Toric Poisson Structures

Lie Theory Seminar

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Poisson Structures

A *Poisson Structure* on M is $\pi \in \Gamma(\bigwedge^2 TM) \simeq \Gamma(\bigwedge^2 T^{**}M)$ such that $\{f,g\} := \pi(df, dg)$ defines a Lie bracket on $C^\infty(M)$ ($[\pi,\pi] = 0$).

Geometry of Poisson Structures:

- Anchor map: $\pi^\# : T^*M \to TM$ defined by $\langle \beta, \pi^\#(\alpha)\rangle = \pi(\alpha, \beta)$.
- π skew-symmetric $\Rightarrow$ image of $\pi^\#$ is even dimensional.
- $\pi^\#(T^*M)$ is a (generalized) distribution $S(M)$ in TM.
- Jacobi identity for $\{\cdot,\cdot\} \Rightarrow \pi^\#(T^*M)$ is integrable and the inverse 2-form along the leaves is closed.

Upshot: (M, π) is foliated by symplectic manifolds.

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Lots of Examples!

- (Trivial PS) $\pi = 0$ on any M.

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- (Std. PLGS) K a cpt. s.s. Lie group. Standard construction $\longrightarrow$ Poisson structure K such that $K \times K \to K$ sends $\pi_K \oplus \pi_K \longrightarrow \pi_K$. Vanishes on a maximal torus T of K.

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Key Point: Interested in special examples.

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Goal of the construction

Desire: Π_Σ should be not-symplectic, yet non-degenerate on an open dense set, and have symmetry.

(M,π)	$\pi = 0$	$(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$	$(K/T, \pi_{K/T})$	π non-deg.
$S(M)$	points	Co-Ad orbits	Bruhat Cells	one leaf = M
	reg.	not reg.	not reg.	reg.

- Flag manifold K/T

  KKS symplectic structure $\leftarrow K/T \rightarrow$ Bruhat Poisson structure

- Smooth toric variety $X(\Sigma)$

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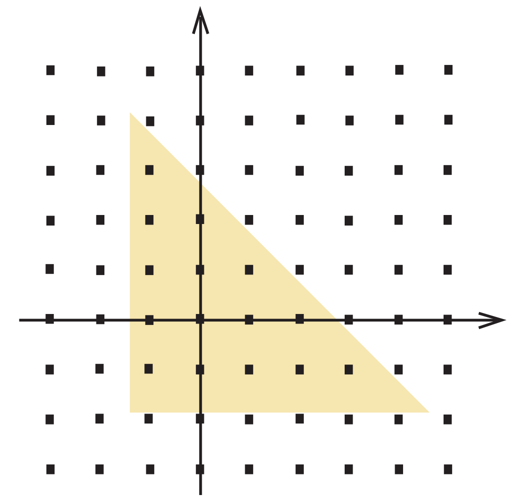
Symplectic vs. Algebraic Geometry

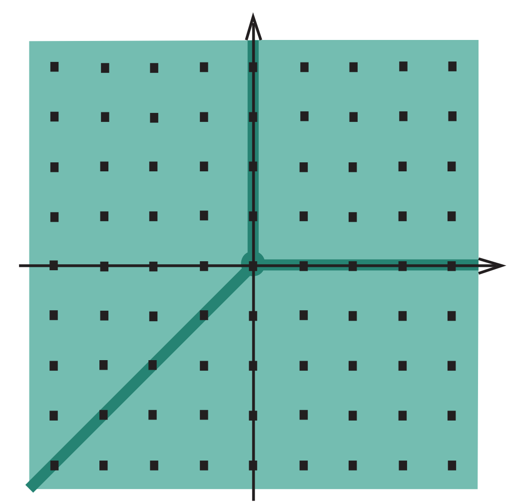
Symplectic Geometry:

- $\Delta \subset \mathfrak{t}^*$ *Delzant polytope* with d facets. $(X_\Delta, \omega_\Delta)$ = symplectic reduction of $(\mathbb{C}^d, \omega_{\text{std}})$ with respect to action of $N \lhd \mathbb{T}^d$.
- Torus N defined by "shape" of Δ.
- Level determined by "size" of Δ.



Algebraic Geometry:

- Σ = the dual fan of Δ ("shape")
- Smooth $\mathbb{C}$-variety $X(\Sigma) = \mathbb{C}^d//N_\mathbb{C}$ with action of $T_\mathbb{C} \simeq \mathbb{T}^d_\mathbb{C}/N_\mathbb{C}$ having an open dense orbit.
- Fact: compact X_Δ is homeomorphic to $X(\Sigma)$.



