\eta = &-\sum_{i,j,k,l} G_{i\bar{j}} \bar{v}^k dv^i \wedge \overline{\Theta}^j_k + \sum_{i,j,k,l} G_{i\bar{j}} v^k \Omega^i_k \wedge d\bar{v}^j \\ &+ \sum_{i,j,k,l} G_{i\bar{j}} v^k \bar{v}^l \Theta^i_k \wedge \overline{\theta}^j_l - \sum_{i,j,k,l} G_{i\bar{j}} v^l \bar{v}^k \theta^i_l \wedge \overline{\Theta}^j_k \\ = &\sum_{i,j,k,l} G_{i\bar{j}} v^k \Theta^i_k \wedge (d\bar{v}^j + \bar{v}^l \overline{\theta}^j_l) - \sum_{i,j,k,l} G_{i\bar{j}} (dv^i + v^l \theta^i_l) \wedge \bar{v}^k \overline{\Theta}^j_k. \end{aligned}$$

Next we observe that

$$\partial\theta^i + \overline{\partial}\theta^i = d\theta^i = \sum_k dv^k \wedge \theta^i_k + \sum_k \partial\theta^i_k v^k + \sum_k \overline{\partial}\theta^i_k v^k$$

and comparing bidegrees yields

$$\partial\theta^i = \sum_k dv^k \wedge \theta^i_k + \sum_k v^k \partial\theta^i_k, \;\; \overline{\partial}\theta^i = \sum_k v^k \overline{\partial}\theta^i_k = \sum_k \Theta^i_k v^k.$$

Thus the identity for $d\eta$ above can also be expressed as:

$$\begin{aligned} d\eta &= \sqrt{-1}(\sum_{i,j,k} G_{i\bar{j}} v^k \overline{\partial}\theta^i_k \wedge \overline{\theta}^j - \sum_{i,j,k} G_{i\bar{j}} \theta^i \wedge \bar{v}^k \partial\overline{\theta}^j_k) \\ &= \sqrt{-1}(\sum_{i,j,k} G_{i\bar{j}} \overline{\partial}(v^k \theta^i_k) \wedge \overline{\theta}^j - \sum_{i,j,k} G_{i\bar{j}} \theta^i \wedge \partial(\bar{v}^k \overline{\theta}^j_k)). \end{aligned}$$

Since $\overline{\partial}(v^k\theta^i_k) = \overline{\partial}(dv^i + v^k\theta^i_k)$ we get

$$d\eta = \sqrt{-1}(\sum_{i,j} G_{i\bar{j}} \overline{\partial}\theta^i \wedge \overline{\theta}^j - \sum_{i,j} G_{i\bar{j}} \theta^i \wedge \partial\overline{\theta}^j).$$

QED:

Corollary 4.8. *With the same assumptions as in Theorem 5.3 the following conditions are equivalent:*

(*i*) *the metric* $< , >_{TE}$ *in Theorem 4.7 is Kähler;*

(*ii*) *the one forms* $\{\theta^i, i = 1, ..., n\}$ *are holomorphic;*

(*iii*) *the curvature of the vertical bundle* $\mathcal{V}$ *satisfies the conditions:*

$$\sum_{1 \le i,k \le r} G_{i\bar{m}} v^k \Theta^i_k = 0, \;\; G_{i\bar{m}}(z,v) = \frac{\partial^2 G}{\partial v^i \partial \bar{v}^m}(z,v)$$

for all m.

Proof. It is clear form the second expression for $d\eta$ in Theorem 5.3 that the holomorphicity of $\{\theta^i, i = 1, ..., n\}$ implies that $d\eta = 0$. For the converse, since $\{\partial_i^{\mathcal{V}}\}$ and $\{\theta_i\}$ are dual frames

$$0 = \iota_{\partial^{\mathcal{V}}_{\bar m}} d\eta = \sqrt{-1}(\sum_{1\le i,j\le n} G_{i\bar j}\{(\iota_{\partial^{\mathcal{V}}_{\bar m}}\overline{\partial}\theta^i)\wedge\overline{\theta}^j + \delta^j_m \overline{\partial}\theta^i\} + \sum_{1\le i,j\le n} G_{i\bar j}\theta^i\wedge(\iota_{\partial^{\mathcal{V}}_{\bar m}}\partial\overline{\theta}^j)$$

where $\iota_{\partial^{\mathcal{V}}_{\bar m}}$ denotes interior product with the vector field $\partial/\partial\bar v^m$. Thus we have

$$\sum_{1\le i\le n} G_{i\bar m}\overline{\partial}\theta^i = -\sum_{1\le i,j\le n} G_{i\bar j}(\iota_{\partial^{\mathcal{V}}_{\bar m}}\overline{\partial}\theta^i)\wedge\overline{\theta}^j + \sum_{1\le i,j\le n} G_{i\bar j}\theta^i\wedge(\iota_{\partial^{\mathcal{V}}_{\bar m}}\partial\overline{\theta}^j)$$

and since the left hand side above is of type (1, 1) we conclude, by comparing types, that

