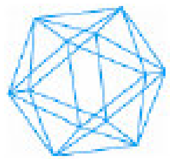
Perspectives from Poisson Geometry

Arlo Caine



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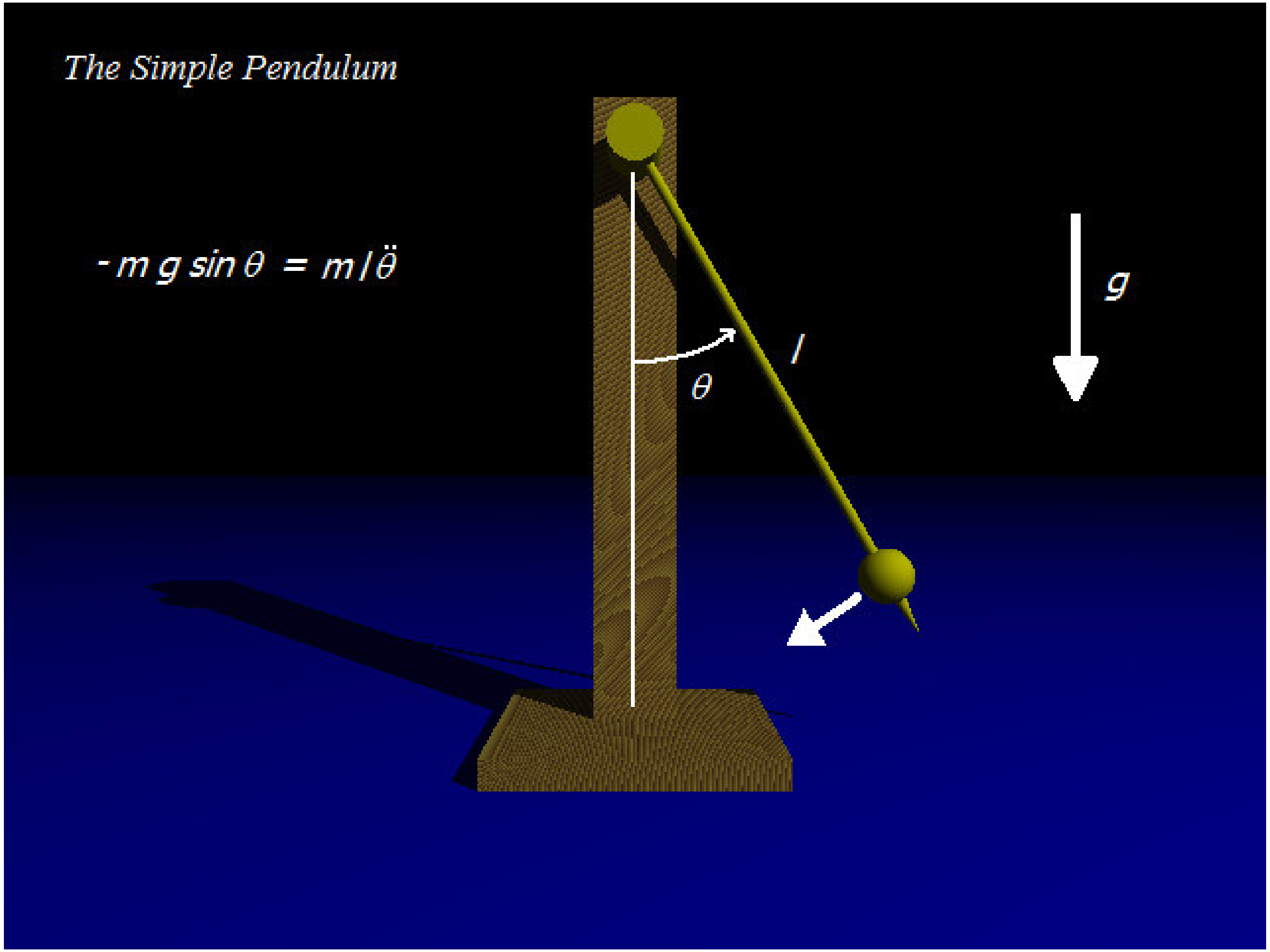
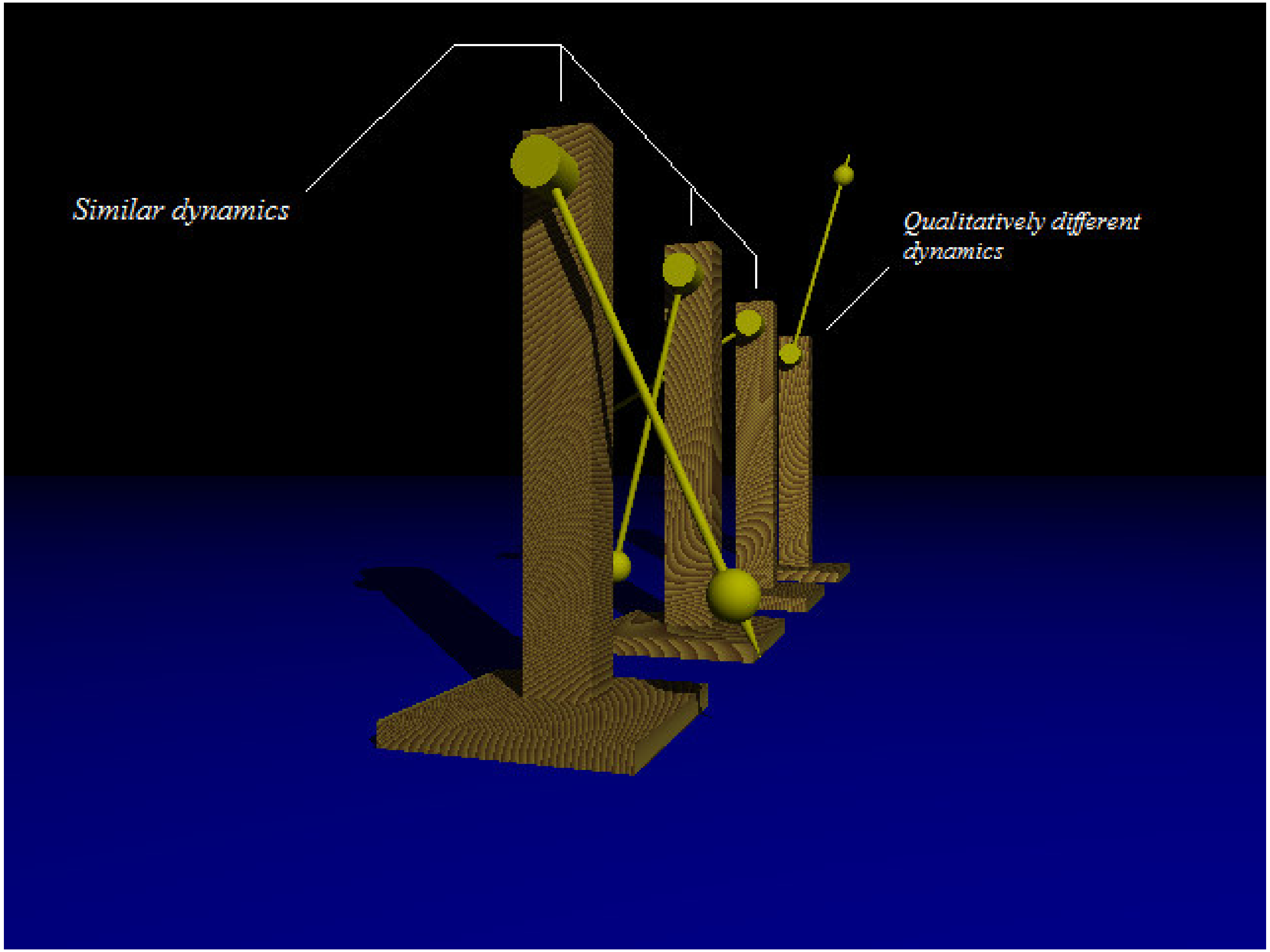
Mechanics of Motion