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Construction of Π_Σ

Poisson Geometry:

- $\pi = \sum_{\ell=1}^{d} i\partial_{\overline{z}_\ell} \wedge \partial_{z_\ell}$ is translation invariant under action of $\mathbb{C}^d$.
- $\Pi := \exp_*(\pi) = \sum_{\ell=1}^{d} i|z_\ell|^2 \partial_{\overline{z}_\ell} \wedge \partial_{z_\ell}$ extends smoothly to all of $\mathbb{C}^d$.
- Π is invariant under action of $\mathbb{T}_{\mathbb{C}}^d$.
- $[\Pi, \Pi] = [\exp_*(\pi), \exp_*(\pi)] = \exp_*([\pi, \pi]) = 0$.
- $\mathbb{C}^d//N_{\mathbb{C}}$ means $\mathcal{U}_\Sigma/N_{\mathbb{C}}$ where $\mathcal{U}_\Sigma \subset \mathbb{C}^d$ is the free locus of $N_{\mathbb{C}}$ action.
- Π_Σ is the Poisson structure on $X(\Sigma)$ co-induced from Π by the quotient map $\mathcal{U}_\Sigma \to \mathcal{U}_\Sigma/N_{\mathbb{C}}$.

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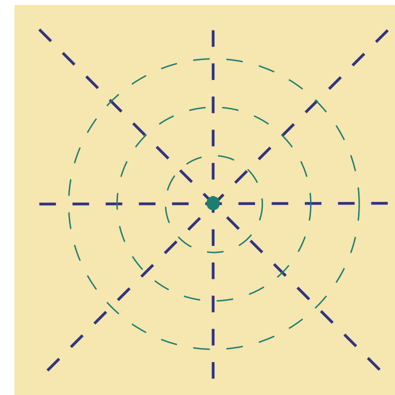
Symplectic Leaves

Theorem 1 *The symplectic leaves of* $(X(\Sigma), \Pi_\Sigma)$ *are the orbits of* $T_{\mathbb{C}} \simeq \mathbb{T}^d / N_{\mathbb{C}}$. *In particular,* Π_Σ *is non-degenerate on an open dense set.*



Local picture when $\dim_{\mathbb{C}} T_{\mathbb{C}} = 1$:

- $\Pi_\Sigma = 2i|w|^2 \partial_{\overline{w}} \wedge \partial_w = (x^2 + y^2)\partial_x \wedge \partial_y$.
- T action Hamiltonian with primitive $\approx \log|w|$.
- $T_{\mathbb{C}}/T$ action not Hamiltonian due to topology of open leaf, local primitive $\approx \operatorname{Arg}(w)$.

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Poisson Cohomology

- Schouten bracket of multi-vector fields has bi-degree -1.
- Jacobi for $\{\cdot,\cdot\} \Leftrightarrow [\pi,\pi]=0 \Rightarrow [\cdot,\pi]$ a differential.
- Homology of this complex $H(M,\pi)$ is the Poisson cohomology of (M,π). Very hard to compute!

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d} & \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) & \xrightarrow{d} & \cdots \\
 & & \downarrow{\scriptstyle \pi^\#} & & \downarrow{\scriptstyle \pi^\#} & & \\
\cdots & \xrightarrow{[\cdot,\pi]} & \mathcal{V}^p(M) & \xrightarrow{[\cdot,\pi]} & \mathcal{V}^{p+1}(M) & \xrightarrow{[\cdot,\pi]} & \cdots
\end{array}$$

$H^1(M,\pi)$ = Poisson vector fields which are not Hamiltonian.

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Some generators of $H^1(X(\Sigma), \Pi_\Sigma)$

Theorem 2 *The action of* $T_\mathbb{C}$ *on* $(X(\Sigma), \Pi_\Sigma)$ *is Poisson but not Hamiltonian, however, each symplectic leaf admits a Hamiltonian action by a sub-torus of* T*.*

- In hol. coordinates $(w_1, \dots, w_n)$ assoc. to an open cone $V \in \Sigma$,

$$\Pi_\Sigma = \sum_{p,q=1}^{n} i B_V^{pq} \overline{w}_p w_q \partial_{\overline{w}_p} \wedge \partial_{w_q}$$

where B_V is a symm. pos. def. integral matrix determined by V.

Corollary 1 $\mathfrak{t} + i\mathfrak{t} \hookrightarrow H^1(X(\Sigma), \Pi_\Sigma)$

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The Modular Class

- (M, π) orientable, μ a volume form on M.
- θ_μ (modular vector field), char. by $\mathcal{L}_{\pi^\#(df)}\mu = \theta_\mu(f)\mu$.
- π non-deg. and $\mu = \pi^{-n} \Rightarrow \theta_\mu = 0$.
- $\mathcal{L}_{\theta_\mu}\pi = 0 \Rightarrow [\theta_\mu] \in H^1(M, \pi)$ (modular class).
- $\theta_{a\mu} = \theta_\mu - \pi^\#(d \log |a|)$ (class independence of μ.)