$$\sum_{1\le i,j\le n} G_{i\bar j}\theta^i\wedge(\iota_{\partial^{\mathcal{V}}_{\bar m}}\partial\overline{\theta}^j) = 0$$

and

$$\sum_{1\le i\le n} G_{i\bar m}\overline{\partial}\theta^i = -\sum_{1\le i,j\le n} G_{i\bar j}(\iota_{\partial^{\mathcal{V}}_{\bar m}}\overline{\partial}\theta^i)\wedge\overline{\theta}^j$$

where $\iota_{\partial^{\mathcal{V}}_{\bar m}}$ denotes interior product with the vector field $\partial/\partial\bar v^m$. These two identities imply that

$$\overline{\partial}\theta^l = -\sum_{1\le i,j\le n} G^{\bar m l}G_{i\bar j}(\iota_{\partial^{\mathcal{V}}_{\bar m}}\overline{\partial}\theta^i)\wedge\overline{\theta}^j = 0.$$

Using the first identity of Theorem 5.3 we get

$$\iota_{\partial_{\bar m}} d\eta = \sqrt{-1}(\sum_{1\le i,k\le r} G_{i\bar m}v^k\Theta^i_k + \sum_{1\le i,j,k\le r} G_{i\bar j}\{v^k(\iota_{\partial_{\bar m}}\Theta^i_k)\wedge\overline{\theta}^j + \theta^i\wedge\bar v^k\iota_{\partial_{\bar m}}\overline{\Theta}^j_k\}$$

for all m and, if the left hand side above is zero then comparison of types yields the identities

$$\sum_{1\le i,j,k\le r} G_{i\bar j}\theta^i\wedge\bar v^k\iota_{\partial_{\bar m}}\overline{\Theta}^j_k = 0,$$

$$\sum_{1\le i,k\le r} G_{i\bar m}v^k\Theta^i_k + \sum_{1\le i,j,k\le r} G_{i\bar j}v^k(\iota_{\partial_{\bar m}}\Theta^i_k)\wedge\overline{\theta}^j = 0$$

for all m.It is clear that these imply that

$$\sum_{1\le i,k\le r} G_{i\bar m}v^k\Theta^i_k = 0$$

for all m. QED

Remark 4.9. Since the curvature froms $\{\Theta^i_k\}$ depend on the base variables $(z^1, ..., z^n)$ and the fiber variables $(v^1, ..., v^r)$ we cannot, as in the case of Corollary 2.2, conclude that the curvature forms $\{\Theta^i_k, 1\le i,k\le r\}$ vanish.

It is well-known that the ampleness of a vector bundle implies that the tensor products $\otimes^k E$ are also amples for any positive integer k. By Theorem 4.5 this can be formulated as saying that the negativity (resp. positivity) of the mixed bisectional curvature of E implies the negativity (resp. positivity) of the mixed bisectional curvature of $\otimes^k E$. In this section we work out the precise relation between the respective curvatures.

For a Hermitian holomorphic vector bundles (E, h) and (F, k) the tensor product $E \otimes F$ is equipped with the Hermitian metric $H = h \otimes k$, i.e.,

$$< a \otimes \alpha, b \otimes \beta >_{h\otimes k} = < a, b >_h < \alpha, \beta >_k, \tag{5.1}$$

more generally, for $s_{i,j}, t_{\mu,\nu} \in \mathbf{C}$,

$$< \sum_i s_{i,j} a_i \otimes \alpha_j, \sum_j t_{\mu,\nu} b_\mu \otimes \beta_\nu >_{h\otimes k} = \sum_{i,j,\mu,\nu} s_{i,j} \bar{t}_{\mu,\nu} < a_i, b_\mu >_h < \alpha_j, \beta_\nu >_k .$$

Let $\{e_1, ..., e_r\}, \{f_1, ..., f_s\}$ be local frames for E and F then $\{e_{ij} = e_i \otimes f_j, 1 \leq i \leq r, 1 \leq j \leq s\}$ is a local frame for $E \otimes F$. For $1 \leq i, j, p, q \leq n$,

$$H_{\{ij\}\{\bar{p}\bar{q}\}} = < e_{ij}, e_{pq} >_H = < e_i, e_p >_h < f_j, f_q >_h = h_{i\bar{p}} k_{j\bar{q}}.$$

If (E, h) and (F, k) are Finsler bundles we cannot, in general, define its tensor product on $E \otimes F$ even though for simple elements $a \otimes b, a \in E, b \in F, H(z, a \otimes b) = h(z, a)k(z, b)$ is well-defined but there is no natural way of extending this definition to the general elements. However, if h and k are strictly pseudoconvex along the fibers we may proceed as follows. Let $\eta = h^2$ and $\kappa = k^2$ then, as was seen in the preceding section,

$$h_{i\bar{j}} = \frac{\partial^2 \eta}{\partial v^i \partial \overline{v}^j}, \; k_{i\bar{j}} = \frac{\partial^2 \kappa}{\partial v^i \partial \overline{v}^j}$$

