

POISSON STRUCTURES ON  $U/K$  AND APPLICATIONS

by

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**DEDICATION**

*to Kat*

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## ABSTRACT

Let  $X$  be a simply connected compact Riemannian symmetric space, let  $U$  be the universal covering group of the identity component of the isometry group of  $X$ , and let  $\mathfrak{g}$  denote the complexification of the Lie algebra of  $U$ ,  $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ . Each  $\mathfrak{u}$ -compatible triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  determines a Poisson Lie group structure  $\pi_U$  on  $U$ . The Evens-Lu construction produces a  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$ . By choosing the basepoint in  $X$  appropriately,  $X$  is presented as  $U/K$  where  $K$  is the fixed point set of an involution which stabilizes the triangular decomposition of  $\mathfrak{g}$ . With this presentation, a connection is established between the symplectic foliation of the Evens-Lu Poisson structure and the Birkhoff decomposition of  $U/K$ . This is done through reinterpretation of results of Pickrell. Each symplectic leaf admits a natural torus action. It is shown that these actions are Hamiltonian and the momentum maps are computed using triangular factorization. Finally, local formulas for the Evens-Lu Poisson structure are displayed in several examples.

## CHAPTER 1

### INTRODUCTION

This dissertation gives a Poisson geometric interpretation of triangular factorization for compact symmetric spaces. The logarithms of certain determinants arising in triangular factorization are shown to be components of momentum maps for the Evens-Lu homogeneous Poisson structure. The basic definitions and conventions of differential geometry used here are those given in [Spi]. The standard reference on symmetric spaces is [Hel78]. This text follows the conventions and notation for Poisson geometry given in [Vai94]. There are, however, a few occasions on which the notation used in this text departs from that established in these references. The purpose of this introduction is preview the main results, outline the flow of this dissertation, and along the way establish the notation which will be used throughout the text.

#### 1.1 Preview of the main results

Let  $X$  denote a compact, connected, and simply connected symmetric space. The basic example of such a space is the round two dimensional sphere. At each point in  $X$ , the local diffeomorphism of reflection along geodesics through that point extends to a global isometry. As a result, the isometry group is quite large, admitting a subgroup which acts transitively on  $X$ , giving  $X$  the structure of a Riemannian homogeneous space. The fundamental work on the subject of symmetric spaces was first carried out by Élie Cartan in the late 1920's. This work, which is quite long and complicated in detail, classified these spaces and determined much about their geometry. Cartan's fundamental idea, which made the classification possible, was

the translation of the problem into a question about certain kinds of coset spaces of Lie groups. It is through this translation that symmetric spaces will be connected with triangular factorization in this dissertation.

The universal covering group of the identity component of the isometry group of  $X$  will be denoted by  $U$ . Cartan established that for each choice of basepoint  $x$  in  $X$ , the map

$$\begin{aligned} U/K &\rightarrow X \\ uK &\mapsto uK \cdot x \end{aligned} \tag{1.1}$$

gives an identification of  $X$  with the coset space  $U/K$ , where  $K$  is the connected subgroup of  $U$  fixing  $x$ . The subgroup  $K$ , known as the stability subgroup (or isotropy subgroup), is always closed, connected, and equal to the fixed point set of an involution  $\theta$  of  $U$ . The coset space  $U/K$  can be given the structure of a Riemannian manifold in such a way that the identification in (1.1) is an isometric isomorphism of Riemannian manifolds which identifies the isometry of  $U/K$  induced by  $\theta$  with the geodesic reflection through  $x$  in  $X$ . For the example of the two-dimensional sphere,  $U$  would be the group  $SU(2)$  and one choice of basepoint would yield the subgroup of diagonal matrices,  $S(U(1) \times U(1))$ . The involution selecting this subgroup is the automorphism of  $SU(2)$  which negates the off-diagonal entries.

In addition to the stability subgroup  $K$ , the involution  $\theta$ , and the presentation of  $X$  as  $U/K$ , another important piece of data is determined by choosing a basepoint. A choice of  $x \in X$  determines a non-compact real form of  $G = U^{\mathbb{C}}$ . Let  $g \mapsto g^{-*}$  denote the Cartan involution of  $G$  selecting  $U$  as a compact real form of  $G$ . The composition of this involution with the holomorphic extension of  $\theta$  to  $G$ , denoted  $g \mapsto g^{\sigma} = g^{-*\theta}$ , defines another involution  $\sigma$  of  $G$ . The connected subgroup of  $G$  fixed by  $\sigma$  will be denoted  $G_0$ . This is the non-compact real form of  $G$  determined by the choice of  $x$ . Often in this text,  $\theta(g)$  will be written as  $g^{\theta}$ . Since  $(\cdot)^{-*}$ ,  $\theta$ , and  $\sigma$  all commute, this practice should not cause confusion.

Every  $n \times n$  complex matrix of trace zero can be uniquely decomposed as a sum of

a strictly lower triangular matrix, a diagonal trace zero matrix, and a strictly upper triangular matrix. The multiplicative analog of this statement for an  $n \times n$  complex matrix  $g$  of determinant one does not always hold. There is an algebraic condition that must be imposed in order to guarantee that  $g$  may be factored uniquely as  $\ell d u$  where  $\ell$  (respectively  $u$ ) is a lower (respectively upper) triangular unipotent matrix, and  $d$  is a diagonal matrix of determinant one. The algorithm that determines this triangular factorization is Gaussian elimination and the condition is that the principal minors of  $g$  cannot vanish. When one or more of the principal minors does vanish, a permutation matrix  $w$  must be introduced leading to a factorization of the form  $\ell w d u$  where  $\ell$ ,  $d$ , and  $u$  are as before. By insisting on this order to the factorization, the diagonal factor  $d$  and the permutation  $w$  are determined uniquely up to a slight ambiguity of signs, but the factors  $\ell$  and  $u$  are uniquely determined only in the case  $w = 1$ . Collecting elements according to the type of factorization they admit, decomposes the smooth variety of  $n \times n$  complex matrices of determinant one into a disjoint union of smooth subvarieties.

As with most topics in linear algebra, this decomposition has been generalized in Lie theory where it is known as the Birkhoff decomposition (alternatively, the triangular decomposition or the LDU decomposition). Given a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$$

of a complex semi-simple Lie algebra  $\mathfrak{g}$ , the corresponding simply connected complex semi-simple Lie group  $G$  decomposes as a disjoint union of submanifolds of  $\Sigma_w^G$  whose codimensions increase with the length of the indexing elements  $w$  of the Weyl group  $W = N_G(H)/H$ .

A generalization of this decomposition for compact symmetric spaces was given in [Pic06], which will now be described. Recall that choosing a basepoint in  $X$  determined a presentation of  $X$  as a coset space  $U/K$ . The involution  $\theta$  fixing  $K$  is

used to define the Cartan embedding,

$$\begin{aligned} \phi: U/K &\rightarrow U \\ uK &\mapsto uu^{-\theta} \end{aligned}$$

of the coset space  $U/K$  into the group  $U$ . The group  $U$  can then be given a natural Riemannian structure so that this map realizes  $U/K$  as a totally geodesic submanifold of  $U$ . In the running example of the two-dimensional sphere, the group  $U = \mathrm{SU}(2)$  with this Riemannian structure can be identified with the round three dimensional sphere and the Cartan embedding realizes  $\mathrm{SU}(2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$  as an equatorial two-sphere.

The group  $U$  is included in its complexification, which is a simply connected complex semi-simple Lie group  $G = U^{\mathbb{C}}$ . The Lie algebra of  $G$ , denoted by  $\mathfrak{g}$ , is the complexification of  $\mathfrak{u}$ , the Lie algebra of  $U$ . The composition

$$X \xrightarrow{\sim} U/K \xrightarrow{\phi} U \hookrightarrow G$$

is what connects symmetric spaces with triangular factorization. A *Birkhoff decomposition of  $X$*  is the decomposition induced by intersecting  $\phi(U/K)$  with a Birkhoff decomposition of  $G$ . Birkhoff decompositions of  $X$  thus depend on the choice of a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  and the choice of a basepoint in  $X$  yielding a presentation of  $X$  as  $U/K$ .

It was proven in [Pic06] that the elements of  $\phi(U/K) \cap \Sigma_1^G$  can be factored uniquely as  $\ell d_\phi \ell^{*\theta}$ . The diagonal  $d_\phi$  can then be considered as a function defined on an dense open subset of  $X$ . In a matrix representation, diagonal elements of  $d$  are ratios of determinants of submatrices of  $g$ , namely the principal minors. These elements are distinguished functions on  $X$ . For example, in the case of the two dimensional sphere,  $G = \mathrm{SL}(2, \mathbb{C})$  and the diagonal elements of  $d_\phi$  are the height function, and its reciprocal.

It is well known that the height function for the two-sphere is a momentum mapping for the invariant symplectic structure on that space and the circle action

which leaves the height function invariant. The main result of this dissertation is that essentially the logarithm of the diagonal can be realized as a momentum map for torus actions on the symplectic leaves of a Poisson structure on  $X$ . The Poisson structure is homogeneous relative to a non-trivial Poisson Lie group structure on  $U$  determined by the triangular decomposition of  $\mathfrak{g}$ .

In chapter 2 it is first observed that each choice of a  $\mathfrak{u}$ -compatible triangular decomposition of  $\mathfrak{g}$  determines a standard example of a non-trivial Poisson Lie group structure  $\pi_U$  on  $U$ . Such structures were first introduced by Lu and Weinstein in [LW90] from a different perspective. These structures have the property that the double of their Lie bi-algebra is isomorphic to  $\mathfrak{g}$  viewed as a real Lie algebra. Write  $\mathfrak{d}$  for  $\mathfrak{g}$  regarded as a real Lie algebra. Using a very general construction due to Evens and Lu ([EL01]), it is possible to introduce a Poisson bivector on the variety  $\mathcal{L}(\mathfrak{d})$  of subalgebras of  $\mathfrak{d}$  which are Lagrangian with respect to the imaginary part of the Killing form on  $\mathfrak{g}$ . The adjoint action of  $G$  on  $\mathfrak{g}$  induces a natural action of  $G$  (and hence all of its subgroups) on  $\mathcal{L}(\mathfrak{d})$ . It is a theorem of Evens and Lu that their Poisson bivector is tangent to each of the  $U$ -orbits in  $\mathcal{L}(\mathfrak{d})$ , and furthermore defines a  $(U, \pi_U)$ -homogeneous Poisson structure on each such orbit. The natural map from  $U/K$  to the  $U$ -orbit through  $\mathfrak{g}_0$  is a finite sheeted covering, and it is through this map that the  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$  considered here is defined.

In chapter 2, the Evens-Lu construction is reviewed. Given a presentation of  $X$  as  $U/K$ , an equivariant formula for the Evens-Lu Poisson structure  $\pi$  is computed. It is here that the role played by the non-compact real form  $G_0$  is made apparent. From the triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  one obtains an Iwasawa decomposition  $\mathfrak{g} = (\mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}}) + \mathfrak{u}$ . Writing  $N^- = \exp(\mathfrak{n}_-)$ , and  $A = \exp(\mathfrak{h}_{\mathbb{R}})$ , there is a corresponding decomposition of the group  $G = N^-AU$ . Thus,  $U$  may be identified with the right coset space  $N^-A \backslash G$ , on which  $G$  acts from the right. Restricting this action to  $G_0$  yields an action of  $G_0$  on  $U$ . Chapter 2 concludes by applying the equivariant

formula for  $\pi$  to prove that the symplectic leaves in  $U/K$  are the projections of the  $G_0$  orbits in  $U$  to  $U/K$ .