Launch External Animation

Mechanics of Motion



Hamilton's Idea

$q = \ell\theta$, $v = \dot{q} = \ell\dot{\theta}$

Look at the E.O.M. in first order form

$$\begin{cases} \dot{v} & = & -g\sin(q/\ell) \\ \dot{q} & = & v \end{cases}$$

Rewrite in terms of linear momentum $p = mv$.

$$\begin{cases} \dot{p} & = & -mg\sin(q/\ell) \\ \dot{q} & = & p/m \end{cases}$$

www.ria.ie/

Key Observation:

$$\begin{cases} \dot{p} & = & -\partial H/\partial q \\ \dot{q} & = & \partial H/\partial p \end{cases}$$

where H is the total Total Energy

$$H = \underbrace{\frac{1}{2m}p^2}_{\text{kinetic energy}} + \underbrace{(-mg\ell\cos(q/\ell) + mg\ell)}_{\text{potential energy}}.$$

Conserved Quantities

H is constant along the flow. (Conservation of Energy)

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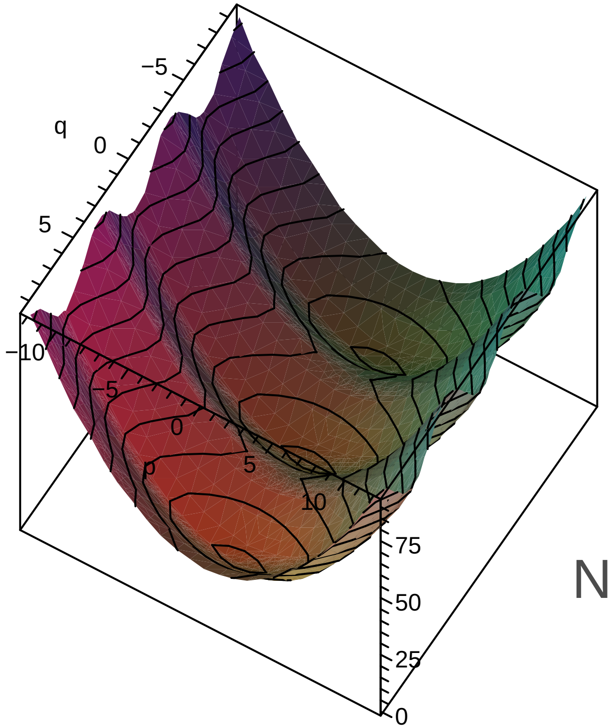


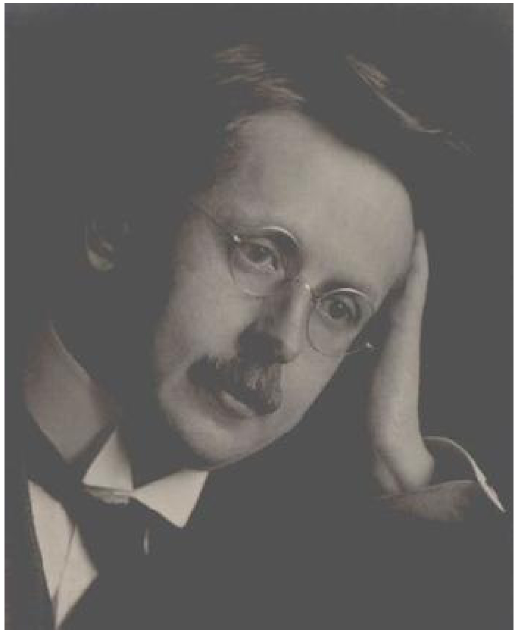
- 1 Dimensional Problem:

  Qualitative view of dynamics obtained by analyzing the level sets of H.
- Higher Dimensions:

Need **independent** conserved quantities.