are hermitian metrics on the vertical bundles $\mathcal{V}_E$ and $\mathcal{V}_F$ respectively. Now the tensor product of these two hermitian metrics is a hermitian metric of the bundle $\mathcal{V}_E \otimes \mathcal{V}_F$. It is easily seen that $\mathcal{V}_E \otimes \mathcal{V}_F$ is the vertical sub-bundle of $T(E \otimes F)$, i.e., $\mathcal{V}_E \otimes \mathcal{V}_F = \mathcal{V}_{E\otimes F}$. In what follows we shall be working with the Hermitian and Finsler cases at the same time with the understanding that, in the later case we are working on $\mathcal{V}_E \otimes \mathcal{V}_F$ instead of $E \otimes F$. The connection of the tensor product is the tensor product of the connections, i.e.,

$$\nabla(a \otimes \alpha) = (\nabla a) \otimes \alpha + a \otimes \nabla \alpha. \tag{5.2}$$

For simplicity of notations the same symbol is used for the 3 different connections. The connection is extended to general elements by enforcing linearity (over $\mathbf{C}$) and Leibnitz rule. We have, by (4.10):

$$\nabla e_{ij} = \nabla(e_i \otimes f_j) = (\nabla e_i) \otimes f_j + e_i \otimes \nabla f_j = \sum_p \theta_i^p(E) e_{pj} + \sum_q \theta_j^q(F) f_{iq}$$

where $\theta_i^p(E)$ and $\theta_j^q(F)$ are the connection forms of E and F respectively (in the Finsler case, $\theta_i^p(\mathcal{V}_E)$ and $\theta_j^q(\mathcal{V}_F)$ are the connection forms of $\mathcal{V}_E$ and $\mathcal{V}_F$). On the other hand, denoting by $\theta_{\{ij\}}^{\{pq\}}$ the connection forms relative to the frame $\{e_{ij}\}$, then

$$\nabla e_{ij} = \sum_{pq} \theta_{\{ij\}}^{\{pq\}} e_{pq}$$

from which we infer that

$$\theta_{\{ij\}}^{\{pq\}} = \begin{cases} \theta_i^p(E)\,(\theta_i^p(\mathcal{V}_E)) & \text{if } q = j, \\ \theta_j^q(F)\,(\theta_i^p(\mathcal{V}_F)) & \text{if } p = i, \\ 0 & \text{otherwise.} \end{cases}$$

For the curvature of the tensor product observe that

$$\begin{aligned} \nabla^2(a \otimes b) &= \nabla((\nabla a) \otimes b + a \otimes \nabla b) \\ &= (\nabla^2 a) \otimes b - (\nabla a) \otimes \nabla b + (\nabla a) \otimes \nabla b + a \otimes \nabla^2 b \\ &= (\nabla^2 a) \otimes b + a \otimes \nabla^2 b \end{aligned}$$

hence we have as in the case of connections,

$$\Theta_{\{ij\}}^{\{pq\}} = \begin{cases} \Theta_i^p(E)\,(\Theta_i^p(\mathcal{V}_E)) & \text{if } q = j, \\ \Theta_j^q(F)\,(\Theta_i^p(\mathcal{V}_F)) & \text{if } i = p, \\ 0 & \text{otherwise.} \end{cases}$$

where $\{\Theta_{\{ij\}}^{\{kl\}}\}$ are the curvature forms relative to the frame $\{e_{ij}\}$ and $\Theta_i^p(E), \Theta_j^q(F)$ $(\Theta_i^p(\mathcal{V}_E)$ and $\Theta_i^p(\mathcal{V}_F))$ are the curvature forms of (E,h) and (F,q) ($\mathcal{V}_E$ and $\mathcal{V}_F$) relative to the frame $\{e_i\}$ and $\{f_j\}$ respectively. For

$$\sigma = \sum_{i,j} a^{ij} e_{ij} = \sum_{i,j} a^{ij} e_i \otimes f_j \in E \otimes F \ (\mathcal{V}_E \otimes \mathcal{V}_F)$$

define for each i_0 and j_0

$$\sigma^{i_0,} = \sum_j a^{i_0 j} f_j \in F \ (\mathcal{V}_F); \ \sigma^{,j_0} = \sum_i a^{ij_0} e_i \in E \ (\mathcal{V}_E)$$

then $\sigma = \sum_i e_i \otimes \sigma^{i,} = \sum_j \sigma^{,j} \otimes f_j$. For the computation that follows we choose the frames $\{e_i\}, \{f_i\}$ to be unitary. For unitary frames it is easily seen that

$$||\sigma||^2_{h\otimes k} = \sum_i ||\sigma^{i,}||_k^2 = \sum_j ||\sigma^{,j}||_h^2.$$