The key observation of this dissertation is that there exists a basepoint in  $X$  (in fact, more than one) such that the given triangular decomposition of  $\mathfrak{g}$  is stable with respect to the involution selecting the stability subgroup in  $U$ . With such a basepoint, and hence a presentation of  $X$  as  $U/K$ , there is a beautiful connection between the symplectic foliation of the Evens-Lu Poisson structure  $\pi$  on  $X$  and the corresponding Birkhoff decomposition of  $U/K$ . This connection enables the explicit description of the geometry of the symplectic foliation.

Recall that the Birkhoff decomposition of  $X$  is obtained by intersecting the image  $\phi(U/K)$  with the Birkhoff decomposition  $G = \coprod_w \Sigma_w^G$ . The intersection  $\phi(U/K) \cap \Sigma_w^G$  will be referred to as the layer of the Birkhoff decomposition of  $X$  corresponding to  $w$ . Each layer may consist of several connected components. For the two-sphere example, there are two preferred basepoints for each given triangular decomposition of  $\mathfrak{sl}(2, \mathbb{C})$ . The corresponding Birkhoff decompositions of the sphere consist of two layers. The layer corresponding to the trivial element of the Weyl group has two connected components, each open hemispheres, whereas the layer corresponding to the non-trivial element is connected, consisting of the great circle which separates them.

In chapter 3, the following theorems are proven.

**Theorem.** *Each connected component of the layer of the Birkhoff decomposition of  $X$  corresponding to  $w$  is foliated by contractible symplectic leaves each diffeomorphic to a space of double cosets of  $G_0$  depending on  $w$ .*

In the two-sphere example, the upper and lower hemispheres are each symplectic leaves of  $\pi$  and the layer corresponding to the non-trivial element of the Weyl group is foliated by zero dimensional symplectic leaves.

It is not always the case that the image of  $U/K$  under the Cartan embedding meets each and every component  $\Sigma_w^G$  of the Birkhoff decomposition of  $G$ . In order for

the intersection  $\phi(U/K) \cap \Sigma_w^G$  to be non-empty, it must be the case that  $\phi(U/K) \cap w$  is non-empty when  $w$  is viewed as a coset of  $T$  in  $N_U(T)$ . For each symplectic leaf foliating the layer indexed by  $w$  there is a unique element  $\mathbf{w}$  of  $\phi(U/K) \cap w$  which the  $\phi$ -image of that leaf passes through. Given  $\mathbf{w}$ , let  $\mathbf{w}_1K$  be the element of  $U/K$  such that  $\phi(\mathbf{w}_1K) = \mathbf{w}$ .

**Theorem.** *Each symplectic leaf foliating the layer corresponding to  $w$  is acted on by the torus*

$$T_w = \exp(\ker\{\text{Ad}(w)\theta|_{\mathfrak{t}} - 1\}) \subset T.$$

*Moreover, the coset  $\mathbf{w}_1K$  represents the unique fixed point for the  $T_w$ -action in the symplectic leaf through  $\mathbf{w}_1K$ .*

In the two-sphere example, the torus  $T$  is actually equal to the stability subgroup  $S(U(1) \times U(1))$ , and is thus fixed by  $\theta$ . Hence,  $T_1 = T$  in this example. The action is by rotation about the axis perpendicular to the equator. This action has one fixed point in each of the open symplectic leaves. The transformation  $\text{Ad}(w) - 1$  induces an isomorphism of  $\mathfrak{t}$  when  $w$  is the non-trivial element of the Weyl group, and hence  $T_w = 1$  in that case.

**Theorem.** *Let  $\mathbf{w} \in \phi(U/K) \cap w$  be an element in the layer corresponding to  $w$ . Let  $S_{\mathbf{w}}$  denote the symplectic leaf of  $(X, \pi)$  for which  $\phi(S_{\mathbf{w}})$  passes through  $\mathbf{w}$  in  $\phi(U/K) \cap w$ . Each element in  $\phi(S_{\mathbf{w}})$  can be factored as  $\ell \mathbf{w} h \ell^{*\theta}$  where  $\ell \in N^-$ , and  $h \in \exp(\ker\{\text{Ad}(w)\sigma|_{\mathfrak{h}} + 1\})$ . Furthermore, the magnitude  $|h| = \sqrt{h^*h}$  is a smooth function on the leaf.*

Each of the previous three theorems are obtained by reinterpreting, in this Poisson geometric context, the results of Pickrell ([Pic06]) characterizing the layers of the Birkhoff decomposition of  $X$ . This reinterpretation is made possible by the results in chapter 2.

The main theoretical result of this dissertation is the following theorem, proven in chapter 3.

**Theorem.** *The action of the torus  $T_w$  on the symplectic leaf  $S_w$  is Hamiltonian with momentum map*

$$\begin{aligned} \mu: S_w &\rightarrow \mathfrak{t}_w^* \\ uK &\mapsto \langle \frac{1}{2}i\theta(\log|h|), \cdot \rangle \end{aligned}$$

where  $h$  is the diagonal part of  $\phi(uK) = uu^{-\theta} \in \phi(S_w)$ .

The remainder of this dissertation is organized as follows. Chapter 4 contains a finer discussion for the cases when  $X$  is a compact Lie group or the involution  $\theta$  is an inner automorphism. In chapter 5, local formulas for the Evens-Lu Poisson structure are computed in several explicit examples. This is done for the complex Grassmannian, complex projective space, and the compact Lie group  $SU(2)$ .

## 1.2 Some comments on homogeneous Poisson geometry

In many branches of geometry, one says that a space is homogeneous if and only if it admits a transitive action by a group of automorphisms. What is meant by a morphism depends on the category of geometric spaces under consideration, e.g. Riemannian, symplectic, Kähler, contact, algebraic, etc. The aim of this section is to illustrate that homogeneous objects in the Poisson category should be treated differently than homogeneous objects in other categories. In particular, it is hoped that the reader will be convinced that, unlike the Riemannian setting, different coset presentations of a homogeneous Poisson space can yield characteristically different views of the homogeneous Poisson structure.

Consider the Riemannian category. Let  $X$  be a connected  $U$ -homogeneous space, meaning that  $X$  is a Riemannian manifold, and  $U$  is a connected Lie group of isometries of  $X$  acting transitively. Fix a point in  $X$  and let  $K$  denote the stability subgroup of  $x$ . The map in (1.1) identifies the set of points in  $X$  with the set of cosets of  $K$  in  $U$ . The subgroup  $K$  is closed and the coset space naturally inherits the structure of a smooth manifold making the map in (1.1) into a diffeomorphism.

Using an Ad-invariant inner product on the Lie algebra of  $U$  one can use left translation to introduce a left invariant Riemannian metric on  $U$ . As  $K$  is necessarily compact, one can average the metric on  $U$  over  $K$  to obtain a left invariant metric on the quotient  $U/K$ . With this metric, the map in (1.1) becomes a  $U$ -equivariant isometry. If one were to choose a different basepoint  $x' \in X$  then the corresponding stability subgroup  $K'$  would be conjugate in  $U$  to  $K$  via an element of  $u$  which maps  $x$  to  $x'$ . Because of the Ad-invariance of the inner product used to construct the metric, there is no difference in choosing to label the points of  $X$  by the space  $U/K$  or  $U/K'$  as the two quotients are naturally  $U$ -equivariantly isometric. The Riemannian structure near  $x$  looks the same as the Riemannian structure near  $x'$ . Hence the name, homogeneous. Some authors define a Riemannian homogeneous space as we did above, while others prefer to define  $U$ -homogeneous Riemannian spaces as coset spaces directly. Either way, the two methods of definition differ simply in perspective.

In the Poisson category a new phenomenon emerges because the Lie group  $U$  itself can have a Poisson structure which respects the multiplication law on the group and this Poisson structure can interact with the Poisson structure on  $X$  in a non-trivial way.

**Definition 1.** *Let  $(U, \pi_U)$  be a Poisson Lie group. A Poisson manifold  $(X, \pi)$  on which  $U$  acts transitively by diffeomorphisms is a  $(U, \pi_U)$ -homogeneous Poisson space if the action map  $U \times X \rightarrow X$  is a Poisson map.*

Suppose that  $\pi_U$  vanishes along a subgroup of  $U$ , then a  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$  will be invariant under the induced action of that subgroup. In particular, when the Poisson Lie group structure on  $U$  is trivial, i.e.,  $\pi_U \equiv 0$ , each  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$  is determined by its value at one point. The Poisson structure would then look the same near each point. However, when  $\pi_U$  is non-trivial, the nature of a homogeneous bivector can be different at different points of  $X$ .

The Poisson Lie group structures considered in this dissertation are non-trivial, yet they vanish on a maximal torus of  $U$ . As a result, the  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$  is invariant under the action of that torus.

Consider again the two-sphere example. View the two-sphere as the complex projective line equipped with the Fubini-Study metric, and denote by  $z$  the local holomorphic coordinate for which the stability subgroup of  $z = 0$  is  $S(U(1) \times U(1))$ .

The Lu-Weinstein Poisson Lie group structure on  $SU(2)$  determined by the standard triangular decomposition of  $\mathfrak{su}(2)^\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$ , vanishes along the maximal torus consisting of the diagonal matrices in  $SU(2)$ . In terms of the  $z$ -coordinate, the Evens-Lu homogeneous Poisson structure on  $\mathbb{C}P^1$  is given by

$$\pi = -i(1 - |z|^4) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}. \quad (1.2)$$

The fact that the coefficient in (1.2) depends only on the modulus of  $z$  reflects the fact that the structure is invariant under the torus action  $z \mapsto e^{i2\phi}z$ . Notice that  $\pi$  vanishes on each point of the torus orbit  $|z| = 1$ , the equatorial circle. The Poisson structure does not look the same in a neighborhood of each point. Thus, not all presentations of  $X$  as  $U/K$  can be considered equivalent. It turns out, in this example, that using a basepoint where the Poisson structure vanishes is the only way to make the natural quotient map  $U \rightarrow U/K$  into a Poisson map.

### 1.3 Relation with other work

The difference of any two  $(U, \pi_U)$ -homogeneous Poisson structures on a  $U$ -homogeneous space  $M$  is a  $U$ -invariant bivector field on  $M$ . If  $M$  is an irreducible compact Hermitian symmetric space, then  $M$  admits a one parameter family of  $U$ -invariant bivector fields. The elements of this family are the scalar multiples of the non-degenerate  $U$ -invariant Poisson structure  $\pi_{KKS}$  that a Hermitian symmetric space carries because it is a coadjoint orbit. The structure  $\pi_{KKS}$  is the contravariant version of the symplectic structure on coadjoint orbits discovered by Kostant,

Kirillov, and Souriau. If  $M$  is a non-Hermitian irreducible symmetric space, then only the trivial bivector field is  $U$ -invariant. Thus, there is exactly one  $(U, \pi_U)$ -homogeneous Poisson structure on an irreducible non-Hermitian symmetric space. For standard Poisson Lie group structures  $\pi_U$ , the  $(U, \pi_U)$ -homogeneous Poisson structures on irreducible Hermitian symmetric spaces were classified in [KRR93].