$[\theta_\mu]$ as an invariant:

(M, π)	$\pi = 0$	$(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$	$(K/T, \pi_{K/T})$	π non-deg.
$[\theta_\mu]$	$\theta_\mu = 0$	$[\theta_\mu] = 0$ iff $\mathfrak{g}$ unimodular	$[\theta_\mu] \neq 0$	$\theta_\mu = 0$

Key Point: $[\theta_\mu]$ is an interesting invariant away from the extremes.

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Modular class is non-trivial

Theorem 3 *If (M, π) is orientable and π is not symplectic, yet is non-degenerate on an open dense set, then the modular class is non-trivial.*

Proof:

- Let μ be a volume form on M and let U be the open set on which π is non-degenerate ($\dim M = 2n$).
- By assumption $\nu = (\pi|_U)^{-n}$ is a volume form on U and thus $\mu|_U = a\nu$ for some $a \in C^\infty(U)$ which does not vanish there.
- Hence, $\theta_\mu|_U = \theta_\nu - \pi^\#(d\log|a|) = -\pi^\#(d\log|a|)$, so θ_μ has a Hamiltonian primitive on U unique up to a locally constant function.
- However, none of these primitives extend continuously to all of M because $a = 0$ on $M \setminus U$. $\square$

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Estimate for $H^1(X(\Sigma), \Pi_\Sigma)$

Theorem 4 $\dim_{\mathbb{R}} H^1(X(\Sigma), \Pi_\Sigma) \geq 2\dim_{\mathbb{R}} T + 1$

Proof:

- In an affine chart $\mathbb{C}^n$ associated an open cone $V \in \Sigma$, $\mu = a\lambda$ where $\lambda = i^n\, dw_1 \wedge d\overline{w}_1 \wedge \cdots \wedge dw_n \wedge d\overline{w}_n$ and $a \in C^\infty(\mathbb{C}^n)$ is non-vanishing.
- Local representation of Π_Σ shows that θ_λ belongs to the range of $\mathfrak{t} + i\mathfrak{t} \hookrightarrow \mathcal{V}^1(\mathbb{C}^n)$.
- Hence $\theta_\mu|_{\mathbb{C}^n} = \theta_\lambda - \Pi_\Sigma^\#(d\log|a|)$ and thus by continuity $[\theta_\mu]$ belongs to the image of $\mathfrak{t} + i\mathfrak{t}$ in $H^1(X(\Sigma), \Pi_\Sigma)$ if and only if $\log|a|$ extends continuously off of the affine chart.
- But $X(\Sigma)$ is compact, so we must have $\lim_{\|w\|\to\infty} |a(w)| = 0$. Thus $[\theta_\mu]$ is independent of the image of $\mathfrak{t} + i\mathfrak{t}$ in $H^1(X(\Sigma), \Pi_\Sigma)$. $\square$

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Bound is tight for $\mathbb{C}\mathrm{P}^1$

Theorem 5 (Nakanishi) *For* $\Pi = 2i|z|^2\partial_{\overline{z}} \wedge \partial_z$ *on* $\mathbb{C}$,

$H^0(\mathbb{C}, \Pi) = \mathbb{R}$,

$H^1(\mathbb{C}, \Pi) = \mathbb{R}\langle R, D\rangle$, *and*

$H^2(\mathbb{C}, \Pi) = \mathbb{R}\langle \pi, \Pi\rangle$.

- Mayer-Vietoris argument yields:

Theorem 6 *For* Π_Σ *on* $\mathbb{C}\mathrm{P}^1$,

$H^0(\mathbb{C}\mathrm{P}^1, \Pi_\Sigma) = \mathbb{R}$,

$H^1(\mathbb{C}\mathrm{P}^1, \Pi_\Sigma) = \mathbb{R}\langle R, D, \theta_\mu\rangle$, *and*

$H^2(\mathbb{C}\mathrm{P}^1, \Pi_\Sigma) \simeq \mathbb{R}^4$.