Denote the curvature operator of the tensor product $E \otimes F\,(\mathcal{V}_E \otimes \mathcal{V}_F)$ by $\hat{K}$ and that of E and F ($\mathcal{V}_E$ and $\mathcal{V}_F$)) by K_E and K_F respectively. By definition we have

$$K_F \sigma^{i,} = \sum_{j,q} a^{ij} \Theta_j^q(F) f_q, \ K_E \sigma^{,j} = \sum_{i,p} a^{ij} \Theta_i^p(E) e_p$$

hence

$$\begin{aligned}< \hat{K}\sigma,\sigma >_{h\otimes k} &=< \sum_{i,j,p,q} a^{ij}\Theta^{\{pq\}}_{\{ij\}} e_{pq},\sigma >_{h\otimes k} \\ &=< \sum_{i,j,p} a^{ij}\Theta^{p}_{i}(E) e_{pj},\sigma >_{h\otimes k} + < \sum_{i,j,q} a^{ij}\Theta^{q}_{j}(F) e_{iq},\sigma >_{h\otimes k} \\ &=< \sum_{j}(K_E\sigma^{,j})\otimes f_j, \sum_{j}\sigma^{,j}\otimes f_j >_{h\otimes k} \\ &\quad + < \sum_{i} e_i\otimes (K_F\sigma^{i,}), \sum_{i} e_i\otimes\sigma^{i,} >_{h\otimes k}\end{aligned}$$

(and the same formula replacing E by $\mathcal{V}_E$ and F by $\mathcal{V}_F$ in the Finsler case). By the definition of $h\otimes k$ the last term on the right above is equal to

$$\sum_{j,p} < K_E\sigma^{,j},\sigma^{,p} >_h < f_j, f_p >_k + \sum_{i,q} < e_i, e_q >_h < K_F\sigma^{i,},\sigma^{q,} >_k$$

and, as the frame is unitary we arrive at the identity:

$$< \hat{K}\sigma,\sigma >_{h\otimes k} = \sum_{j} < K_E\sigma^{,j},\sigma^{,j} >_h + \sum_{i} < K_F\sigma^{i,},\sigma^{i,} >_k .$$

This implies that (replacing E by $\mathcal{V}_E$ and F by $\mathcal{V}_F$ in the case of Finsler metrics)

$$\begin{aligned}\frac{< \hat{K}\sigma,\sigma >_{h\otimes k}}{||\sigma||^2_{h\otimes k}} &= \sum_{i} \frac{||\sigma^{i,}||^2_k}{||\sigma||^2_{h\otimes k}} \frac{< K_F\sigma^{i,},\sigma^{i,} >_k}{||\sigma^{i,}||^2_k} \\ &\quad + \sum_{j} \frac{||\sigma^{,j}||^2_h}{||\sigma||^2_{h\otimes k}} \frac{< K_E\sigma^{,j},\sigma^{,j} >_h}{||\sigma^{,j}||^2_h}.\end{aligned}$$

It is clear from the preceding formula that the mixed holomorphic bisectional curvature of $(E\otimes F, h\otimes k)$ is ≤ 0 (resp. < 0) if the mixed holomorphic bisectional curvatures of (E,h) and (F,k) are both ≤ 0 (resp. < 0). By induction, we have:

Proposition 5.1. *Let $(E_i,h_i), i = 1,...,m$ be Hermitian (resp. Finslerwhichispseudoconvexalongthefib holomorphic vector bundles over a complex manifold M. If the holomorphic bisectional curvature of (E_i,h_i) are ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0) for $i = 1,...,m$ then the holomorphic bisectional curvature of $(\otimes_{i=1}^m E_i, \otimes_{i=1}^m h_i)$ $((\otimes_{i=1}^m \mathcal{V}_{E_i}, \otimes_{i=1}^m h_i)$ for Finsler metrics) is ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0).*

The case of exterior product is similar. For a hermitian holomorphic vector bundles (E,h) the wedge product $\wedge^m E (m \leq r = \text{rank } E)$ is equipped with the metric $H = \wedge^m h$, i.e.,

$$< a_1\wedge ...\wedge a_m, b_1\wedge ...\wedge b_m >_{\wedge^m h} = \det(< a_i, b_j >_h)_{1\leq i,j\leq m}. \tag{5.2}$$

If h is only a Finsler metric which is pseudoconvex along the fibers of E then it defines a hermitian metric on $\mathcal{V}_E$ and hence also a hermitian metric along the fibers of $\wedge^m \mathcal{V}_E$. In the following we shall work with the case of hermitian metric with the understanding that the Finsler case is analogous and is obtained by replacing E by $\mathcal{V}_E$ in the following discussions. Denote the set of increasing indices by:

$$\mathcal{I}_{m,r} = \{I = (i_1, ..., i_m) \,|\, 1 \leq i_1 < ... < i_m \leq r\}.$$

Let $\{e_1, ..., e_r\}$ be a local frame for E then $\{e_I \,|\, I \in \mathcal{I}_{m,r}\}$ is a local frame for $\wedge^m E$. In terms of these frames the components of the metric H are given by

$$H_{I\bar{J}} =< e_I, e_J >_{\wedge^m h} = \det(h_{i_k \bar{j}_l})_{1 \leq i,j \leq m} = \det(< e_{i_k}, e_{j_l} >_h)_{1 \leq k,l \leq m}.$$

The associated connection ∇ on $\wedge^m E$ (which will be denoted by the same notation) is given as follows:

$$\nabla(a_1 \wedge ... \wedge a_m) = \sum_{i=1}^{m} a_1 \wedge ... \wedge \nabla a_i \wedge ... \wedge a_m. \tag{5.3}$$