What is referred to in this dissertation as the Evens-Lu Poisson structure on compact symmetric spaces was also studied by Foth and Lu in [FL04]. They gave an alternate construction which one may interpret as follows. It is possible to choose the basepoint of  $X$ , determining a different stability subgroup  $K'$  and corresponding model point  $\mathfrak{g}'_0$ , in such a way that the Borel subalgebra  $\mathfrak{h} + \mathfrak{n}_+$  is Iwasawa relative to the noncompact real form  $\mathfrak{g}'_0$  of  $\mathfrak{g}$ . This means that the intersection  $(\mathfrak{h} + \mathfrak{n}_+) \cap \mathfrak{g}'_0$  contains  $\mathfrak{a}'_0 + \mathfrak{n}'_0$  for some Iwasawa decomposition  $\mathfrak{g}'_0 = \mathfrak{k}' + \mathfrak{a}'_0 + \mathfrak{n}'_0$  of  $\mathfrak{g}'_0$ . An alternative characterization of this property is to say that the basepoint is chosen in such a way that the triangular decomposition is  $\sigma'$ -stable. In such a presentation, the push-forward of  $\pi_U$  under the projection map  $U \rightarrow U/K'$  gives the Evens-Lu Poisson structure. This method of construction has the advantage that the natural quotient map is Poisson but the drawback that it is difficult to explicitly calculate examples. Also, it is not clear that there are torus actions on the symplectic leaves.

It is important to note that the stability subgroup  $K'$  is not necessarily a Poisson Lie subgroup of  $(U, \pi_U)$ , even though the projection of  $\pi_U$  defines a  $(U, \pi_U)$ -homogeneous Poisson structure on  $U/K'$ . Using results of Lu and Weinstein ([LW90]), the authors Khoroshkin, Radul, and Rubtsov proved the existence in the Hermitian symmetric case of a parabolic subgroup  $P$  of  $G$  such that  $U \cap P$  is a Poisson Lie subgroup of  $(U, \pi_U)$ . With  $X$  presented as  $U/(U \cap P)$ , the natural projection of  $\pi_U$  to  $U/(U \cap P)$  defines another  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$ . This structure will be denoted  $\pi_{PL}$  as in [KRR93]. Khoroshkin, Radul, and Rubtsov proved that the Schouten bracket  $[\pi_{PL}, \pi_{KKS}]$  vanishes. Thus,  $\pi_{PL} + \lambda \pi_{KKS}$  is the one parameter family of  $(U, \pi_U)$ -homogeneous Poisson structures on a Her-

mitian symmetric space  $X$ . This completely determines the Poisson structures on an irreducible Hermitian symmetric space  $U/K$  that are homogeneous relative to a given Lu-Weinstein Poisson Lie group structure on  $U$ .

The main point of [Pic06] was to prove an integral formula for a generalization of the  $c$ -function of Harish Chandra to symmetric spaces. Suppose that  $K$  is a simply connected compact Lie group. Given a triangular decomposition of  $\mathfrak{k}^{\mathbb{C}} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ , generic elements  $g \in K$  have a unique factorization  $g = \ell du$  where  $\ell \in N^- = \exp(\mathfrak{n}_-)$ ,  $d \in H = \exp(\mathfrak{h})$ , and  $u \in N^+ = \exp(\mathfrak{n}_+)$ . Write  $|d|$  for  $\sqrt{d^*d}$ . A formula of Harish-Chandra asserts that, for  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ ,

$$\int_K |d|^{-i\lambda} = c(2\delta - i\lambda) = \prod_{\alpha > 0} \frac{\langle 2\delta, \alpha \rangle}{\langle 2\delta - i\lambda, \alpha \rangle} \quad (1.3)$$

where the integral is with respect to normalized Haar measure, the product is over the positive complex roots,  $\langle \cdot, \cdot \rangle$  denotes the Killing form (on the dual), and  $2\delta$  is the sum of the positive complex roots.

One interpretation of (1.3) is that the formula computes the Fourier transform of the push-forward of the invariant measure on  $K$  under the map  $\log |d|: \Sigma_1^K \rightarrow \mathfrak{h}_{\mathbb{R}}$ . In the language of probability,  $|d|$  can be viewed as an  $\exp(\mathfrak{h}_{\mathbb{R}})$ -valued random variable with respect to the normalized Haar measure on  $K$ . The push-forward of the invariant measure under  $|d|: \Sigma_1^K \rightarrow \exp(\mathfrak{h}_{\mathbb{R}})$  gives the distribution of this random variable. The space  $\mathfrak{h}_{\mathbb{R}}$  is, in fact, diffeomorphic to  $\exp(\mathfrak{h}_{\mathbb{R}})$  via the exponential map, thus  $\log: \exp(\mathfrak{h}_{\mathbb{R}}) \rightarrow \mathfrak{h}_{\mathbb{R}}$  serves as a global coordinate. On a linear space, such as  $\mathfrak{h}_{\mathbb{R}}$ , knowing the Fourier transform of such a measure is equivalent to understanding the measure itself. In this sense, formula (1.3) completely characterizes the distribution of the diagonal  $|d|$ .

The generalization of (1.3) considered by Pickrell in [Pic06] involved a special kind of Birkhoff decomposition of  $X$ , namely one determined by a choice of basepoint in  $X$  making the triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  stable with respect to the involution  $\theta$ . Let  $K$  now denote the subgroup of  $U$  fixed by  $\theta$ . Generic elements

of the image of  $U/K$  under the Cartan embedding  $\phi: U/K \rightarrow U$  can be factored uniquely as  $\ell d_\phi \ell^{*\theta}$  where  $\ell \in N^-$  and  $d_\phi \in \exp(\mathfrak{h})$ . Unlike the group case, the generic part of  $\phi(U/K)$  often consists of several connected components. In the special case that  $\theta$  is an inner automorphism,  $d_\phi$  can be factored further as  $\mathbf{w}|d_\phi|$  where  $\mathbf{w}$  is an element of order 2 in the torus  $H \cap K$ . The element  $\mathbf{w}$  indexes the connected component of the generic part of  $\phi(U/K)$  that  $\ell d_\phi \ell^{*\theta}$  lies in. Pickrell's generalization of (1.3) takes the form

$$\int_{\phi(U/K)} |d_\phi|^{-i\lambda} = \frac{1}{Z} \sum_{\mathbf{w}} \prod_{\alpha} \frac{\langle \delta, \alpha \rangle}{\langle \delta - i\lambda, \alpha \rangle} \quad (1.4)$$

where the integral is with respect to an invariant measure on  $U/K$ ,  $Z$  is a normalization, and, given  $\mathbf{w}$ , the product is over positive roots  $\alpha$  which are of noncompact type for the involution  $\text{Ad}(\mathbf{w})\theta$ . A similar formula holds in the general case.

In the same vein as before, the integral formula in (1.4) can be understood as determining the distribution of the diagonal  $|d_\phi|$  as a random variable on the symmetric space  $X$ . In the inner case, each of the connected components of the generic part of  $\phi(U/K)$  is diffeomorphic to the non-compact symmetric space  $G_0/K$  dual to  $U/K$ . The diffeomorphism is induced from the  $G_0$  action on  $U$  through the Iwasawa decomposition. After restricting the integral on the right hand side of (1.4) to a component and then pulling back to  $G_0/K$ , Pickrell recognized that the transformed integral was of a form to which the Duistermaat-Heckman exact stationary phase formula applied.

The relevant symplectic structure on  $G_0/K$  was that induced by the Poisson structure resulting from the Evens-Lu construction applied to the  $G_0$ -orbit through  $\mathbf{u}$  in the variety of Lagrangian subalgebras. This structure is, in some sense, dual to the Poisson structure on  $U/K$  considered in this dissertation. Surprisingly, the momentum map for the  $G_0/K$  structure is eerily similar to that found in this paper for the  $U/K$  structure. It is hoped that the momentum map found in this dissertation can be used to produce an alternate proof of the formula in (1.4). In addition, the

momentum mapping determined in this dissertation concerns layers other than the top layer of the Birkhoff decomposition. This presents the possibility of computing the associated integrals over the other parts of the Birkhoff decomposition.

## CHAPTER 2

### THE EVENS-LU POISSON STRUCTURE

Recall that  $X$  denotes a connected and simply connected compact symmetric space. For simplicity, further assume that  $X$  is irreducible. The universal covering group of the identity component of the isometry group of  $X$  is denoted by  $U$  with complexification  $G$  and corresponding Lie algebra  $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ .

Each choice of a  $\mathfrak{u}$ -compatible triangular decomposition of  $\mathfrak{g}$  determines a Poisson Lie group structure on  $U$  as follows. Fix a  $\mathfrak{u}$ -compatible triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ . This means that:  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$ ; a set of positive roots for the adjoint action of  $\mathfrak{h}$  on  $\mathfrak{g}$  has been chosen;  $\mathfrak{n}_{\pm}$  is the direct sum of the positive (respectively negative) root spaces; and  $-(\mathfrak{n}_{\pm})^* = \mathfrak{n}_{\mp}$  where  $-(\cdot)^*$  denotes the Cartan involution fixing  $\mathfrak{u}$  in  $\mathfrak{g}$ . The sum  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  is a direct sum of vector spaces. As every sum of vector spaces that will need to be written down in this dissertation will be direct, the notation  $+$  will be used instead of the more cumbersome  $\oplus$  to denote direct sum. Set  $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{u}$  and  $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$ . Define a  $\mathbb{C}$ -linear transformation  $\mathcal{H}: \mathfrak{g} \rightarrow \mathfrak{g}$  relative to the given triangular decomposition by

$$\mathcal{H}(Z_- + Z_{\mathfrak{h}} + Z_+) = -iZ_- + iZ_+ \tag{2.1}$$

for each  $Z = Z_- + Z_{\mathfrak{h}} + Z_+ \in \mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ . The real subspace  $\mathfrak{u}$  of  $\mathfrak{g}$  is stable under  $\mathcal{H}$ . Thus  $\mathcal{H}$  is the complexification of a  $\mathbb{R}$ -linear transformation  $\mathcal{H}_{\mathbb{R}}: \mathfrak{u} \rightarrow \mathfrak{u}$  which is skew-symmetric relative to the Killing form on  $\mathfrak{u}$ . Using the Killing form, denoted  $\langle \cdot, \cdot \rangle$ , identify the dual of  $\mathfrak{u}$  with  $\mathfrak{u}$  itself, and then view  $\mathcal{H}_{\mathbb{R}}$  as an element of  $\mathfrak{u} \wedge \mathfrak{u}$ . The bivector field

$$\pi_U|_g = r_{g*}\mathcal{H}_{\mathbb{R}} - \ell_{g*}\mathcal{H}_{\mathbb{R}} \tag{2.2}$$

defines a Poisson Lie group structure on  $U$ . (Here  $\ell_g$  and  $r_g$  denote left and right translation by  $g \in U$ ). The Lie algebra structure induced by  $\pi_U$  on the dual of  $\mathfrak{u}$  is isomorphic to the real Lie algebra  $\mathfrak{n}_- + \mathfrak{h}_\mathbb{R}$ . The identification is given by the imaginary part of the Killing form. The double of this Lie bialgebra can then be identified with  $\mathfrak{g}$  (regarded as a real Lie algebra) via the Iwasawa decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h}_\mathbb{R} + \mathfrak{u}$ . The Poisson Lie group structure  $\pi_U$  corresponds to the Manin triple  $(\mathfrak{g}, \mathfrak{u}, \mathfrak{n}_- + \mathfrak{h}_\mathbb{R})$ . Following terminology common in the literature, the Poisson structure in (2.2) will be referred to as a *standard Poisson Lie group structure* on  $U$ .