Weyl's Observations

- Velocities are *tangent vectors*.
- Physics $\Rightarrow$ Momentum is a *1-form*.
- Why? Pair with velocity $\longrightarrow$ kinetic energy.
- View H as a function $H : T^*Q \to \mathbb{R}$.
- Hamilton's E.O.M. $\Leftrightarrow$ equality of 1-forms



www.weylmann.com

$$\omega(X, \cdot) = dH(\cdot)$$

where $X = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$ and $\omega = dp \wedge dq$.

$$\begin{pmatrix} \dot{q} & \dot{p} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J} = \begin{pmatrix} \frac{\partial H}{\partial q} & \frac{\partial H}{\partial p} \end{pmatrix} \rightsquigarrow J^2 = -1$$

complex $\leftrightsquigarrow$ symplectic

Poisson's Viewpoint

- Hamilton's E.O.M are equivalent to:

$$X = \pi(\cdot, dH)$$

where $\pi = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$ (bivector on T^*Q).

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Key Point:

$$\{\{F_1, F_2\}, H\} + \{\{F_2, H\}, F_1\} + \{\{H, F_1\}, F_2\} = 0.$$

Thus, F_1 and F_2 conserved $\Rightarrow \{F_1, F_2\}$ is conserved.

Geometric Abstraction

Geometric structures on M.

- **Definition 1:** A *symplectic structure* on M is a 2-form ω on M such that:
  1. ω is non-degenerate on T_pM for each $p \in M$,
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2 & 3 are equivalent: $\{f, g\} := \pi(df, dg)$

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Symplectic Foliation

Anchor Map:

$$\begin{array}{rccc} \pi^{\#}: & T^*M & \rightarrow & TM \\ & \alpha & \mapsto & \pi(\cdot,\alpha) \end{array}$$

- $\mathrm{Image}(\pi^{\#}|_p) \subset T_pM$ is even dimensional for each $p \in M$.
- $\mathrm{Rank}(\pi^{\#}|_p)$ is not necessarily constant in p.
- $\mathrm{Image}(\pi^{\#})$ defines a general distribution Δ in TM.
- π is tangent to $\mathrm{Image}(\pi^{\#})$ and non-degenerate there.
- $[\pi,\pi]_{Sch} = 0 \overset{SS}{\Rightarrow} \Delta$ is integrable and π^{-1} is symplectic on each leaf.

Sussmann-Stefan Frobenius Theorem

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Poisson manifolds are foliated by symplectic submanifolds

Sussmann-Stefan Frobenius Theorem

General Examples

- World's most uninteresting example: $\pi = 0$ on any M.
- (M^{2k}, ω) symplectic $\Rightarrow \pi = \omega^{-1}$ is a Poisson structure on M and Δ has exactly one symplectic leaf.
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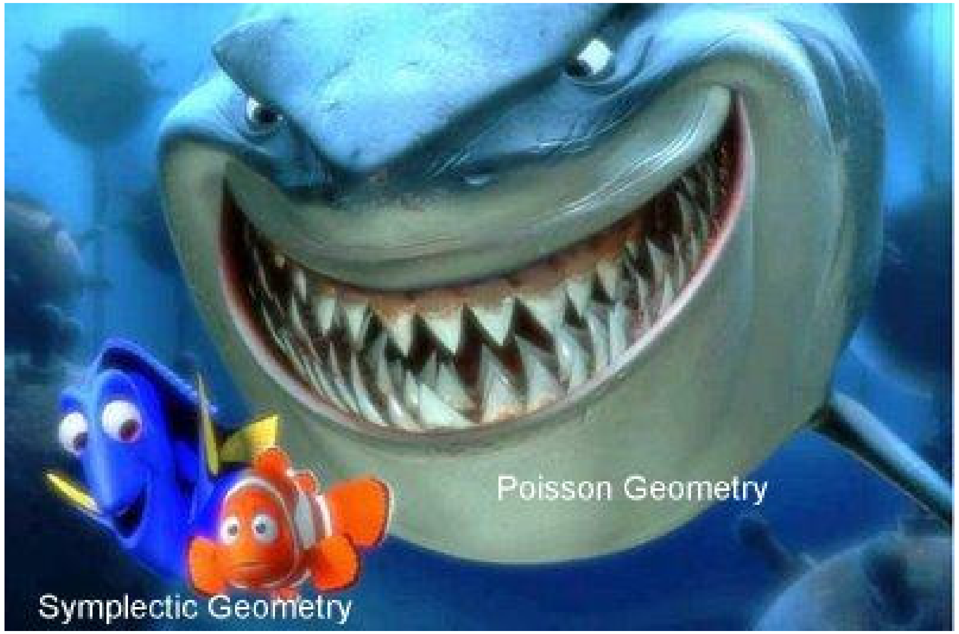