The preceding defining identity can be verified by skew symmetrizing the connection for tensor products. For example if $m = 2$ then $a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a)$ hence

$$\begin{aligned} \nabla(a \wedge b) &= \frac{1}{2}(\nabla(a \otimes b) - \nabla(b \otimes a)) \\ &= \frac{1}{2}\{(\nabla a) \otimes b + a \otimes \nabla b - (\nabla b) \otimes a - b \otimes \nabla a\} \\ &= (\nabla a) \wedge b + a \wedge \nabla b. \end{aligned}$$

The general case is verified in an analogously way using the identity

$$a_1 \wedge ... \wedge a_m = \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\, a_{\sigma(1)} \otimes ... \otimes a_{\sigma(m)}$$

where σ ranges over the symmetric group on m elements. In terms of the given frame the connection forms are given by

$$\nabla e_I = \sum_J \theta_I^J e_J$$

and, on the other hand, by

$$\begin{aligned} \nabla e_I &= \sum_{j=1}^{m} e_{i_1} \wedge ... \wedge \nabla e_{i_j} \wedge ... \wedge e_{i_m} \\ &= \sum_{j=1}^{m} \sum_{k=1}^{r} \theta_{i_j}^k e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m} \end{aligned}$$

where θ_i^k are connection forms for (E, h). The notation $e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m}$ indicates that e_{i_j} is replaced by e_k. Since $e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m} = 0$ if $k \in I_j = \{i_1, ..., i_m\} \backslash \{i_j\}$ it is clear that, in the expression above, we may sum only those indices that are not in I_j:

$$\nabla e_I = \sum_{j=1}^{m} \sum_{1 \leq k \leq r, k \notin I_j} \theta_{i_j}^k e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m}.$$

The indices $(i_1, ..., (k)_j, ..., i_m)$ where i_j is replaced by $k \notin I_j$ may not be increasing but we can always, by rearrangement, arrived at an unique $I_{k,j} \in \mathcal{I}_{m,r}$ containing the same *set* of indices as $(i_1, ..., (k)_j, ..., i_m)$ and is increasing. It is clear that $e_{I_{k,j}}$ and $e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m}$ are equal modulo a $\pm$ sign, i.e.,

$$e_{I_{k,j}} = \operatorname{sgn}(i_1, ..., (k)_j, ..., i_m)\, e_{i_1} \wedge ... \wedge (e_k)_j \wedge ... \wedge e_{i_m}.$$

We infer from the above that

$$\theta_I^J = \begin{cases} \operatorname{sgn}(i_1, ..., (k)_j, ..., i_m)\, \theta_{i_j}^k, & \text{if } J = I_{k,j}, \\ 0, & \text{otherwise.} \end{cases}$$

An analogous comutation as in (5.2) yields

$$\nabla^2(a_1 \wedge ... \wedge a_m) = \sum_{i=1}^m a_1 \wedge ... \wedge \nabla^2 a_i \wedge ... \wedge a_m. \tag{5.4}$$

hence the formulas for curvature forms are analogous to those for the connection forms:

$$\Theta_I^J = \begin{cases} \operatorname{sgn}(i_1, ..., (k)_j, ..., i_m)\, \Theta_{i_j}^k, & \text{if } J = I_{k,j}, \\ 0, & \text{otherwise.} \end{cases}$$

For example if $m = 2$ then

$$\begin{aligned} &\nabla(e_{i_1} \wedge e_{i_2}) \\ &= \sum_{k=1}^r (\theta_{i_1}^k e_k \wedge e_{i_2} + \theta_{i_2}^k e_{i_1} \wedge e_k) \\ &= \sum_{k=1, k\neq i_2}^r \operatorname{sgn}(k, i_2)\, \theta_{i_1}^k e_k \wedge e_{i_2} + \sum_{k=1, k\neq i_1}^r \operatorname{sgn}(i_1, k)\, \theta_{i_2}^k e_{i_1} \wedge e_k \end{aligned}$$

hence

$$\theta_{i_1 i_2}^{j_1 j_2} = \begin{cases} \operatorname{sgn}(k, i_2)\, \theta_{i_1}^k, & \text{if } (j_1, j_2) = \operatorname{sgn}(k, i_2)\, (k, i_2), \\ \operatorname{sgn}(i_1, k)\, \theta_{i_2}^k, & \text{if } (j_1, j_2) = \operatorname{sgn}(i_1, k)\, (i_1, k), \\ 0, & \text{otherwise} \end{cases}$$

with sgn $(i, j) = 1$ (resp. -1) if $i < j$ (resp. $i > j$) and analogously

$$\Theta_{i_1 i_2}^{j_1 j_2} = \begin{cases} \operatorname{sgn}(k, i_2)\, \Theta_{i_1}^k, & \text{if } (j_1, j_2) = \operatorname{sgn}(k, i_2)\, (k, i_2), \\ \operatorname{sgn}(i_1, k)\, \Theta_{i_2}^k, & \text{if } (j_1, j_2) = \operatorname{sgn}(i_1, k)\, (i_1, k), \\ 0, & \text{otherwise.} \end{cases}$$