More often, this Poisson structure is presented in the literature in terms of a basis. For each positive root  $\alpha$  of the action of  $\mathfrak{h}$  on  $\mathfrak{g}$  choose a root vector  $E_\alpha$  such that  $\langle E_\alpha, -(E_\alpha)^* \rangle = -1$ . The  $\mathfrak{u}$ -compatibility of the triangular decomposition implies that  $E_\alpha^*$  is a root vector for  $-\alpha$ . Set  $E_{-\alpha} = E_\alpha^*$ . Then  $X_\alpha = E_\alpha - E_{-\alpha}$  and  $Y_\alpha = i(E_\alpha + E_{-\alpha})$  are in  $\mathfrak{u}$  for each positive root and  $\mathcal{H}_\mathbb{R} = \frac{1}{2} \sum_{\alpha > 0} X_\alpha \wedge Y_\alpha$ . It is shown in [LW90] that  $\mathcal{H}_\mathbb{R}$  satisfies  $[\mathcal{H}_\mathbb{R}, \mathcal{H}_\mathbb{R}] = 0$ . Thus, by a theorem of Drinfeld, the bivector field  $\pi_U$  in (2.2) defines a Poisson Lie group structure on the compact group  $U$ .

For the remainder of this chapter and its sequel, fix a triangular decomposition of  $\mathfrak{g}$  and thus a standard Poisson Lie group structure  $\pi_U$  on  $U$ .

Write  $\mathfrak{d}$  for  $\mathfrak{g}$  regarded as a real Lie algebra. The imaginary part of the Killing form for  $\mathfrak{g}$ , denoted  $\text{Im}\langle \cdot, \cdot \rangle$ , gives a real bilinear form on  $\mathfrak{d}$ . A Lie subalgebra  $\mathfrak{l}$  of  $\mathfrak{d}$  is said to be *Lagrangian* with respect to  $\text{Im}\langle \cdot, \cdot \rangle$ , if  $2 \dim_\mathbb{R} \mathfrak{l} = \dim_\mathbb{R} \mathfrak{g}$ , and  $\text{Im}\langle a, b \rangle = 0$  for all  $a, b \in \mathfrak{l}$ . The set of all Lie subalgebras of  $\mathfrak{d}$  which are Lagrangian with respect to  $\text{Im}\langle \cdot, \cdot \rangle$  will be denoted  $\mathcal{L}(\mathfrak{d})$ . This is naturally a subvariety of the real Grassmannian  $Gr(d, \mathfrak{d})$  of  $d$ -dimensional subspaces of  $\mathfrak{d}$ , where  $2d = \dim_\mathbb{R} \mathfrak{d}$ . The adjoint action of  $G$  on  $\mathfrak{g}$  induces an action of  $G$  (and therefore any subgroup of  $G$ ) on  $\mathcal{L}(\mathfrak{d})$ . Each  $U$ -orbit in  $\mathcal{L}(\mathfrak{d})$  is smooth.

Using  $\pi_U$ , Evens and Lu ([EL01]) construct a smooth bivector field  $\Pi$  on  $Gr(d, \mathfrak{d})$

with the property that the Schouten bracket  $[\Pi, \Pi]$  vanishes at each point of the subvariety  $\mathcal{L}(\mathfrak{d})$ . Furthermore, they show that  $\Pi$  is tangent to each  $U$ -orbit in  $\mathcal{L}(\mathfrak{d})$ , so that each  $U$ -orbit is a Poisson manifold. The construction is carried out in such a way that each  $U$ -orbit in  $\mathcal{L}(\mathfrak{d})$  becomes a  $(U, \pi_U)$ -homogenous Poisson space.

Each choice of basepoint in  $X$  determines a model point in  $\mathcal{L}(\mathfrak{d})$  as follows. Let  $K$  be the stability subgroup in  $U$  of a basepoint in  $X$ . The subgroup  $K$  is closed and is the fixed point set of an involution  $\theta$  of  $U$ . As in the introduction, let  $\theta$  also denote the complex extension of the involution from  $U$  to all of  $G$  and write  $g^\theta$  for  $\theta(g)$ . The Cartan involution of  $G$  fixing  $U$  will be denoted  $g \mapsto g^{-*}$ . Write  $G_0$  for the connected subgroup of  $G$  which is fixed by the involution  $g \mapsto g^\sigma = g^{-*\theta}$ . The intersection of  $U$  and  $G_0$  in  $G$  is  $K$ . The Lie algebra of  $G_0$ , denoted  $\mathfrak{g}_0$ , is a real form of  $\mathfrak{g}$  and thus a subalgebra of  $\mathfrak{d}$  which is Lagrangian with respect to  $\text{Im}\langle \cdot, \cdot \rangle$ . The coset space  $U/K$  is a finite sheeted covering of the  $U$  orbit through  $\mathfrak{g}_0$  in  $\mathcal{L}(\mathfrak{d})$ . This is why  $\mathfrak{g}_0$  is called a model point for  $X \simeq U/K$  in  $\mathcal{L}(\mathfrak{d})$ .

The diagram of groups and Lie algebras shown in figure 1 lists this information for reference. The upward arrows are inclusions in both diagrams. In the group diagram the quotients are listed for each leg. Each quotient is also a symmetric space. The quotient  $G_0/K$  is a model for the non-compact symmetric space dual to  $X$  presented as  $U/K$ . Also shown in the diagram is the decomposition of the Lie algebras  $\mathfrak{u}$  and  $\mathfrak{g}_0$  into the eigenspaces of  $\theta$  as  $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$  and  $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ . At times in this dissertation it will be convenient to write  $X^g$  for  $\text{Ad}(g)(X)$  to compactify notation. Unfortunately, the adjoint action of  $g$  does not necessarily commute with the involutions which are also being written as superscripts. Thus one notation or the other will be used at different points in the dissertation depending on the situation.

The Poisson structure  $\Pi$  on the orbit  $U \cdot \mathfrak{g}_0$  may be lifted to a Poisson structure  $\pi$  on  $X$ , and it is this structure, making  $X$  into a  $(U, \pi_U)$ -homogeneous Poisson space, that is of interest in this dissertation. It will be referred to as the *Evens-Lu Poisson*

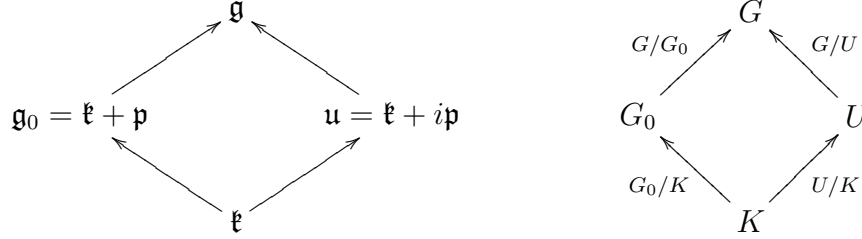


Figure 2.1: Algebras, groups, and quotients.

structure on  $X$ . The Evens-Lu construction will now be reviewed in order to derive an expression for  $\pi$  amenable to the discussion in later sections.

Let  $(\cdot, \cdot)$  denote the non-degenerate  $\mathbb{R}$ -bilinear form on  $\mathfrak{d} = (\mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}}) + \mathfrak{u}$  defined by

$$(\xi_1 + x_1, \xi_2 + x_2) = \text{Im}\langle \xi_1, x_2 \rangle + \text{Im}\langle \xi_2, x_1 \rangle$$

for each  $\xi_k \in (\mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}})$  and  $x_k \in \mathfrak{u}$ ,  $k = 1, 2$ . Identify the dual and double dual of  $\mathfrak{d}$  with  $\mathfrak{d}$  itself using this non-degenerate pairing. This allows one to define an element of  $\bigwedge^2 \mathfrak{d}$  by its action on a pair of elements of  $\mathfrak{d}$ . Evens and Lu define  $R \in \bigwedge^2 \mathfrak{d}$  by

$$R(\xi_1 + x_1, \xi_2 + x_2) = \text{Im}\langle \xi_2, x_1 \rangle - \text{Im}\langle \xi_1, x_2 \rangle \quad (2.3)$$

and use it to generate a bivector field on the Grassmannian of  $d$ -dimensional subspaces of  $\mathfrak{d}$ . The adjoint action of  $G$  on  $\mathfrak{g}$  induces a  $\mathbb{R}$ -Lie algebra anti-homomorphism

$$\kappa: \mathfrak{d} \rightarrow \Gamma(TGr(d, \mathfrak{d}))$$

whose multi-linear extension  $\bigwedge \mathfrak{d} \rightarrow \Gamma(\bigwedge TGr(d, \mathfrak{d}))$  will also be denoted by  $\kappa$ . The bivector field  $\Pi$  is then defined by  $\Pi = \frac{1}{2} \kappa(R)$ .

To do calculations, identify  $T(U/K)$  with  $U \times_K i\mathfrak{p}$  using right translation. Further, identify  $i\mathfrak{p}$  with  $(i\mathfrak{p})^*$  using the Killing form so that  $T^*(U/K)$  may also be represented by  $U \times_K i\mathfrak{p}$ . In this setting, the action of  $\pi$  on a pair of cotangent vectors represented by classes  $[u, X]$  and  $[u, Y]$  in  $U \times_K i\mathfrak{p}$  may be computed by

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y \rangle$$

where  $\Omega_u: \mathfrak{ip} \rightarrow \mathfrak{ip}$  is a skew-symmetric  $\mathbb{R}$ -linear transformation which is  $K$ -equivariant in its dependence on  $u$ .

**Theorem 2.1.** *The Evens-Lu Poisson bivector  $\pi$  can be expressed as*

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y \rangle$$

where  $X, Y \in \mathfrak{ip}$ , the transformation  $\Omega_u: \mathfrak{ip} \rightarrow \mathfrak{ip}$  is given by

$$\Omega_u(X) = \{\text{Ad}(u^{-1}) \circ \mathcal{H} \circ \text{Ad}(u)(X)\}_{\mathfrak{ip}} \quad (2.4)$$

and  $\{\cdot\}_{\mathfrak{ip}}$  denotes the projection to  $\mathfrak{ip}$  along the decomposition  $\mathfrak{g} = \mathfrak{iu} + \mathfrak{k} + \mathfrak{ip}$ .

