C^*-ALGEBRAS AND THE GELFAND-NAIMARK-SEGAL CONSTRUCTION

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Abstract. These are lecture notes for an survey talk in the Math Physics seminar at Notre Dame on the topic in the title. Detailed proofs of all results not mentioned here can be found in the three references. I tried to be long on perspective but short on technicalities in the discussion.

1. Topological Preliminaries

The following notation will be used throughout the talk:

- $\mathcal{H}$: a complex Hilbert space.
- $(\cdot,\cdot)_{\mathcal{H}}$: the Hermitian inner product on $\mathcal{H}$, conjugate linear in first argument, linear in second.
- $\|\cdot\|_{\mathcal{H}}$: norm on $\mathcal{H}$ induced from inner product.
- $\mathcal{B}(\mathcal{H})$: complex vector space of continuous linear endomorphisms of $\mathcal{H}$.

There are many topologies on the set $\mathcal{B}(\mathcal{H})$. Two will be of importance in this talk. They are defined in terms of convergence of nets $\alpha \mapsto T_\alpha$ as follows:

Uniform Operator Topology: $T_\alpha \to T$ iff $\|T_\alpha - T\| \to 0$ where

$$\|T\| := \sup_{\phi\neq 0} \frac{\|T\phi\|_{\mathcal{H}}}{\|\phi\|_{\mathcal{H}}} \text{ for each } T \in \mathcal{B}(\mathcal{H})$$

is the operator norm.

Weak Operator Topology: $T_\alpha \to T$ iff $(\phi, T_\alpha\psi) \to (\phi, T\psi)$ for each $\phi, \psi \in \mathcal{H}$.

Remark 1. If $\mathcal{H}$ is separable, then one can use sequences instead of nets to define these topologies on $\mathcal{B}(\mathcal{H})$.

I. The topologies are comparable and in general the uniform topology has more open sets than the weak topology. The topologies coincide if and only if $\dim \mathcal{H} < \infty$.
   a. uniform convergence $\Rightarrow$ weak convergence.
   b. closed in the uniform operator topology $\Leftarrow$ closed in the weak operator topology.

II. $\mathcal{B}(\mathcal{H})$ is an associative unital complex algebra with respect to composition and point-wise addition and scalar multiplication with $\mathrm{id}_{\mathcal{H}}$ acting as 1.

III. $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is a Banach space.

2. $*$-SUBALGEBRAS OF $\mathcal{B}(\mathcal{H})$

Definition 2.1. A *$*$-subalgebra of* $\mathcal{B}(\mathcal{H})$ is a sub-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ such that $A^* \in \mathcal{A}$ for each $A \in \mathcal{A}$ where A^* is the Hilbert adjoint operator of A.

Example 2.1. The following constructions exhibit families of $*$-subalgebras of $\mathcal{B}(\mathcal{H})$.

I. $T \in \mathcal{B}(\mathcal{H}) \Rightarrow T^* \in \mathcal{B}(\mathcal{H})$. Thus $\mathcal{B}(\mathcal{H})$ is a $*$-subalgebra of itself. It is non-commutative if and only $\dim \mathcal{H} > 1$.

II. If $T_1, \dots, T_n \in \mathcal{B}(\mathcal{H})$ is a commuting family of normal operators then

$$\{p(T_1, \dots, T_n, T_1^*, \dots, T_n^*) \colon p \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]\}$$

is a commutative unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

III. If $T \colon \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator with spectrum $\sigma(T) \subset \mathbb{C}$, then $\{f(T) \colon f \in C_0(\sigma(T))\}$ is a commutative $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is unital if and only if T is bounded.

IV. Let $\mathcal{S}$ be a subset of $\mathcal{B}(\mathcal{H})$ such that $S \in \mathcal{S} \Rightarrow S^* \in \mathcal{S}$ and set $\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) \colon [T, S] = 0 \ \forall\ S \in \mathcal{S}\}$. Then $\mathcal{S}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ called the *commutant* of $\mathcal{S}$.

V. The set of finite rank operators $\mathcal{F}(\mathcal{H})$ forms a $*$-subalgebra and a non-zero ideal of $\mathcal{B}(\mathcal{H})$ which is unital iff $\dim \mathcal{H} < \infty$. The closure $\mathcal{K}(\mathcal{H})$ of $\mathcal{F}(\mathcal{H})$ in the uniform topology is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. It is a proper ideal of $\mathcal{B}(\mathcal{H})$ if and only if $\dim \mathcal{H} < \infty$.