An element $\sigma \in \wedge^r E$ is of the form

$$\sigma = \sum_I a^I e_I = \sum_I \sum_{j=1}^m (-1)^{j-1} \sigma_I^j \wedge e_{I_j}$$

where $I_j = (i_1, ..., i_{j-1}, i_{j+1}, ..., i_m)$ and σ^j is defined by

$$\sigma_I^j = a^{i_1 ... i_m} e_{i_j} \in E.$$

To simplify the computation we choose the frame $\{e_i\}$ to be unitary so that

$$||\sigma||^2_{\wedge^r h} = \sum_I \sum_{j=1}^m ||\sigma_I^j||_h^2.$$

Denote the curvature operator of $\wedge^r E$ by $\hat{K}$ and that of E by K. By definition we have

$$K\sigma_I^j = \sum_k a^{i_1 \dots i_m} \Theta_{i_j}^k e_k,$$

hence

$$\begin{aligned}
< \hat{K}\sigma, \sigma >_{\wedge^r h} &=< \sum_{I,J} a^I \Theta_I^J e_J, \sigma >_{\wedge^r h} \\
&=< \sum_{I=(i_1,\dots,i_m)} \sum_j \sum_{k \in I \setminus \{i_j\}} a^I \mathrm{sgn}(I_{k,j}) \Theta_{i_j}^k e_{I_{k,j}}, \sigma >_{\wedge^r h} \\
&=< \sum_{I=(i_1,\dots,i_m)} \sum_j \sum_{k \in I \setminus \{i_j\}} a^I \Theta_{i_j}^k e_{I(k)_j}, \sigma >_{\wedge^r h} \\
&=< \sum_{I=(i_1,\dots,i_m)} \sum_{k=1}^m \sum_j a^I \Theta_{i_j}^k e_{I(k)_j}, \sigma >_{\wedge^r h} \\
&=< \sum_I \sum_{k=1}^m (-1)^{k-1} (K\sigma_I^k) \wedge e_{I_{k,j}}, \sum_I \sum_{l=1}^m (-1)^{l-1} \sigma_I^l \wedge e_{I_l} >_{\wedge^r h} .
\end{aligned}$$

where $I(k)_j = (i_1, \dots, (k)_j, \dots, i_m)$ and $I_{k,j}$ is the rearrangement of $I(k)_j$ with increasing indices. Since the frame is unitary, we see from the definition (see (4.10)) of $\wedge^m h$ that the last term on the right above is simply

$$\sum_I \sum_{k=1}^m < K\sigma_I^k, \sigma_I^k >_h$$

hence

$$< \hat{K}\sigma, \sigma >_{\wedge^r h} = \sum_I \sum_{k=1}^m < K\sigma_I^k, \sigma_I^k >_h .$$

This implies that

$$\frac{< \hat{K}\sigma, \sigma >_{\wedge^r h}}{||\sigma||^2_{\wedge^r h}} = \sum_I \sum_{k=1}^m \frac{||\sigma^k||_h^2}{||\sigma||^2_{\wedge^r h}} \frac{< K\sigma_I^k, \sigma_I^k >_h}{||\sigma_I^k||_h^2}$$

and that

Proposition 5.2. *Let (E, h) be an hermitian (resp. Finsler which is pseudoconvex along the fibers) holomorphic vector bundle over a complex manifold M. If the holomorphic bisectional curvature of (E, h) is ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0) then, for $1 \leq m \leq r$, the mixed holomorphic bisectional curvature of $(\wedge^m E, \wedge^m h)$ (resp. $(\wedge^m \mathcal{V}_E, \wedge^m h)$ is ≤ 0 (resp. < 0 resp. ≥ 0, resp. > 0).*

For a hermitian holomorphic vector bundle (E, h) the symmetric product $\odot^2 E$ is equipped with the metric $H = h \odot h = \odot^2 h$:

$$< a \odot \alpha, b \odot \beta >_{h\odot h} = \frac{1}{2}(< a, b >_h < \alpha, \beta >_h + < a, \beta >_h < \alpha, b >_h). \tag{5.5}$$

More generally, for $s_{i,j}, t_{\mu,\nu} \in \mathbf{C}$,

$$< \sum_i s_{i,j} a_i \odot \alpha_j, \sum_j t_{\mu,\nu} b_\mu \odot \beta_\nu >_{h\otimes h} = \sum_{i,j,\mu,\nu} s_{i,j} \bar{t}_{\mu,\nu} < a_i \odot \alpha_\mu, b_j \odot \beta_\nu >_{h\odot h} .$$

If h is Finsler metric which is pseudoconvex along the fibers the same argumnt applies th the bundle $\odot^m \mathcal{V}_E$. Let $\{e_1, ..., e_r\}$ be a local frame for E then $\{e_{ij} = e_i \odot e_j, 1 \leq i \leq j \leq r\}$ is a local frame for $E \odot F$. For $1 \leq i \leq j \leq r, 1 \leq p, \leq q \leq r$, the components $H_{\{ij\}\{\bar{p}\bar{q}\}}$ of the metric $H = h \odot h$ are given by

$$\begin{aligned} H_{\{ij\}\{\bar{p}\bar{q}\}} &= < e_{ij}, e_{pq} >_H \\ &= \frac{1}{2}(< e_i, e_p >_h < e_j, e_q >_h + < e_i, e_q >_h < e_j, e_p >_h) \\ &= \frac{1}{2}(h_{i\bar{p}} h_{j\bar{q}} + h_{i\bar{q}} h_{j\bar{p}}). \end{aligned}$$