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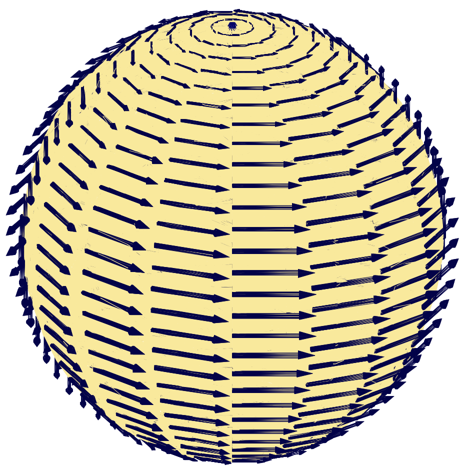
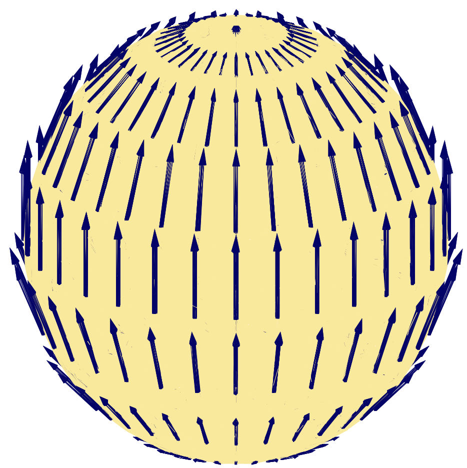
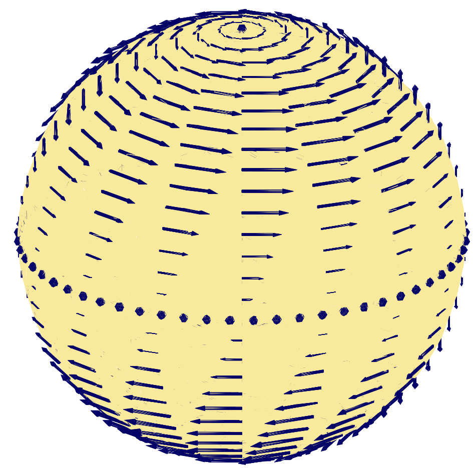
Special Representatives for $H^1(\mathbb{C}\mathrm{P}^1, \Pi_\Sigma)$

$R = iw\partial_w + c.c.$	$D = w\partial_w + c.c.$	θ_μ
		

θ_μ for $(\mathbb{C}P^n, \Pi_\Sigma)$ for $\mu = \frac{1}{n!}\omega_\Delta^n$

- The standard n-simplex Δ' is a Delzant polytope in $\mathfrak{t}^*$.
- Set $\Delta = \Delta' - \nu$ where ν is the center of mass of Δ' and let Σ be the dual fan of Δ.
- Apply Delzant construction to get $(\mathbb{C}P^n, \omega_\Delta, \Phi_\Delta)$ and $\mu = \frac{1}{n!}\omega_\Delta^n$.

Theorem 7 *For each open cone V in Σ*

$$\theta_\mu = \frac{(n+1)}{2\pi}\sum_{\ell=1}^{n}\langle\Phi_\Delta, (uB_V)_\ell\rangle R_\ell$$

where $R_1, \ldots, R_n$ are the vector fields corresponding to the basis $u = (u_1, \ldots, u_n)$ of $\Lambda \subset \mathfrak{t}$ generating the cone V.

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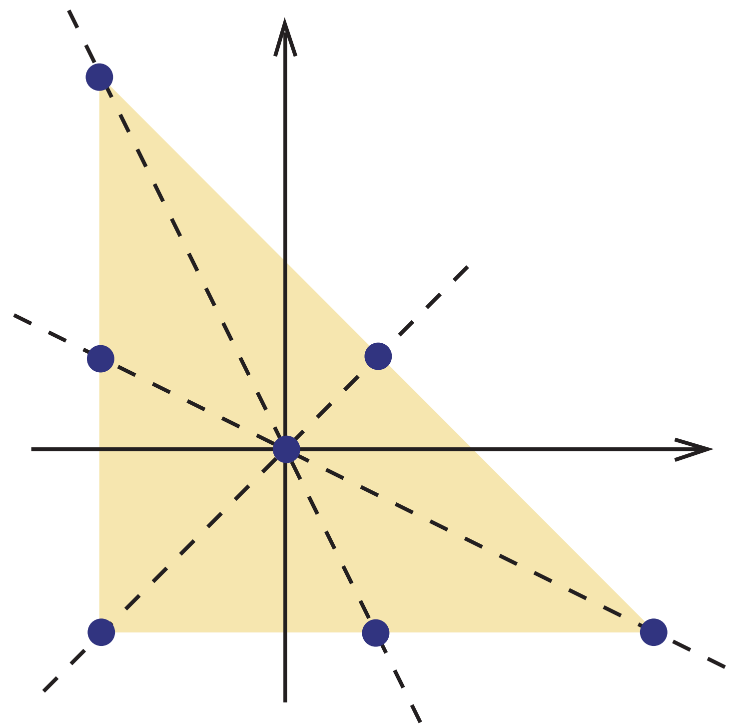
Zeroes of θ_μ on $\mathbb{CP}^n$

Theorem 8 *If $Z = \{\theta_\mu = 0\} \subset \mathbb{CP}^n$, then $\Phi_\Delta(Z)$ is the set of centroids of the faces of Δ.*



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