Definition 2.2. A *von Neumann algebra on* $\mathcal{H}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology.

Definition 2.3. A *concrete C^*-algebra on* $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the uniform operator topology.

Remark 2. By Remark 1.I.b, every von Neumann algebra on $\mathcal{H}$ is, in particular, a unital concrete C^*-algebra on $\mathcal{H}$.

I. $\mathcal{B}(\mathcal{H})$ is both a von Neumann algebra on $\mathcal{H}$ and a concrete C^*-algebra on $\mathcal{H}$.

II. Example 1.II is neither a von Neumann algebra nor a concrete C^*-algebra on $\mathcal{H}$ because it is not closed in either topology.

III. When $\mathcal{H}$ is not finite dimensional, Example 1.III is a concrete C^*-algebra on $\mathcal{H}$ which is not a von Neumann algebra.

IV. It is a theorem of von Neumann that if $\mathcal{S}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ then its bicommutant $\mathcal{S}'' = (\mathcal{S}')'$ is equal to the closure of $\mathcal{S}$ in the weak operator topology.

IV. When $\mathcal{H}$ is not finite dimensional, $\mathcal{K}(\mathcal{H})$ is a concrete C^*-algebra which is not a von Neumann algebra on $\mathcal{H}$ because it is not unital.

3. C^*-ALGEBRAS

Of the properties of the operator norm $\|\cdot\|$ and the Hilbert adjoint operator $T \mapsto T^*$ on $\mathcal{B}(\mathcal{H})$, two are important here.

Banach Inequality: $\|T_1T_2\| \leq \|T_1\|\|T_2\|$ for each T_1 and T_2,

C^*-identity: $\|T^*T\| = \|T\|^2$ for each T.

Definition 3.1. A *(abstract) C^*-algebra* is a complex algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|_{\mathcal{A}}$ and a conjugate linear anti-involution $T \mapsto T^*$ (i.e., $(\lambda T_1 + T_2)^* = \bar{\lambda}T_1^* + T_2^*$, $(T_1T_2)^* = T_2^*T_1^*$, and $(T^*)^* = T$) for which the Banach inequality and the C^*-identity hold and which is complete as a normed linear space.

Definition 3.2. A *morphism of C^*-algebras* $\phi\colon \mathcal{A} \to \mathcal{A}'$ is a complex algebra homomorphism such that $\phi(T^*) = (\phi(T))^*$ $\forall\ T \in \mathcal{A}$. If $\mathcal{A}' = \mathcal{B}(\mathcal{H}_\phi)$ for some Hilbert space $\mathcal{H}_\phi$, then ϕ is said to be a *$*$-representation of $\mathcal{A}$ on $\mathcal{H}_\phi$.*

Remark 3. It is a theorem that a morphism $\phi\colon \mathcal{A} \to \mathcal{A}'$ of C^*-algebras is automatically continuous. Moreover, if ϕ is non-zero, then ϕ has norm 1 as a linear map between Banach spaces.

Example 3.1 (Group Algebras)**.** Let Γ be a group. Define $\delta_\gamma \in \mathrm{Maps}(\Gamma, \mathbb{C})$ $\forall\ \gamma \in \Gamma$ by

$$\delta_\gamma(\gamma') = \begin{cases} 1 & \text{if } \gamma' = \gamma \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ \gamma' \in \Gamma.$$

Set $C_0(\Gamma) = \mathrm{span}_{\mathbb{C}}\{\delta_\gamma \colon \gamma \in \Gamma\}$. The assignments $\delta_\gamma \star \delta_{\gamma'} \mapsto \delta_{\gamma\gamma'}$ for each $\gamma, \gamma' \in \Gamma$ determine an associative unital algebra structure on $C_0(\Gamma)$ which is commutative if and only if Γ is abelian. Denoted by $\mathbb{C}\Gamma$, this algebra is called the *complex group algebra of* $\mathbb{C}$.