*Proof.* Let  $\text{pr}_{\mathfrak{u}}: \mathfrak{g} \rightarrow \mathfrak{u}$  denote the projection onto  $\mathfrak{u}$  along the Iwasawa decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}} + \mathfrak{u}$ . In terms of the triangular decomposition  $Z = Z_- + Z_{\mathfrak{h}} + Z_+ \in \mathfrak{g}$ , this projection is given by the formula

$$\text{pr}_{\mathfrak{u}}(Z) = -(Z_+)^* + Z_{\mathfrak{t}} + Z_+.$$

The composition  $\mathfrak{u} \xrightarrow{i} \mathfrak{iu} \xrightarrow{\text{pr}_{\mathfrak{u}}} \mathfrak{u}$  agrees with the transformation  $\mathcal{H}$  when restricted to  $\mathfrak{u}$  (recall the definition of  $\mathcal{H}$  in (2.1)). Indeed, each  $Z \in \mathfrak{u}$  satisfies  $Z_- = -(Z_+)^*$  and  $Z_{\mathfrak{h}} = Z_{\mathfrak{t}}$ ; hence  $(iZ)_{\mathfrak{t}} = 0$ , and  $-((iZ)_+)^* = i(Z_+)^* = iZ_-$ . It follows that

$$\text{pr}_{\mathfrak{u}}(iZ) = -iZ_- + iZ_+ = \mathcal{H}(Z). \quad (2.5)$$

In equation (2.4), the equivalence class  $[u, X]$  represents a linear functional on the tangent space to  $U/K$  at  $uK$ . Using the covering map to identify this space with the tangent space to the  $U$ -orbit through  $\mathfrak{g}_0$  at  $u \cdot \mathfrak{g}_0$ , each such tangent vector corresponds to an element  $\chi \in \mathfrak{u}/\mathfrak{k}^u$  via the map  $\chi \mapsto \left. \frac{d}{dt} \right|_{t=0} e^{t\chi} u \cdot \mathfrak{g}_0$ . The corresponding class in  $U \times_K \mathfrak{ip}$  is  $[u, \chi^{u^{-1}}]$ . Moreover, the action of  $[u, X]$  as a linear functional on  $[u, \chi^{u^{-1}}]$  is given by

$$[u, X] \left( [u, \chi^{u^{-1}}] \right) = \langle X, \chi^{u^{-1}} \rangle = \text{Im} \langle iX^u, \chi \rangle.$$

By the Evens-Lu construction

$$\pi([u, X], [u, Y]) = \frac{1}{2} \kappa(R)|_{uK} ([u, X], [u, Y]) = \frac{1}{2} R(iX^u, iY^u).$$

Combining the definition of  $R$  in (2.3), the result in (2.5), and skew-symmetry of  $\mathcal{H}$ , it follows that

$$\begin{aligned} R(iX^u, iY^u) &= \operatorname{Im}\langle \operatorname{pr}_u(iX^u), iY^u \rangle - \operatorname{Im}\langle iX^u, \operatorname{pr}_u(iY^u) \rangle \\ &= \langle \mathcal{H}(X^u), Y^u \rangle - \langle X^u, \mathcal{H}(Y^u) \rangle \\ &= 2\langle (\mathcal{H}(X^u))^{u^{-1}}, Y \rangle. \end{aligned}$$

This completes the proof of the theorem.  $\square$

Using the equivariant formula in (2.4) we can directly verify that the Evens-Lu construction produced a  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$ .

**Proposition 2.2.** *The formula (2.4) defines a  $(U, \pi_U)$ -homogeneous Poisson structure on  $U/K$ .*

*Proof.* It needs to be shown that the push-forward of the bivector  $\pi_U + \pi$  at  $(u_1, u_2K)$  under the action map

$$\begin{aligned} a: U \times (U/K) &\rightarrow U/K \\ (u_1, u_2K) &\mapsto u_1u_2K \end{aligned} \tag{2.6}$$

equals  $\pi$  evaluated at  $u_1u_2K$ . Use the Killing form to identify the dual of  $\mathfrak{u}$  with  $\mathfrak{u}$  itself and further identify  $TU$  and  $T^*U$  with  $U \times \mathfrak{u}$  using right translation. With this presentation, the Lu-Weinstein Poisson Lie group structure  $\pi_U$  can be computed by the formula

$$\pi_U((u, V), (u, W)) = \langle (\mathcal{H} - \operatorname{Ad}(u) \circ \mathcal{H} \circ \operatorname{Ad}(u^{-1}))(V), W \rangle \tag{2.7}$$

where  $V$  and  $W$  are elements of  $\mathfrak{u}$  representing cotangent vectors at  $u$ . Given the present identifications, a tangent vector to  $U \times U/K$  at the point  $(u_1, u_2K)$ , represented by  $(V, X) \in \mathfrak{u} \times i\mathfrak{p}$ , corresponds to the curve  $t \mapsto (e^{tV}u_1, u_2e^{tX})$ . Applying

the action map (2.6), this curve is sent to the curve

$$t \mapsto e^{tV} u_1 u_2 e^{tX} = u_1 u_2 e^{tV(u_1 u_2)^{-1}} e^{tX}. \quad (2.8)$$

The derivative of the right hand side of (2.8) at  $t = 0$  then corresponds to the class

$$[u_1 u_2, \{\text{Ad}(u_1 u_2)^{-1}(V)\}_{i\mathfrak{p}} + X]. \quad (2.9)$$

In order to compute the push forward of the bivector  $\pi_U + \pi$  under the action map, one needs to compute the transpose of the linear transformation

$$a_*|_{(u_1, u_2 K)}: T_{u_1} U \times T_{u_2 K}(U/K) \rightarrow T_{u_1 u_2 K}(U/K). \quad (2.10)$$

Given the present identifications, the value of the transpose of (2.10) on a linear functional represented by  $\langle Y, \cdot \rangle$  can be computed by

$$\begin{aligned} (a_*|_{(u_1, u_2 K)})^{\text{tr}}(\langle Y, \cdot \rangle)((u_1, V), [u_2, X]) &= \langle Y, \{\text{Ad}(u_1 u_2)^{-1}(V)\}_{i\mathfrak{p}} + X \rangle \\ &= \langle \text{Ad}(u_1 u_2)(Y), V \rangle + \langle Y, X \rangle. \end{aligned}$$

The following formula (2.11) describes this map in terms of the identifications.

$$\begin{aligned} (a_*|_{(u_1, u_2 K)})^{\text{tr}}: T_{u_1 u_2 K}^*(U/K) &\rightarrow T_{u_1}^* U \times T_{u_2 K}^*(U/K) \\ [u_1 u_2, Y] &\mapsto ((u_1, \text{Ad}(u_1 u_2)(Y)), [u_2, Y]) \end{aligned} \quad (2.11)$$

We can now compute the push-forward of the bivector. The value of the bivector  $a_*|_{(u_1, u_2 K)}((\pi_U + \pi))$  on two cotangent vectors at  $u_1 u_2 K$  represented by  $[u_1 u_2, X]$  and  $[u_1 u_2, Y]$  is given by

$$\pi_U((u_1, \text{Ad}(u_1 u_2)(X)), (u_1, \text{Ad}(u_1 u_2)(Y))) + \pi([u_2, X], [u_2, Y]) \quad (2.12)$$

using the map in (2.11). By (2.7), the first term in (2.12) is given by

$$\langle (\mathcal{H} - \text{Ad}(u_1) \circ \mathcal{H} \circ \text{Ad}(u_1)^{-1})(\text{Ad}(u_1 u_2)(X)), \text{Ad}(u_1 u_2)(Y) \rangle \quad (2.13)$$

while the equivariant formula for  $\pi$  in theorem 2.1 yields that the second term in (2.12) is given by

$$\langle \text{Ad}(u_2)^{-1} \circ \mathcal{H} \circ \text{Ad}(u_2)(X), Y \rangle. \quad (2.14)$$

Using that  $\text{Ad}$  is a homomorphism and that  $\text{Ad}(u)^* = \text{Ad}(u^{-1})$ , it follows that the quantity in (2.13) is equal to

$$\langle \text{Ad}(u_1 u_2)^{-1} \circ \mathcal{H} \circ \text{Ad}(u_1 u_2)(X), Y \rangle - \langle \text{Ad}(u_2)^{-1} \circ \mathcal{H} \circ \text{Ad}(u_2)(X), Y \rangle. \quad (2.15)$$

Combining the results of (2.15) with (2.14), one obtains that the value of  $a_*|_{(u_1, u_2 K)}((\pi_U + \pi))$  on two cotangent vectors at  $u_1 u_2 K$  represented by  $[u_1 u_2, X]$  and  $[u_1 u_2, Y]$  is

$$\langle \text{Ad}(u_1 u_2)^{-1} \circ \mathcal{H} \circ \text{Ad}(u_1 u_2)(X), Y \rangle = \pi([u_1 u_2, X], [u_1 u_2, Y])$$

as was to be shown.  $\square$

From the formula in theorem 2.1, the connection with triangular factorization is evident and a group theoretic interpretation of the symplectic foliation can be given. At the group level, write the Iwasawa decomposition for  $G$  corresponding to  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h}_{\mathbb{R}} + \mathfrak{u}$  as

$$\begin{aligned} G &\simeq N^- \times A \times U. \\ g &\mapsto (\mathbf{l}(g), \mathbf{a}(g), \mathbf{u}(g)) \end{aligned}$$

where  $A = \exp(\mathfrak{h}_{\mathbb{R}})$ . There is a natural action of  $G$  (and therefore any subgroup of  $G$ ) on  $U$  coming from the identification of  $U$  with the right coset space  $N^- A \backslash G$ .

$$\begin{aligned} U \times G &\rightarrow U \\ u \cdot g &\mapsto \mathbf{u}(ug) \end{aligned}$$

As observed in [LW90], the symplectic leaves of the Poisson Lie group structure  $\pi_U$  on  $U$  are precisely the  $N^- A$  orbits in  $U$ .

**Proposition 2.3.** *For the Evens-Lu Poisson structure  $\pi$  the leaves of the symplectic foliation are the projections of the  $G_0$ -orbits in  $U$  to  $U/K$ .*

*Proof.* The symplectic foliation of  $U/K$  is generated by the distribution in  $T(U/K)$  which is image of the natural map  $\pi^\sharp: T^*(U/K) \rightarrow T(U/K)$  given by  $\pi^\sharp(\alpha) = \pi(\alpha, \cdot)$ .

In terms of the identifications  $T^*(U/K) \simeq U \times_K \mathfrak{ip} \simeq T(U/K)$ ,

$$\pi^{\natural}([u, X]) = [u, \{(\mathcal{H}(X^u))^{u^{-1}}\}_{\mathfrak{ip}}] = [u, \{(\mathrm{pr}_{\mathbf{u}}(iX^u))^{u^{-1}}\}_{\mathfrak{ip}}].$$

Fix  $u \in U$ . The map  $G_0 \rightarrow U$  given by  $g_0 \mapsto \mathbf{u}(ug_0)$  is equivariant for the right  $K$ -action on both  $G_0$  and  $U$  and thus descends to a map  $G_0/K \rightarrow U/K$ . Now, suppose that  $iX \in \mathfrak{p}$ , so that  $X \in \mathfrak{ip}$ . Consider the curve

$$u \cdot e^{tiX} = \mathbf{u}(ue^{tiX}) = \mathbf{u}(e^{tiX^u} u).$$

Differentiating at  $t = 0$ , one obtains the tangent vector  $(r_u)_*(\mathrm{pr}_{\mathbf{u}}(iX^u))$ . It follows that the distribution tangent to the projection of the  $G_0$ -orbit at  $uK$  is spanned by vectors of the form

$$\kappa(\mathrm{pr}_{\mathbf{u}}(iX^u))|_{uK} = [u, \{(\mathrm{pr}_{\mathbf{u}}(iX^u))^{u^{-1}}\}_{\mathfrak{ip}}]$$

which is exactly the image of  $\pi^{\natural}$ . □

This proposition has been established in several contexts for this Poisson structure. Using their construction, Foth and Lu give an alternate proof using general principles (cf. Proposition 1.1 in [FL04]).