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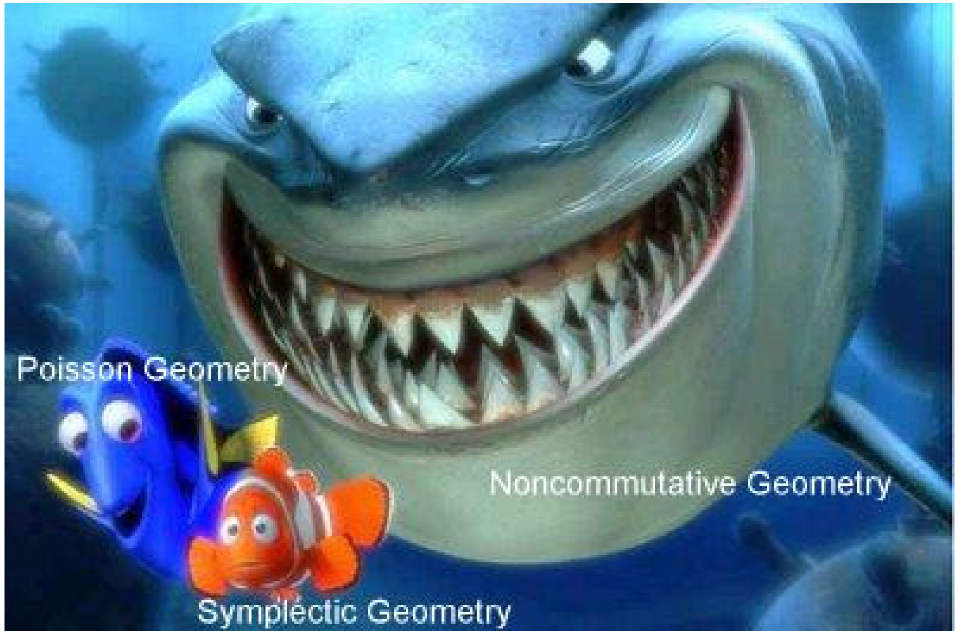
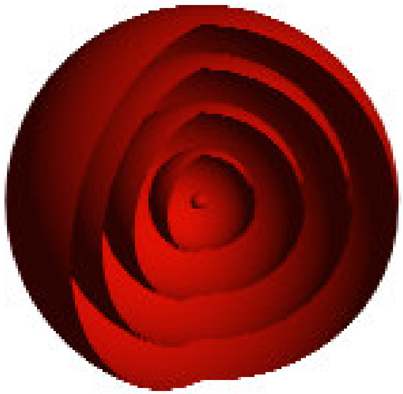


Poisson Structures on $\mathbb{R}^3$

$\{f, g\} := (\nabla f \times \vec{F}) \cdot \nabla g$ where $\vec{F} \cdot (\nabla \times \vec{F}) = 0$.



$\vec{F} = \nabla(\frac{1}{2}(x^2 + y^2 + z^2))$

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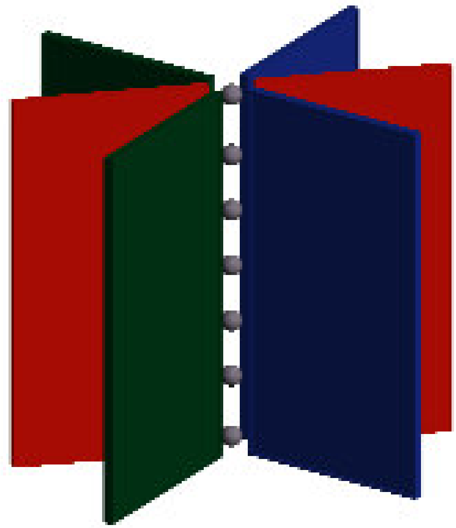

$\vec{F} = -y\vec{i} + x\vec{j}$

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} \rightsquigarrow \pi = P\partial_y \wedge \partial_z + Q\partial_z \wedge \partial_x + R\partial_x \wedge \partial_y$$

$C^\infty(\mathbb{R}^3)/\mathbb{R} \hookrightarrow$ {Poisson Structures on $\mathbb{R}^3$}

Lots of variety!