If $f = \sum_{\gamma\in\Gamma} f(\gamma)\delta_\gamma$ and $g = \sum_{\gamma\in\Gamma} g(\gamma)\delta_\gamma$ (finite support) then

$$f \star g = \sum_{\gamma\in\Gamma}\left(\sum_{\tau\in\Gamma} f(\gamma\tau^{-1})g(\tau)\right)\delta_\gamma.$$

The assignments $f \mapsto (g \mapsto f \star g \, \forall \, g \in C_0(\Gamma)$ for each $f \in \mathbb{C}\Gamma$ define a faithful representation of $\lambda\colon \mathbb{C}\Gamma \to \mathrm{End}(C_0(\Gamma))$ by finite rank endomorphisms of $C_0(\Gamma)$. By continuity this extends uniquely in the codomain to a faithful representation $\lambda\colon \mathbb{C}\Gamma \to \mathcal{B}(\ell^2(\Gamma))$ where $(f,g)_{\ell^2(\Gamma)} := \sum_{\gamma\in\Gamma} \overline{f(\gamma)} g(\gamma)$. Define a $*$-operation by

$$f = \sum_{\gamma\in\Gamma} f(\gamma)\delta_\gamma \mapsto f^* = \sum_{\gamma\in\Gamma} \overline{f(\gamma^{-1})}\delta_\gamma$$

then λ is a $*$-representation of $\mathbb{C}\Gamma$.

The uniform operator closure of $\lambda(\mathbb{C}\Gamma) \subset \mathcal{B}(\ell^2(\Gamma))$, denoted $C^*_r(\Gamma)$, is called the *reduced group C^*-algebra of* Γ. The weak operator closure is denoted $vN(\Gamma)$ and is called the *group von Neumann algebra of* Γ.

FACT: If $\phi\colon \mathbb{C}\Gamma \to \mathcal{B}(\mathcal{H}_\phi)$ is a $*$-representation of $\mathbb{C}\Gamma$ then $\|\phi(f)\| \le \|f\|_1 := \sum_{\gamma\in\Gamma} |f(\gamma)|$ for each $f \in \mathbb{C}\Gamma$. Therefore

$$\|f\|_{\sup} := \sup\{\|\phi(f)\| \colon \phi\colon \mathbb{C}\Gamma \to \mathcal{B}(\mathcal{H}_\phi) \text{ is a } *\text{-representation}\}$$

defines a norm on $\mathbb{C}\Gamma$ satisfying the Banach inequality and C^*-identity. The abstract completion of $\mathbb{C}\Gamma$ with respect to $\|\cdot\|_{\sup}$, denoted $C^*(\Gamma)$, is called the *group C^*-algebra of* Γ.

Remark 4 (Group C^*-algebras for topological groups)*.* Group C^*-algebras are completely understood in the case of finite groups, but little is known in general for infinite groups. In that case it often is better to find a topology with respect to which the group operations are continuous and which has fewer open sets than the discrete topology. For locally compact Hausdorff topological groups G there exists a left-invariant regular Borel measure μ on G, known as left Haar measure, which is unique up to a scalar multiple. Then $L^1(G, d\mu)$ with $f \star g$ defined by

$$(f \star g)(x) = \int_G f(xy^{-1}) g(y) d\mu(y) \ \ \forall \ x \in G$$

and $f \mapsto f^*$ defined by

$$f(x) = \overline{f(x^{-1})} \ \forall \ x \in G$$

is a $*$-algebra which is unital if and only if G has the discrete topology. As above, one obtains the reduced (topological) group C^*-algebra $C^*_r(G)$ as the uniform closure $\lambda(L^1(G,d\mu))$ in $\mathcal{B}(L^2(G,d\mu))$, and $C^*(G)$ as the abstract completion of $L^1(G,d\mu)$ with respect to $\|\cdot\|_{\sup}$.

Remark 5*.* The assignment $\Gamma \mapsto C^*(\Gamma)$ induces a functor from the category of groups to the category of C^*-algebras. However, it is not faithful because $C^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq C^*(\mathbb{Z}/4\mathbb{Z})$ for example.

Example 3.2. Let X be a locally compact Hausdorff topological space. Then $C_0(X)$ with the operations of point-wise addition, multiplication, and scalar multiplication is a commutative C^*-algebra when equipped with point-wise complex conjugation and the sup norm, $\|f\|_\infty := \sup_{x\in X}|f(x)|$. $C_0(X)$ is unital if and only if X is compact. Write $C(X)$ in that case. Example 2.1.III gives is a $*$-representation of $C_0(\sigma(T))$ where $\sigma(T)$ is the spectrum of a self-adjoint operator $T\colon D(T)\subset\mathcal{H}\to\mathcal{H}$.