The connection of the symmetric product is the symmetric product of the connections:

$$\nabla(a \odot \alpha) = (\nabla a) \odot \alpha + a \odot \nabla \alpha. \tag{5.6}$$

The connection is extended to general elements by enforcing linearity (over $\mathbf{C}$) and Leibnitz rule. For the frame e_{ij}, we have:

$$\nabla e_{ij} = \nabla(e_i \odot e_j) = (\nabla e_i) \odot e_j + e_i \odot \nabla e_j = \sum_p \theta_i^p(E) e_{[pj]} + \sum_q \theta_j^q(E) e_{[iq]}$$

where $\theta_i^p(E)$ is the connection forms of E and the notations $e_{[ip]}$ means that

$$e_{[ip]} = \begin{cases} e_{ip}, & \text{if } i \leq p, \\ e_{pi}, & \text{if } p \leq i. \end{cases}$$

In other words,

$$\nabla e_{ij} = \sum_{1 \leq q \leq i} \theta_j^q(E) e_{qi} + \sum_{i+1 \leq q \leq r} \theta_j^q(E) e_{iq} + \sum_{1 \leq p \leq j} \theta_i^p(E) e_{pj} + \sum_{j+1 \leq p \leq r} \theta_i^p(E) e_{jp}$$

On the other hand, denoting by $\theta_{\{ij\}}^{\{pq\}}$ the connection forms relative to the frame $\{e_{ij}\}_{1 \leq i \leq j \leq r}$, then

$$\nabla e_{ij} = \sum_{1 \leq p \leq q \leq r} \theta_{\{ij\}}^{\{pq\}} e_{pq}$$

from which we infer that, for $1 \leq i \leq j \leq r$ and $1 \leq p \leq q \leq r$,

$$\theta_{\{ij\}}^{\{pq\}} = \begin{cases} \theta_j^q(E) & \text{if } p = i \leq q, \\ \theta_i^p(E) & \text{if } p \leq j = q, \\ 0 & \text{otherwise.} \end{cases}$$

For the curvature of the symmetric product observe that

$$\begin{aligned}\nabla^2(a\odot b) &= \nabla((\nabla a)\odot b + a\odot \nabla b)\\ &= (\nabla^2 a)\odot b - (\nabla a)\odot\nabla b + (\nabla a)\odot \nabla b + a\odot\nabla^2 b\\ &= (\nabla^2 a)\odot b + a\odot \nabla^2 b\end{aligned}$$

hence we have as in the case of connections,

$$\Theta^{\{pq\}}_{\{ij\}} = \begin{cases}\Theta^q_j(E) & \text{if } p=i\le q,\\ \Theta^p_i(E) & \text{if } p\le j=q,\\ 0 & \text{otherwise.}\end{cases}$$

Choose unitary local frame $\{e_i\}$ for E. It is clear that the frame $\{e_{\{ij\}}, 1\le i\le j\le r\}$ is also unitary. Let $\sigma = \sum_{1\le i\le j\le r} a^{ij} e_{\{ij\}}$ and define, for j fixed,

$$\sigma^{,j} = \sum_{1\le i\le j} a^{ij} e_i \in E$$

and for i fixed,

$$\sigma^{i,} = \sum_{1\le i\le j} a^{ij} e_j \in E$$

so that $\sigma = \sum_j \sigma^{,j}\odot e_j$. Denote by $\hat{K}$ (resp. K) the curvature operator on $\odot^2 E$ (resp. E) then

$$\begin{aligned}< \hat{K}\sigma,\sigma >_{h\odot h} &= \sum_{i,j,p,q} a^{ij}\overline{a^{pq}}\Theta^{\{pq\}}_{\{ij\}}\\ &= \sum_{1\le i\le j,q\le r} a^{ij}\overline{a^{iq}}\Theta^q_j + \sum_{1\le i,p\le j\le r} a^{ij}\overline{a^{pj}}\Theta^p_i\\ &= \sum_j < K\sigma^{,j},\sigma^{,j} >_{h\odot h} + \sum_i < K\sigma^{i,},\sigma^{i,} >_{h\odot h} .\end{aligned}$$

Thus the mixed bisectional curvature:

$$\begin{aligned}\frac{< \hat{K}\sigma,\sigma >_{h\odot h}}{||X||^2||\sigma||^2} &= \sum_j \frac{< K\sigma^{,j},\sigma^{,j} >_{h\odot h}}{||X||^2||\sigma||^2} + \sum_i \frac{< K\sigma^{i,},\sigma^{i,} >_{h\odot h}}{||X||^2||\sigma||^2}\\ &= \sum_j \frac{< K\sigma^{,j},\sigma^{,j} >_{h\odot h}}{||X||^2||\sigma^{j,}||^2_h}\frac{||\sigma^{j,}||^2_h}{||\sigma||^2_{h\odot h}} + \sum_i \frac{< K\sigma^{i,},\sigma^{i,} >_{h\odot h}}{||X||^2||\sigma^{i,}||^2_h}\frac{||\sigma^{i,}||^2_h}{||\sigma||^2_{h\odot h}}.\end{aligned}$$