Non-gradient examples $\leftrightarrow$ Triply orthogonal coordinate systems

Lie Poisson Structures

Recall $\{f,g\} = \pi(df, dg)$.

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- Enough to specify its action on *coordinate functions*.

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Let $\mathfrak{g}$ be a finite dimensional real Lie algebra.

- View elements of $\mathfrak{g}$ as linear functions on $\mathfrak{g}^*$.
- Basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ determines linear coordinates on $\mathfrak{g}^*$.
- Define $\{\cdot, \cdot\}$ on $\mathfrak{g}^*$ by

$$\underbrace{\{X_k, X_\ell\}}_{\text{as lin. func. on } \mathfrak{g}^*} = \underbrace{[X_k, X_\ell]}_{\text{as elts of } \mathfrak{g}}$$

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Theorem: (Weinstein) Symplectic leaves are coadjoint orbits.

Poisson Lie Groups

New feature!

- **Definition:** A Poisson map $f : (M, \pi_M) \to (N, \pi_N)$ is a smooth map such that $f_* \pi_M = \pi_N$.
- **Definition:** A Poisson Lie group is a Lie group U with a Poisson structure π_U such that the multiplication map $m \colon U \times U \to U$ is a Poisson map, i.e. $m_*(\pi_U \oplus \pi_U) = \pi_U$.

Notes:

- $\pi_U = 0$ is a (boring) Poisson Lie group structure on U.
- $(\cdot)^{-1} \colon U \to U$ is Poisson if and only if $\pi_U = 0$.

Interesting Examples?

Standard Example

Theorem:(Drinfel'd) $\pi_U = \Lambda^r - \Lambda^\ell$ is a Poisson Lie group structure if and only if $[\Lambda, \Lambda] \in \bigwedge^3 \mathfrak{u}$ is Ad-invariant.

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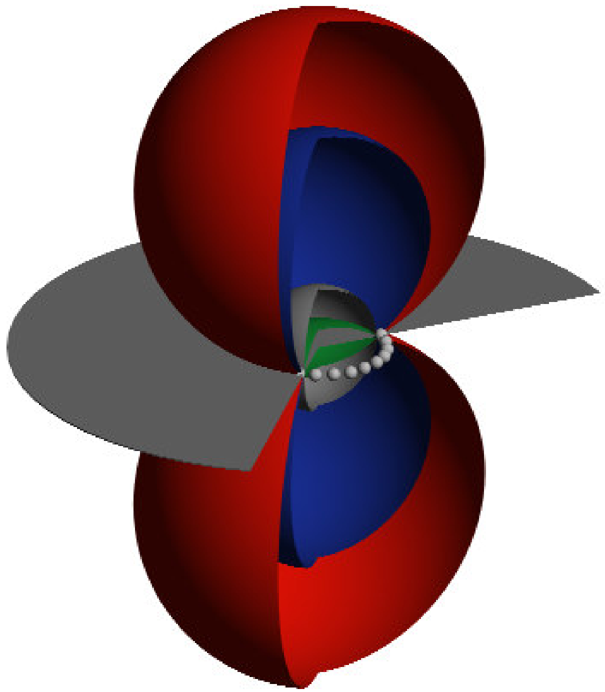
Theorem: $[\mathcal{H}, \mathcal{H}]_{Sch}$ is an ad-invariant element of $\wedge^3 \mathfrak{u}$.

Many equivalent versions, observed by many authors.