Theorem 3.1 (Gelfand)**.** *Let $\mathcal{A}$ be a commutative unital C^*-algebra. The set* $\operatorname{Specm}(\mathcal{A})$ *of complex algebra homomorphisms $\mathcal{A}\to\mathbb{C}$ is a compact (Hausdorff) subset of the dual $\mathcal{A}^\vee$ with the weak-$*$ topology. The map $a\mapsto\hat{a}$ where $\hat{a}(\rho):=\rho(a)$ for each $\rho\in\mathcal{A}^\vee$ is an isomorphism of C^*-algebras* $\mathcal{A}\to C(\operatorname{Specm}(\mathcal{A}))$*, i.e., every commutative C^*-algebra is isomorphic to a C^*-algebra of continuous functions on a compact set.*

Remark 6. In the proof, one shows that non-zero complex algebra homomorphisms $\mathcal{A}\to\mathbb{C}$ are: 1) automatically bounded and 2) forced to be (non-zero) morphisms of C^*-algebras, and thus have norm 1.

4. Positive Elements, Positive Functionals, and States

Definition 4.1. Let $\mathcal{A}$ be a C^*-algebra, then $a\in\mathcal{A}$ is a *positive element* iff $a=b^*b$ for some $b\in\mathcal{A}$.

Definition 4.2. A linear functional $\rho\colon\mathcal{A}\to\mathbb{C}$ is *positive* iff $\rho(a)\geq 0$ for each $a\in\mathcal{A}^+$.

Definition 4.3. A positive linear functional with norm 1 is called a *state*. Write $\mathcal{S}(\mathcal{A})$ for the set of all states on $\mathcal{A}$.

Remark 7. The following are some important facts to know.

I. The unit ball of $\mathcal{A}^\vee$ is compact in the weak-* topology (Banach-Alaoglu Theorem).
II. $\mathcal{S}(\mathcal{A})$ is a weak-* closed subset of the unit ball in $\mathcal{A}^\vee$ and therefore compact.
III. $\mathcal{S}(\mathcal{A})$ is a convex set.
IV. If ρ is a positive linear functional then
    a. $\overline{\rho(a)}=\rho(a^*)\ \forall\ a\in\mathcal{A}^+$, which $\Rightarrow$
    b. "Cauchy-Schwarz Inequality" $|\rho(a^*b)|^2\leq\rho(a^*a)\rho(b^*b)\ \forall\ a,b\in\mathcal{A}$, which $\Rightarrow$
    c. ρ is bounded and therefore continuous.

Definition 4.4. A *pure state* on $\mathcal{A}$ is an extremal point of $\mathcal{S}(\mathcal{A})$. Write $\mathcal{PS}(\mathcal{A})$ for the set of all pure states on $\mathcal{A}$.

Example 4.1. Suppose $\dim \mathcal{H} = 2$. Choose an ON basis and identify $\mathcal{B}(\mathcal{H}) \simeq M_2(\mathbb{C})$. Define $\|A\|_{\mathrm{tr}} := \mathrm{tr}((A^*A)^{1/2}) \; \forall \; A \in M_2(\mathbb{C})$. The map $A \mapsto (B \mapsto \mathrm{tr}(AB))$ is an isometry $(M_2(\mathbb{C}), \|\cdot\|_{\mathrm{tr}}) \to (M_2(\mathbb{C})^\vee, \|\cdot\|)$ which furthermore identifies $M_2(\mathbb{C})^+$ with $\{\rho \in M_2(\mathbb{C})^\vee \colon \rho(a) \geq 0 \, \forall \, a \in \mathcal{A}^+\}$. Since $M_2(\mathbb{C})$ is finite dimensional, the topology induced by $\|\cdot\|_{\mathrm{tr}}$ and the weak-* topology coincide. $\mathcal{S}(M_2(\mathbb{C}))$ is identified with the non-negative 2×2 matrices of trace equal to 1. The determinant of such a matrix must also be non-negative so $\mathcal{S}(\mathcal{A})$ is identified with

$$\left\{ \begin{pmatrix} a & x+iy \\ x-iy & 1-a \end{pmatrix} : a(1-a) - x^2 - y^2 \geq 0 \right\}.$$

By completing the square, $a(1-a) - x^2 - y^2 \geq 0$ is equivalent to $(a - \frac{1}{2})^2 + x^2 + y^2 \leq \frac{1}{4}$. In other words, $\mathcal{S}(\mathcal{A})$ is a ball of radius $\frac{1}{2}$ centered on the scalar matrix $\frac{1}{2}I_2$. Since the boundary points of a Euclidean ball are precisely the extremal points, we see that $\mathcal{PS}(M_2(\mathbb{C}))$ is S^2 as a topological space. This is precisely the U_2-orbit through the matrix

$$r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

under conjugation.