## CHAPTER 3

### SYMPLECTIC LEAVES AND THE MOMENTUM MAP

Let  $G$  be a simply connected complex semi-simple Lie group with Lie algebra  $\mathfrak{g}$ . Choose a Cartan subalgebra  $\mathfrak{h}$ , and a set of positive roots for the adjoint action of  $\mathfrak{h}$  on  $\mathfrak{g}$ . Let  $\mathfrak{n}_{\pm}$  denote the sum of the positive (resp. negative) root spaces. This data gives a triangular decomposition of  $\mathfrak{g}$ .

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$$

Set  $H = \exp(\mathfrak{h})$ , and  $N^{\pm} = \exp(\mathfrak{n}_{\pm})$ . Corresponding to this decomposition is a Birkhoff decomposition (a.k.a. triangular decomposition a.k.a LDU decomposition) of the group

$$G = \coprod_{w \in W} \Sigma_w^G, \text{ where } \Sigma_w^G = N^- w H N^+$$

and  $W$  denotes the Weyl group of  $G$ ; i.e.,  $W = N_G(H)/H$ . Each component  $\Sigma_w^G$  is a manifold diffeomorphic to  $(N^- \cap w N^- w^{-1}) \times H \times N^+$ . The codimension of  $\Sigma_w^G$  increases with the length of  $w$  in the Weyl group and  $\Sigma_1^G$  is a Zariski open subset of  $G$ . Each element in  $\Sigma_w^G$  can be factored as a product of: an element of  $N^-$ , an element of  $w \subset N_G(H)$ , an element of  $H$ , and an element of  $N^+$ . For  $\text{SL}(n, \mathbb{C})$  in an appropriate representation this would correspond to the factorization of an  $n \times n$  complex matrix of determinant one into a product of a lower triangular unipotent matrix, a unitary permutation matrix, a diagonal matrix of determinant one, and an upper triangular unipotent matrix. Each element of  $\Sigma_1^G$  admits a unique factorization of this form. If  $w$  is a non-trivial element of the Weyl group then the elements in  $\Sigma_w^G$  admit several such factorizations. Further conditions are required to guarantee uniqueness in those cases.

This chapter concerns a generalization of this decomposition for symmetric spaces and, in particular, its relationship to the Poisson geometry of the Evens-Lu Poisson structure. Recall the setup considered in this dissertation:  $X$  is a compact, connected, and simply connected Riemannian symmetric space;  $U$  is the universal covering group of the identity component of the isometry group of  $X$ ;  $G$  is the complexification of  $U$ . The Poisson Lie group structure  $\pi_U$  on  $U$  was defined by a triangular decomposition of  $\mathfrak{g}$  which had the additional property of being  $\mathfrak{u}$ -compatible, i.e.  $-(\mathfrak{n}_\pm)^* = \mathfrak{n}_\mp$ .

For each choice of basepoint in  $X$ , Cartan defined an embedding of  $X$  into  $U \subset G$ .

$$\begin{aligned} \phi: U/K &\rightarrow U \hookrightarrow G \\ uK &\mapsto uu^{-\theta} \end{aligned} \tag{3.1}$$

In (3.1),  $\theta$  is the involution fixing the stability subgroup  $K$  of the basepoint in  $X$ . If  $U$  is viewed as a Riemannian manifold with the metric induced from the Killing form, then this map gives a totally geodesic embedding of  $X$  into  $U \subset G$ . The intersection of this image with the decomposition of  $G$  induces a decomposition of  $X$  called a *Birkhoff decomposition of  $X$* . Such a decomposition depends on the choice of basepoint in  $X$  and the triangular decomposition of  $\mathfrak{g}$ . In this dissertation, the triangular decomposition of  $\mathfrak{g}$  is regarded as fixed, having defined  $\pi_U$ .

**Lemma 3.1.** *There exists a basepoint in  $X$  such that the given triangular decomposition of  $\mathfrak{g}$  is stable with respect to the involution selecting the stability subgroup of  $x$  in  $U$ .*

*Proof.* Fix a point  $x \in X$ . This determines the data:  $K$ , the stability subgroup of  $x$ ;  $\theta$ , the involution of  $U$  fixing  $K$ ;  $G_0$ , a non-compact real form  $G$ ;  $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ , a decomposition of  $\mathfrak{u}$  into the eigenspaces of  $\theta$ . The Lie algebra of  $G_0$  is  $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ . First, it will be shown that a triangular decomposition of  $\mathfrak{g}$ , which is both  $\mathfrak{u}$ -compatible and  $\theta$ -stable, exists.

Let  $\mathfrak{t}_0$  be a Cartan subalgebra of  $\mathfrak{k}$ , and  $\mathfrak{h}_0$  denote the centralizer of  $\mathfrak{t}_0$  in  $\mathfrak{g}_0$ . Theorem 6.60 of [Kna02] shows that  $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$  is a Cartan subalgebra of  $\mathfrak{g}_0$ , where  $\mathfrak{a}_0 \subset \mathfrak{p}$ . Thus, the subalgebras

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0, \mathfrak{t} = \mathfrak{t}_0 + i\mathfrak{a}_0, \text{ and } \mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$$

are Cartan subalgebras of  $\mathfrak{g}_0$ ,  $\mathfrak{u}$ , and  $\mathfrak{g}$  (respectively). Selecting a Weyl chamber in  $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$ , chooses a set of positive roots for the adjoint action of  $\mathfrak{h}$  on  $\mathfrak{g}$ , and thus determines a triangular decomposition for  $\mathfrak{g}$ . For each choice, the resulting decomposition will be  $\mathfrak{u}$ -compatible. The Cartan subalgebra  $\mathfrak{h}$  is then  $\theta$ -stable by construction. However,  $\theta$  will stabilize the positive root spaces only if the chosen Weyl chamber contains an element of  $i\mathfrak{t}_0$ . The fact that  $i\mathfrak{t}_0$  contains a regular element is equivalent to the fact that  $\mathfrak{h}_0$  is a Cartan subalgebra of  $\mathfrak{g}_0$ . Thus, a  $\mathfrak{u}$ -compatible and  $\theta$ -stable triangular decomposition of  $\mathfrak{g}$  exists.

To finish the proof, note that all  $\mathfrak{u}$ -compatible triangular decompositions of  $\mathfrak{g}$  are  $U$ -conjugate. Therefore, one may conjugate the constructed triangular decomposition to the given  $\mathfrak{u}$ -compatible triangular decomposition of  $\mathfrak{g}$  by an element  $u \in U$ . Conjugating the stability subgroup  $K$  by the same element selects a new basepoint in  $X$  with corresponding involution  $\theta' = \text{Ad}(u) \circ \theta \circ \text{Ad}(u^{-1})$ . The given  $\mathfrak{u}$ -compatible decomposition  $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$  is stable with respect to the involution  $\theta'$ .  $\square$

For the remainder of this chapter, fix a presentation of the symmetric space  $X$  as  $U/K$  such that the triangular decomposition of  $\mathfrak{g}$  defining  $\pi_U$  is  $\theta$ -stable. Under the assumption of  $\theta$ -stability, Pickrell was able to characterize the connected components of the resulting Birkhoff decomposition of  $X$  (cf. [Pic06]). It is through reinterpretation of those results that an explicit description of the geometry of the Evens-Lu Poisson structure on  $X$  can be given. For the convenience of the reader, the same notation as that in [Pic06] has been adopted in this dissertation.

The first observation is that  $\phi(U/K)$  does not necessarily intersect each component of the Birkhoff decomposition of  $G$ . In fact,  $\phi(U/K) \cap \Sigma_w^G$  is non-empty if and

only if  $\phi(U/K) \cap w$  is non-empty where  $w \subset N_U(T)$  represents the Weyl group element  $w \in W = N_G(H)/H \simeq N_U(T)/T$  (cf. Theorem 2 (a) combined with Theorem 1 (a) in [Pic06]).

One should think of a Birkhoff decomposition of  $X$  as consisting of a number of layers indexed by the possible elements  $w$ . In this dissertation, *the layer corresponding to  $w$* , will refer to the set  $\phi(U/K) \cap \Sigma_w^G$ , or its equivalent in  $U/K$  or  $X$  when the context is clear. The reader should understand that when this terminology is used they are to implicitly assume that  $w$  is such that  $\phi(U/K) \cap w$  is non-empty.

Each layer consists of a number of connected components. For a given  $w \in N(T)/T$  the connected components of the layer corresponding to  $w$  are indexed by the elements of the set

$$\{\mathbf{w} \in \phi(U/K) \cap w\}/T$$

where  $T$  acts on the right by  $\mathbf{w} \cdot t = t^{-1}\mathbf{w}t^\theta$  (cf. Theorem 2 (c) in [Pic06]). This characterization uses  $\theta$ -stability. The elements  $\mathbf{w} \in \phi(U/K) \cap T$  are the images under the Cartan embedding of the preferred basepoints in  $X$  in the sense of lemma 3.1.

**Notation.** Given an element  $\mathbf{w}$  in the layer corresponding to  $w$ , write  $\Sigma_{\mathbf{w}}^{\phi(U/K)}$  for the connected component of  $\phi(U/K) \cap \Sigma_w^G$  containing  $\mathbf{w}$ .

In proposition 2.3 it was shown that the symplectic leaves of the Evens-Lu Poisson structure on  $X \simeq U/K$  are the projections modulo  $K$  of the  $G_0$ -orbits in  $U$ . The right action of  $G_0$  on  $U$  was that induced by the identification  $U \simeq N^-A \backslash G$  coming from the Iwasawa decomposition  $G \simeq N^-AU$ . By combining this with an action of the torus  $T = \exp(\mathfrak{h} \cap \mathfrak{u})$  as

$$\begin{aligned} U \times (T \times G_0) &\rightarrow U \\ u \cdot (t, g_0) &\mapsto t^{-1}\mathbf{u}(ug_0) \end{aligned}$$

Pickrell was able to characterize each component  $\Sigma_{\mathbf{w}}^{\phi(U/K)}$ . The following proposition is Theorem 4 (a) in [Pic06].