On SU(2)

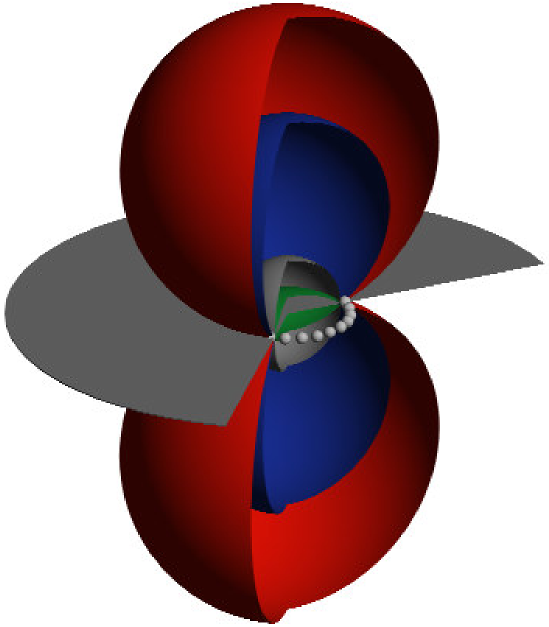
$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto \pi_U = (1-|a|^2+|b|^2)X\wedge Y + 2\mathrm{Im}(a\bar{b})Y\wedge H - 2\mathrm{Re}(a\bar{b})H\wedge X.$$

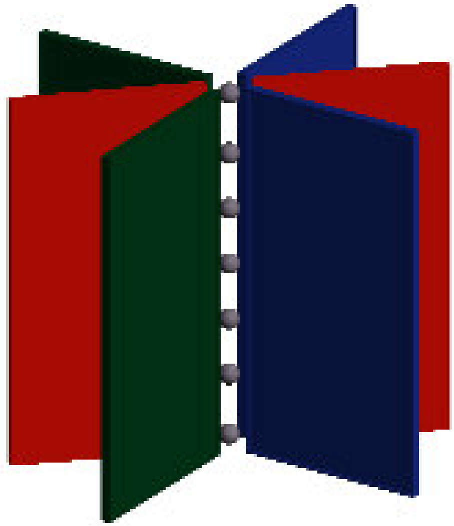
- $\bigwedge^2 T\mathrm{SU}(2) \simeq \mathrm{SU}(2) \times \bigwedge^2 \mathfrak{su}(2)$
- $\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}}\{H, X, Y\}$ ($i\cdot$Pauli spin matrices)
- π_U vanishes on $T = \{b = 0\}$.
- Δ has zero dimensional leaves along T.
- 2-d leaves are disks whose boundary meets T.



Stereographic picture in S^3

From another viewpoint





$$\infty \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \infty \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Generates a non-gradient example of 3-dimensional Poisson structure!

Homogeneous Poisson Structures

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Example:

- $\mathrm{SU}(2)/T \simeq \mathbb{C}P^1$
- The two symplectic leaves give a cell decomposition of $\mathbb{C}P^1$.

Bruhat Poisson Structure on U

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{c}{1} & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$$

$$G = \coprod_{w \in W} N^+ w H N^+$$

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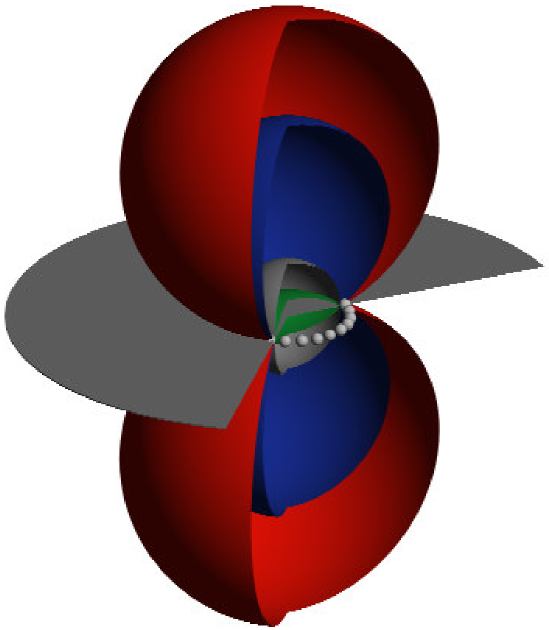