5. The GNS Construction

The Gelfand-Naimark-Segal (GNS) construction builds a $*$-representation of a C^*-algebra $\mathcal{A}$ for each positive linear functional ρ on $\mathcal{A}$. Here is the idea:

1. If I is a left ideal of an associative algebra A, then A/I is a left A-module. This gives an algebra representation $\phi \colon A \to \mathrm{End}(A/I)$.
2. Remark 7.IV $\Rightarrow$ if ρ is a positive linear functional then $I_\rho = \{a \in \mathcal{A} \colon \rho(a^*a) = 0\}$ is a left ideal of $\mathcal{A}$.
3. The formula $\langle a + I_\rho, b + I_\rho \rangle_{\mathcal{H}_\rho} := \rho(a^*b)$ defines a Hermitian inner product on $\mathcal{A}/I_\rho$.
4. Define $\mathcal{H}_\rho$ to be the completion of $\mathcal{A}/I_\rho$ with respect to the induced norm $\|\cdot\|_{\mathcal{H}_\rho}$.
5. USEFUL FACT: $\|a\|_{\mathcal{A}}^2 b^*b - b^*a^*ab \in \mathcal{A}^+$ for each $a, b \in \mathcal{A}$ which $\Rightarrow$
6. $\|a(b + I_\rho)\|_{\mathcal{H}_\rho}^2 = \|ab + I_\rho\|_{\mathcal{H}_\rho}^2 = \rho(b^*a^*ab) \leq \|a\|_{\mathcal{A}}^2 \rho(b^*b) = \|a\|_{\mathcal{A}}^2 \|b + I_\rho\|_{\mathcal{H}_\rho}^2 \; \forall \; a, b \in \mathcal{A}$ which $\Rightarrow$
7. representation induced by left action extends uniquely to a representation $\phi_\rho \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$.

8. ϕ_ρ is a $*$-representation by construction because

$$\langle a+I_\rho, c(b+I_\rho)\rangle_{\mathcal{H}_\rho} = \langle a+I_\rho, cb+I_\rho\rangle_{\mathcal{H}_\rho} = \rho(a^*cb) = \rho((c^*a)^*b) = \langle c^*(a+I_\rho), b+I_\rho\rangle_{\mathcal{H}_\rho}.$$

Remark 8. GNS representations enjoy the following properties:

I. (Cyclicity) $\exists\ \xi_\rho \in \mathcal{H}_\rho$ such that
   a. $\phi_\rho(\mathfrak{A})\xi_\rho$ is dense in $\mathcal{H}_\rho$ and
   b. $\langle \xi_\rho, \phi_\rho(a)\xi_\rho\rangle_{\mathcal{H}_\rho} = \rho(a)\ \forall\ a \in A$.

   When $\mathcal{A}$ is unital, one can take $\xi_\rho = 1 + I_\rho \in \mathcal{A}/I_\rho \subset \mathcal{H}_\rho$. If, in addition, ρ is a state, the ξ_ρ is a unit vector. In the non-unital case, one uses then existence of approximate identities and analysis to construct ξ. Every $*$-representation is unitarily equivalent to a direct sum of cyclic representations by a Zorn's Lemma argument. If $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a cyclic $*$-representation with cyclic vector ξ, then ϕ is unitarily equivalent to the GNS representation ϕ_{ρ_ξ} defined by the positive linear functional $\rho_\xi(a) = (\xi, \phi(a)\xi)\ \forall\ a \in \mathcal{A}$.

II. (Pure States and Irreducibility) The GNS representation ϕ_ρ is irreducible if and only if ρ is a pure state. Moreover, every irreducible $*$-representation of $\mathcal{A}$ can be constructed this way (up to unitary equivalence).

III. (Tracial States and Unitary Equivalence) Left multiplication by $u \in \mathfrak{A}$ such that $u^{-1} = u^*$ induces a unitary transformation of $\mathcal{H}_\rho$. If ρ is a trace, then right multiplication by u such that $u^{-1} = u^*$ also induces a unitary transformation of $\mathcal{H}_\rho$. Thus, if two tracial states ρ and ρ' are related by pre-composition with conjugation by u then ϕ_ρ and $\phi_{\rho'}$ are unitarily equivalent.