**Proposition 3.2.** *Consider the layer of the Birkhoff decomposition of  $X$  corresponding to  $w \in N_U(T)/T$ . Let  $\mathbf{w} \in \phi(U/K) \cap w$  and fix a choice of  $\mathbf{w}_1 \in U$  such that  $\phi(\mathbf{w}_1 K) = \mathbf{w}$ . The map*

$$\begin{aligned} T \times G_0 &\rightarrow \phi(U/K) \cap \Sigma_{\mathbf{w}}^{\phi(U/K)} \\ (t, g_0) &\mapsto \phi(t^{-1}\mathbf{u}(\mathbf{w}_1 g_0)) \end{aligned}$$

*is surjective and induces a diffeomorphism*

$$T \times_{\exp(\ker\{\text{Ad}(w)\theta|_t - 1\})} R \backslash G_0 / K \rightarrow \Sigma_{\mathbf{w}}^{\phi(U/K)}$$

*where  $R = (N^-A)^{\mathbf{w}_1^{-1}} \cap G_0$  is a contractible subgroup of  $G_0$  and  $\lambda \in \exp(\ker\{\text{Ad}(w)\theta|_t - 1\})$  is identified with the pair  $(\lambda, \lambda^{\mathbf{w}_1^{-1}})$ .*

The proof is fairly involved and will not be reproduced here. The key point is that the theorem can be reinterpreted as characterizing the symplectic leaves of the Evens-Lu Poisson structure. The projection modulo  $K$  of the  $G_0$ -orbit through  $\mathbf{w}_1$  maps to a sub-manifold of  $\Sigma_{\mathbf{w}}^{\phi(U/K)}$  passing through  $\mathbf{w}$ . This shows that with the appropriate choice of basepoint in  $X$ , the symplectic foliation of the Evens-Lu Poisson structure aligns with the Birkhoff decomposition of  $X$ .

**Corollary 3.3.** *Each connected component of the layer corresponding to  $w$  is foliated by contractible symplectic leaves, each diffeomorphic to the double coset space  $R \backslash G_0 / K$ .*

More can be seen from proposition 3.2. Each leaf admits a natural torus action. The acting torus is a sub-torus of  $T$  determined by the layer of the Birkhoff decomposition in which the leaf is contained.

**Corollary 3.4.** *Each symplectic leaf foliating the layer corresponding to  $w$  is acted on by the torus*

$$T_w = \exp(\ker\{\text{Ad}(w)\theta|_t - 1\}) \subset T.$$

*Moreover, the coset  $\mathbf{w}_1 K$  represents the unique fixed point for the  $T_w$ -action in the symplectic leaf through  $\mathbf{w}_1 K$ .*

At the end of this chapter, it will be shown that the action of  $T_w$  on each symplectic leaf foliating the layer indexed by  $w$  is Hamiltonian. The following proposition and corollary will be needed in order to compute the momentum map. First, notice that an element  $g$  in  $\phi(U/K) \cap G$  first satisfies the equation  $g^{-1} = g^\theta$  and the further condition that  $g \in U$  so that  $g^{-1} = g^*$ . In fact,  $\phi(U/K)$  can be realized as the identity component of the set  $\{g^* = g^\theta\} \cap U$  (cf. Theorem 1 (a) in [Pic06]). The following proposition is Theorem 2 (d) in [Pic06].

**Proposition 3.5.** *Fix  $w \in W$ . Suppose  $\mathbf{w} \in w \subset N_G(H)$  with  $\mathbf{w}^* = \mathbf{w}^\theta$ . For brevity, write  $D_{\mathbf{w}}$  for the set*

$$\{(h, L) \in H \times (N^- \cap (N^+)^w) : \theta(h^{w^{-1}}) = h^*, (\theta(L^{-*}))^{\mathbf{w}h} = L^{-1}\}.$$

The map

$$\begin{aligned} (N^- \cap (N^+)^w) \times D_{\mathbf{w}} &\rightarrow \{g^* = g^\theta\} \cap \Sigma_w^G \\ (l, (h, L)) &\mapsto lL^{-1}\mathbf{w}h(lL^{-1})^{*\theta} \end{aligned}$$

is a diffeomorphism onto the connected component of  $\{g^* = g^\theta\} \cap \Sigma_w^G$  containing  $\mathbf{w}$ .

Applied to  $\phi(U/K) \cap \Sigma_w^G$ , this proposition provides further conditions to guarantee the uniqueness of the triangular factorization of elements in the layers indexed by non-trivial elements  $w$ . The following corollary reinterprets this result.

**Corollary 3.6.** *Let  $\mathbf{w} \in \phi(U/K) \cap w$  be an element in the layer corresponding to  $w$ . Let  $S_{\mathbf{w}}$  denote the symplectic leaf of  $(X, \pi)$  for which  $\phi(S_{\mathbf{w}})$  passes through  $\mathbf{w}$  in  $\phi(U/K) \cap w$ . Each element in  $\phi(S_{\mathbf{w}})$  can be factored as  $\ell\mathbf{w}h\ell^{*\theta}$  where  $\ell \in N^-$ , and  $h \in \exp(\ker\{Ad(w)\sigma|_{\mathfrak{h}} + 1\})$ . Furthermore, the magnitude  $|h| = \sqrt{h^*h}$  is a smooth function on the leaf.*

*Proof.* Existence of the factorization is guaranteed by proposition 3.5 with the observation that  $h$  satisfies the condition  $\theta(h^{w^{-1}}) = h^*$  if and only if  $h \in \exp(\ker\{Ad(w)\sigma|_{\mathfrak{h}} + 1\})$ . Note that proposition 3.5 also guarantees that  $h$  is a smooth function on the leaf and thus  $|h| = \sqrt{h^*h} \in \exp(\mathfrak{h}_{\mathbb{R}})$  is smooth as well.  $\square$

**Theorem 3.7.** *The action of the torus  $T_w$  on the symplectic leaf  $S_{\mathbf{w}}$  is Hamiltonian with momentum map*

$$\begin{aligned} \mu: S_{\mathbf{w}} &\rightarrow \mathfrak{t}_w^* \\ uK &\mapsto \langle \tfrac{1}{2}i\theta(\log|h|), \cdot \rangle \end{aligned}$$

where  $h$  is the diagonal part of  $\phi(uK) = uu^{-\theta} \in \phi(S_{\mathbf{w}})$ .

*Proof.* Let  $X \in \mathfrak{t}_w$ . Practically, this means that  $X$  satisfies the equation  $\mathbf{w}X^\theta\mathbf{w}^{-1} = X$  for each  $\mathbf{w} \in w$ . The vector field  $\tilde{X}$  induced by the action  $T_w$  on  $U/K$  is represented by the class  $[u, \{-X^{u^{-1}}\}_{i\mathfrak{p}}]$  in  $U \times_K i\mathfrak{p}$ . It needs to be shown that  $\mu_X: S_{\mathbf{w}} \rightarrow \mathbb{R}$  given by  $\mu_X(uK) = \langle \tfrac{1}{2}i\theta(\log|h|), X \rangle$  is a Hamiltonian function for the vector field  $\tilde{X}$ . This will follow if  $\pi^\flat(d\mu_X) = \tilde{X}$ .

Let  $[u, Y] \in U \times_K i\mathfrak{p}$  be a tangent vector to  $U/K$  at  $uK$ . This tangent vector is represented by the curve  $ue^{tY}K$  which passes through  $uK$  at  $t = 0$ . Decompose  $\phi(uK) = uu^{-\theta}$  as  $\ell\mathbf{w}h\ell^{*\theta}$  using corollary 3.6. Then

$$\begin{aligned} ue^{tY}(ue^{tY})^{-\theta} &= e^{2tY^u}uu^{-\theta} \\ &= e^{2tY^u}\ell\mathbf{w}h\ell^{*\theta} \\ &= \ell\mathbf{w}e^{2t(\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w}}h\ell^{*\theta} \end{aligned}$$

and thus  $\frac{d}{dt}\big|_{t=0} \log |d_\phi(ue^{tY}K)| = 2 \operatorname{pr}_{\mathfrak{h}_{\mathbb{R}}}((\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w})$ . With this calculation it follows that

$$\begin{aligned} d\mu_X([u, Y]) &= \langle i\theta(\operatorname{pr}_{\mathfrak{h}_{\mathbb{R}}}((\ell\mathbf{w})^{-1}Y^u\ell\mathbf{w})), X \rangle \\ &= \langle Y, \operatorname{Ad}(u^{-1})((i\mathbf{w}X^\theta\mathbf{w}^{-1})^\ell) \rangle \\ &= \langle Y, \operatorname{Ad}(u^{-1})(iX^\ell) \rangle. \end{aligned}$$

Hence,  $d\mu_X$  corresponds to the class  $[u, \{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{i\mathfrak{p}}]$ .

**Assertion 1:**  $\{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{i\mathfrak{p}} = \frac{1}{2}\{\operatorname{Ad}(u^{-1})(iX^\ell)\}_{\mathfrak{u}}$  where  $\{\cdot\}_{\mathfrak{u}}$  denotes the projection to  $\mathfrak{u}$  along the decomposition  $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ .

Given  $Z \in \mathfrak{g}$ , the projection to  $i\mathfrak{p}$  is given by

$$\{Z\}_{i\mathfrak{p}} = \frac{1}{4} (Z + Z^{*\theta} - (Z + Z^{*\theta})^*) = \frac{1}{2} \{Z + Z^{*\theta}\}_{\mathfrak{u}}. \quad (3.2)$$

The equation  $uu^{-\theta} = \ell \mathbf{w} h \ell^{*\theta}$  implies that  $u^{-\theta} \ell^\sigma = u^{-1} \ell \mathbf{w} h$ . Recalling that  $(\cdot)^*$  reverses the order of products, this equality gives

$$(\text{Ad}(u^{-1})(iX^\ell))^{*\theta} = u^{-\theta} \ell^\sigma iX^\theta \ell^{-\sigma} u^\theta \quad (3.3)$$

$$= u^{-1} \ell \mathbf{w} iX^\theta \mathbf{w}^{-1} \ell^{-1} u \quad (3.4)$$

$$= \text{Ad}(u^{-1})(iX^\ell) \quad (3.5)$$

where (3.4) follows from (3.3) after substituting for  $u^{-\theta} \ell^\sigma$  and noting the fact that  $h \in H$  acts trivially on  $iX^\theta$ . The assertion follows.

In terms of the identifications of  $T(U/K)$  and  $T^*(U/K)$  with  $U \times_K i\mathfrak{p}$  the map  $\pi^\natural$  is given by

$$[u, Y] \mapsto [u, \{(\mathcal{H}(Y^u))^{u^{-1}}\}_{i\mathfrak{p}}] \quad (3.6)$$

where  $\mathcal{H}(Y) = -i(Y)_- + i(Y)_+$  is the transformation from (2.1). Using assertion 1 and the fact that  $\text{Ad}(u)$  commutes with the projection to  $\mathfrak{u}$  one can see that  $\pi^\natural(d\mu_X) = [u, \{\text{Ad}(u^{-1}) \circ \mathcal{H}(\{iX^\ell\}_{\mathfrak{u}})\}_{i\mathfrak{p}}]$ .

**Assertion 2:**  $\mathcal{H}(\{iX^\ell\}_{\mathfrak{u}}) = \frac{1}{2} \{X^\ell\}_{\mathfrak{u}} - X$ .

The key point is that  $iX^\ell = (iX^\ell)_- + iX$  since  $\ell \in N^-$  and  $X \in H$ . Hence,

$$2\{iX^\ell\}_{\mathfrak{u}} = ((iX^\ell)_- - ((iX^\ell)_-)^*) = ((\ell iX\ell^{-1})_- - (\ell^{-*} iX\ell^*)_+).$$

Applying  $\mathcal{H}$  to this expression gives

$$\begin{aligned} 2\mathcal{H}(\{iX^\ell\}_{\mathfrak{u}}) &= (\ell X\ell^{-1})_- + (\ell^{-*} X\ell^*)_+ \\ &= \ell X\ell^{-1} - X + \ell^{-*} X\ell^* - X \\ &= \{X^\ell\}_{\mathfrak{u}} - 2X. \end{aligned}$$

This proves assertion 2.