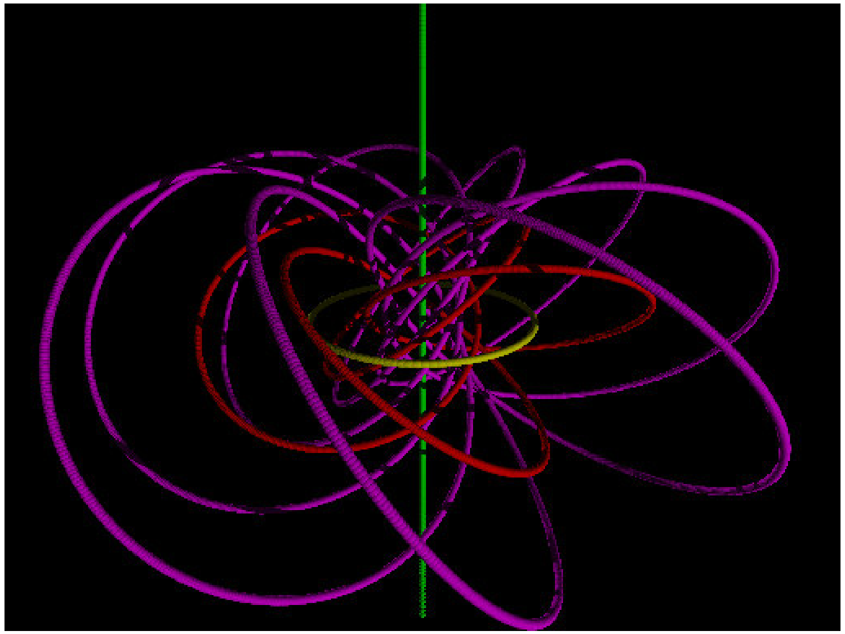

Birkhoff Decomposition of U

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{\sigma_1} & 1 & 0 \\ \frac{a_{31}}{\sigma_1} & \frac{M_{23}}{\sigma_2} & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_1}{1} & 0 & 0 \\ 0 & \frac{\sigma_2}{\sigma_1} & 0 \\ 0 & 0 & \frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{a_{12}}{\sigma_1} & \frac{a_{13}}{\sigma_1} \\ 0 & 1 & \frac{M_{32}}{\sigma_2} \\ 0 & 0 & 1 \end{pmatrix}$$

- $A = (a_{ij}) \in SL(3, \mathbb{C})$ with non-trivial principal minors σ_1, σ_2.
- Gaussian elimination $\Rightarrow$ factorization exists and is unique for $\sigma_1 \sigma_2 \neq 0$.
- Introduction of permutation factor necessary for higher codimension components (e.g. $\sigma_1 = 0$.)
- Birkhoff Decomposition

$$G = \coprod_{w \in W} \Sigma_w^G \text{ where } \Sigma_w^G = N^- w H N^+$$

- Induced decomposition $U = \coprod_{w \in W} (U \cap \Sigma_w^G)$.

Birkhoff Poisson Structure

Theorem:(C) The symplectic foliation of the Poisson structure $\Pi_U = \mathcal{H}^r + \mathcal{H}^\ell$ on U aligns with the Birkhoff decomposition.

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- Π_U arises by applying the Evens-Lu construction to generate a $(U \times U, \pi_U \oplus \pi_U)$-homogeneous Poisson structure on U (viewed as a symmetric space).

Birkhoff Poisson Structure

Theorem:(C) The symplectic foliation of the Poisson structure $\Pi_U = \mathcal{H}^r + \mathcal{H}^\ell$ on U aligns with the Birkhoff decomposition.

- Π_U arises by applying the Evens-Lu construction to generate a $(U \times U, \pi_U \oplus \pi_U)$-homogeneous Poisson structure on U (viewed as a symmetric space).

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Final Comment:

- The first theorem can be translated in to the infinite dimensional setting of the loop group of U, but the second result is finite dimensional.