An entirely similar calculation can be carried out for the symmetric product $\odot^m E$ for any positive integer m and we have:

Proposition 5.3. *Let (E,h) be an hermitian (resp. Finsler which is pseudoconvex along the fibers) holomorphic vector bundle over a complex manifold M. If the holomorphic bisectional curvature of (E,h) is ≤ 0 (resp. <0 resp. ≥ 0, resp. >0) then, for $1\le m\le r$ the holomorphic bisectional curvature of $(\odot^m E,\odot^m h)$ (resp. $(\odot^m \mathcal{V}_E,\odot^m h)$) is ≤ 0 (resp. <0 resp. ≥ 0, resp. >0).*

For the symmetric bundle the converse is also true. This can be seen as follows. The bundle E is embedded in $\odot^2 E$ via the diagonal map:

$$\iota : E \cong \Delta \subset \odot^2 E, \;\; s \mapsto \odot^2 s$$

thus $\iota_* : TE \cong T\Delta \subset T(\odot^2 E) = \odot^2 TE$. Let H be a hermitian metric on $\odot^m E$ and $h = H|_\Delta$ be the induced metric. Choose an unitary basis $f_1, .., f_N$ of $\odot^2 E$ such that $f_1 = e_1 \odot e_1, ..., f_r = e_r \odot e_r$ is an unitary basis of Δ. Let $\hat{D}$ (resp. D) be the hermitian connection of H (resp. h) with respect to the basis chosen

$$\hat{D} f_i = \sum_{i=1}^{N} \hat{\theta}_i^j f_j$$

for $1 \leq i \leq N$ and the connection on E:

$$D f_i = \sum_{i=1}^{r} \theta_i^j f_j, \; \theta_i^j = \iota^* \hat{\theta}_i^j$$

for $1 \leq i \leq r$. Let $Q = \odot^2 E/\Delta$ be the quotient bundle with the quotient metric and the quotient connection and define an operator

$$A = \hat{D}|_\Delta - D : \Gamma(\Delta) \to \Gamma(Q)$$

then the connection matrices are related by

$$\hat{\theta} = \begin{pmatrix} \theta_\Delta & {}^t\overline{A} \\ A & \theta_Q \end{pmatrix}$$

where A is represented, with respect to the chosen frame, by a matrix $(a_i^j)_{r+1 \leq i \leq N, 1 \leq j \leq r}$ of 1-forms. The curvature forms are given by:

$$\hat{D}^2 f_i = \sum_{i,k=1}^{N} \hat{\Theta}_i^k \otimes f_k, \; \hat{\Theta}_i^k = d\hat{\theta}_i^k - \sum_{j=1}^{N} \hat{\theta}_i^j \wedge \hat{\theta}_j^k$$

for $1 \leq i \leq N$ and

$$D^2 f_i = \sum_{i,k=1}^{r} \Theta_i^k \otimes f_k, \; \Theta_i^k = d\theta_i^k - \sum_{j=1}^{r} \theta_i^j \wedge \theta_j^k.$$

The curvature matrices are related by (see [])

$$\hat{\Theta} = \begin{pmatrix} \Theta_\Delta -{}^t \overline{A} \wedge A & * \\ * & \Theta_Q + A \wedge {}^t \overline{A} \end{pmatrix}$$

where ${}^t\overline{A} \wedge A = (A_i^k = \sum_{r+1\le j\le N} \bar{a}_i^j \wedge a_j^k)_{1\le i,k\le r}$. Thus, for $\sigma = \sum_{j=1}^r a^j f_j \in \Delta$,

$$
\begin{aligned}
< \hat{K}\sigma, \sigma >_H &=< \sum_{i,j=1}^r a^i \hat{\Theta}_i^j f_j, \sum_{k=1}^r a^k f_k >_H \\
&=< \sum_{j=1}^r a^j (\Theta_i^j - A_i^k) f_j, \sum_{k=1}^r a^k f_k >_h \\
&=< K\sigma, \sigma >_h - < \sum_{j=1}^r a^j A_i^k f_j, \sum_{k=1}^r a^k f_k >_h .
\end{aligned}
$$

The second term on the right above is positive (see []) hence we have a partial converse of Proposition 4.12:

Proposition 5.4. *Let E be a holomorphic vector bundle and suppose that, for some positive integer m there exists a hermitian metric H on $\odot^m E$ such that the mixed holomorphic bisectional curvature of $(\odot^m E, H)$ is ≤ 0 (resp. < 0), resp. > 0) then the mixed holomorphic bisectional curvature of (E, h) is ≤ 0 (resp. < 0) where h is the hermitian metric induced by H via the diagonal embedding of E in $\odot^m E$.*

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