Example 5.1. In Example 4.1 $\mathcal{PS}(M_2(\mathbb{C}))$ was identified with the U_2-orbit through the matrix $r = \mathrm{diag}(1,0)$ in $M_2(\mathbb{C})$ under conjugation. Let ρ be the linear functional $b \mapsto \mathrm{tr}(rb)$ for each $b \in M_2(\mathbb{C})$. Then

$$I_\rho = \left\{ \begin{pmatrix} 0 & \beta \\ 0 & \delta \end{pmatrix} : \beta, \delta \in \mathbb{C} \right\} \subset M_2(\mathbb{C}) \text{ and } \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ \gamma & 0 \end{pmatrix} + I_\rho$$

gives an isometry of $\mathbb{C}^2$ with the standard Hermitian inner product onto $\mathcal{H}_\rho$. The representation is the defining representation. Since ρ and its orbits under the co-adjoint action of U_2 are all tracial states, every pure state of $M_2(\mathbb{C})$ gives rise to the defining representation up to unitary equivalence.

For comparison, $s = \operatorname{diag}(\frac{1}{2}, \frac{1}{2})$ was the center of the Euclidean ball in $M_2(\mathbb{C})$ that was identified with $\mathcal{S}(M_2(\mathbb{C}))$. For its associated functional, I_ρ is the trivial ideal. In the GNS representation $\mathcal{H}_\rho$ is $M_2(\mathbb{C})$ with the inner product $(A, B) = \frac{1}{2}\operatorname{tr}(A^*B)$ which breaks up as a direct sum of the (unitarily equivalent) irreducible representations corresponding to $\operatorname{diag}(1, 0)$ and $\operatorname{diag}(0, 1)$.

Example 5.2. If X is a locally compact Hausdorff space, then $\mathcal{S}(C_0(X))$ is identified with the set of Borel probability measures on X via the Riesz representation theorem and $\mathcal{PS}(C_0(X))$ corresponds to the set of Dirac delta measures on X. Applied to μ, the GNS construction yields the multiplication representation of $C_0(X)$ on $L^2(X, \mu)$.

Example 5.3. If Γ is a group, then $\rho(f) := f(1)\ \forall\ f \in C^*(\Gamma)$ defines a state on $C^*(\Gamma)$. The GNS construction produces the left regular representation of $C^*(\Gamma)$ from ρ.

Theorem 5.1 (Gelfand-Naimark)**.** *Every C^*-algebra admits a faithful $*$-representation. If the C^*-algebra is separable, then the Hilbert space can be taken to be separable.*

Strategy of the Proof. If $\mathcal{A}$ is unital, then the "Cauchy-Schwarz" inequality implies that if ρ a positive linear functional and $\rho(a) = 0$ then $\rho(a^*a) = 0$. A similar statement is proved in the non-unital case using existence of approximate identities. If ρ is a positive linear functional, then $0 = \rho(a^*a) = \langle \xi_\rho, \phi_\rho(a^*a)\xi_\rho\rangle_{\mathcal{H}_\rho} = \langle \phi_\rho(a)\xi_\rho, \phi_\rho(a)\xi_\rho\rangle_{\mathcal{H}_\rho} = \|\phi_\rho(a)\xi_\rho\|^2_{\mathcal{H}_\rho}$, whence ϕ_ρ is not faithful.

If $F \subset \mathcal{S}(\mathcal{A})$ has the property that $\forall\ a \in \mathcal{A}^+ \setminus \{0\}\ \exists\ \rho \in F$ such that $\rho(a) \neq 0$, then $\phi_F := \oplus_{\rho\in F}\phi_\rho \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_F)$ where $\mathcal{H}_F := \bigoplus_{\rho\in F}\mathcal{H}_\rho$ is a faithful representation. Hence, we need to show the existence of such a family. The crux is the proof of the following assertion.

Assertion: For each $a \in \mathcal{A}^+\ \exists\ \rho \in \mathcal{S}(\mathcal{A})$ such that $\rho(a) = \|a\|_{\mathcal{A}}$.

In the separable case, the GNS Hilbert space $\mathcal{H}_\rho$ is separable, so we can obtain a separable $\mathcal{H}_F$ as above if the family F is countable. But, $\mathcal{A}$ separable implies the existence of a sequence $\{a_n\}$ which is dense in $\mathcal{A}^+ \cap \{a \in \mathcal{A} \colon \|a\|_{\mathcal{A}} = 1\}$. For each $n \in \mathbb{N}^*$, choose $\rho_n \in \mathcal{S}(\mathcal{A})$ such that $\rho_n(a_n) = 1$, and let $F = \{\rho_n\}_{n=1}^\infty$. Then F has the desired property and $\mathcal{H}_F$ is separable. □

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