To finish the calculation of  $\pi^{\natural}(d\mu_X)$ , observe that  $\text{Ad}(u^{-1})$  preserves  $\mathfrak{u}$ , and use assertion 2 to conclude that the right hand side of (3.6) is equal to

$$[u, \frac{1}{2}\{\text{Ad}(u^{-1})(X^\ell)\}_{i\mathfrak{p}} - \{X^{u^{-1}}\}_{i\mathfrak{p}}].$$

The same calculation in (3.3) through (3.5) shows that  $(\text{Ad}(u^{-1})(X^\ell))^{*\theta} = -\text{Ad}(u^{-1})(X^\ell)$ . Combining this observation with the formula in (3.2) yields that the  $i\mathfrak{p}$  part of  $\text{Ad}(u^{-1})(X^\ell)$  is zero. Thus

$$\pi^{\natural}(d\mu_X) = [u, -\{X^{u^{-1}}\}_{i\mathfrak{p}}],$$

which completes the proof of the theorem. □

## CHAPTER 4

### FURTHER COMMENTS

This section addresses several special cases and sets the stage for the explicit examples in chapter 5.

#### 4.1 The group case

Let  $K$  be a connected and simply connected compact Lie group. Fix a triangular decomposition

$$\mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+. \quad (4.1)$$

Write  $\mathcal{H}_{\mathfrak{k}}$  for the linear transformation in (2.1) relative to the decomposition (4.1). Trivialize the tangent bundle to  $K$  using right translation. This identifies  $TK$  with  $K \times \mathfrak{k}$ . Further identify  $\mathfrak{k}$  with its dual using the Killing form so that  $T^*K$  is also identified with  $K \times \mathfrak{k}$ . Using the definition of  $\pi_K$  from (2.2), a short calculation yields the following formula.

**Proposition 4.1.** *The Lu-Weinstein Poisson Lie group structure  $\pi_K$  can be expressed by*

$$\pi_K((k, P), (k, Q)) = \langle (\mathcal{H}_{\mathfrak{k}} - \text{Ad}(k) \circ \mathcal{H}_{\mathfrak{k}} \circ \text{Ad}(k^{-1}))(P), Q \rangle. \quad (4.2)$$

for each  $(k, P), (k, Q) \in K \times \mathfrak{k}$ .

With this formula, one can see that the maximal torus  $\mathfrak{t} = \tilde{\mathfrak{h}} \cap \mathfrak{k}$  is a Poisson Lie subgroup of  $K$ , as  $\pi_K$  vanishes identically there. Thus, the push-forward of  $\pi_K$  under the natural projection map  $K \rightarrow K/T$  defines a  $(K, \pi_K)$ -homogeneous Poisson structure on the flag manifold  $K/T$ . The symplectic leaves of this induced structure

are precisely the corresponding Bruhat cells in  $K/T$  because the symplectic leaves of  $\pi_K$  foliate the components of the Bruhat decomposition of  $K$ . This was shown in [LW90].

When equipped with the invariant metric induced by the Killing form,  $K$  is also a Riemannian symmetric space. The isometries of  $K$ , connected to the identity transformation, are given by either left or right translation by elements of  $K$ . In this case,  $U = K \times K$ ,  $G = K^{\mathbb{C}} \times K^{\mathbb{C}}$ , and the Cartan involution selecting  $U$  in  $G$  is  $(g_1, g_2) \mapsto (g_1^{-*}, g_2^{-*})$  where  $g \mapsto g^{-*}$  is the Cartan involution selecting  $K$  inside of  $K^{\mathbb{C}}$ . The left action of  $u = (k_1, k_2)$  on  $k \in K$  is given by  $(k_1, k_2) \cdot k = k_1 k k_2^{-1}$ . The decomposition

$$\mathfrak{g} = \underbrace{(\tilde{\mathfrak{n}}_- \times \tilde{\mathfrak{n}}_-)}_{\mathfrak{n}_-} + \underbrace{(\tilde{\mathfrak{h}} \times \tilde{\mathfrak{h}})}_{\mathfrak{h}} + \underbrace{(\tilde{\mathfrak{n}}_+ \times \tilde{\mathfrak{n}}_+)}_{\mathfrak{n}_+} \quad (4.3)$$

built using (4.1) is a  $\mathfrak{u}$ -compatible triangular decomposition of  $\mathfrak{g}$ .

Using the identity in  $K$  as a basepoint gives a presentation of  $K$  as the coset space  $U/\Delta$  where the stability subgroup  $\Delta = \{(k, k) : k \in K\}$  is the diagonal image of  $K$  in  $U$ . The involution  $\theta$  fixing  $\Delta$  in  $U$  is the automorphism which interchanges the two factors of  $U$ . The triangular decomposition in (4.3) is stable with respect to this outer automorphism.

The image of the Cartan embedding of  $U/\Delta$  into  $U$  is the anti-diagonal image of  $K$  in  $U$ , namely  $\phi(U/\Delta) = \{(k, k^{-1}) : k \in K\} \subset U$ . The corresponding Birkhoff decomposition of  $K$  as a symmetric space is identical to the decomposition of  $K$  induced by the Birkhoff decomposition of  $K^{\mathbb{C}}$  with respect to  $\mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{n}}_- + \tilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$ .

**Theorem 4.2.** *The Evens-Lu Poisson structure  $\pi$  on  $K$  can be expressed as*

$$\pi((k, P), (k, Q)) = \langle (\mathcal{H}_{\mathfrak{k}} + \text{Ad}(k) \circ \mathcal{H}_{\mathfrak{k}} \circ \text{Ad}(k^{-1}))(P), Q \rangle. \quad (4.4)$$

*Proof.* The isometry  $\psi: U/\Delta \rightarrow K$  given by  $(k_1, k_2)\Delta \mapsto k_1 k_2^{-1}$  identifies  $U/\Delta$  with  $K$ . Use right translation to identify the tangent bundle to  $U/\Delta$  with  $U \times_{\Delta} \mathfrak{ip}$  and  $TK$  with  $K \times \mathfrak{k}$ . Given  $(k_1, k_2) \in U$  write  $k = \psi(k_1, k_2) = k_1 k_2^{-1}$ . In this setting  $\mathfrak{ip} =$

$\{(X, -X) : X \in \mathfrak{k}\}$ . A curve representing  $(X, -X)$  in  $i\mathfrak{p}$  passing through  $(k_1, k_2)\Delta$  is given by  $t \mapsto (k_1 e^{tX}, k_2 e^{-tX})\Delta$  and its image under  $\psi$  is  $t \mapsto k_1 e^{tX} e^{tX} k_2^{-1} = e^{2tX k_1} k$ . Thus,  $\psi_*[(k_1, k_2), (X, -X)]$  is represented by  $(k, 2X^{k_1}) \in K \times \mathfrak{k}$ . Using the Killing form to identify the dual of  $\mathfrak{k}$  with  $\mathfrak{k}$ , the cotangent bundle can also be represented by  $K \times \mathfrak{k}$  using right translation. Applying the transpose of the map  $\psi_*$  to a cotangent vector at  $k$ , represented by  $(k, P)$ , gives the class  $[(k_1, k_2), (P^{k_1^{-1}}, -P^{k_1^{-1}})]$ . Let  $\pi_{EL}$  denote the Evens-Lu Poisson structure  $U/\Delta$ . From theorem 2.3 it follows that

$$(\psi_* \pi_{EL})((k, P), (k, Q)) = \langle \Omega_{(k_1, k_2)}(P^{k_1^{-1}}, -P^{k_1^{-1}}), (Q^{k_1^{-1}}, -Q^{k_1^{-1}}) \rangle_{\mathfrak{u}} \quad (4.5)$$

where  $\Omega_{(k_1, k_2)}(X, -X) = \{\text{Ad}(k_1^{-1}, k_2^{-1}) \circ \mathcal{H}_{\mathfrak{k} \times \mathfrak{k}} \circ \text{Ad}(k_1, k_2)(X, -X)\}_{i\mathfrak{p}}$ . The result of applying the transformation  $\Omega_{(k_1, k_2)}$  to  $(P^{k_1^{-1}}, -P^{k_1^{-1}})$  is equivalent to the  $i\mathfrak{p}$  part of

$$\text{Ad}(k_1^{-1}, k_2^{-1}) \circ \mathcal{H}_{\mathfrak{k} \times \mathfrak{k}}(P, -P^{k_1^{-1}}) = ((\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}}, -(\mathcal{H}_{\mathfrak{k}}(P^{k_1^{-1}}))^{k_2^{-1}}). \quad (4.6)$$

Note that the left and right hand side of (4.6) are in  $\mathfrak{u} = \mathfrak{k} \times \mathfrak{k}$ . Temporarily denote the right hand side of (4.6) by  $Z$ . Since  $Z \in \mathfrak{u}$ , the projection to  $i\mathfrak{p}$  is given by  $\frac{1}{2}(Z - Z^\theta)$  or

$$\frac{1}{2} \left( (\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k_1^{-1}}))^{k_2^{-1}}, -((\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k_1^{-1}}))^{k_2^{-1}}) \right). \quad (4.7)$$

Substituting (4.7) into the right hand side of (4.5) yields the expression

$$\langle (\mathcal{H}_{\mathfrak{k}}(P))^{k_1^{-1}} + (\mathcal{H}_{\mathfrak{k}}(P^{k_1^{-1}}))^{k_2^{-1}}, Q^{k_1^{-1}} \rangle$$

from which it follows that

$$(\psi_* \pi_{EL})((k, P), (k, Q)) = \langle (\mathcal{H}_{\mathfrak{k}} + \text{Ad}(k) \circ \mathcal{H}_{\mathfrak{k}} \circ \text{Ad}(k^{-1}))(P), Q \rangle.$$

completing the proof of the theorem.  $\square$

For a compact group  $K$  there are essentially two Poisson structures intimately related to the Lie theory of  $K$ . On the one hand there is the Lu-Weinstein Poisson

Lie group structure whose symplectic foliation respects the Bruhat decomposition of  $K$ . This is given by the difference of the right and left invariant bivector fields generated by  $\mathcal{H}_{\mathbb{R}}$ . On the other hand there is the Evens-Lu homogeneous Poisson structure on  $K$  whose symplectic foliation respects the Birkhoff decomposition of  $K$ . This is given by the sum of the left and right invariant bivector fields generated by  $\mathcal{H}_{\mathbb{R}}$ .

#### 4.2 When $\theta$ is an inner automorphism

Returning to the general case where  $X$  is not necessarily a group, note that all stability subgroups of  $U$  corresponding to points in  $X$  are conjugate in  $U$ . Thus, the class of  $\theta$  in the outer automorphism group of  $U$  is an invariant of  $X$ . When this class is trivial, the Evens-Lu Poisson structure is non-degenerate on an open dense subset of points of  $X$ , as will be shown below. This stems from the fact that when  $\theta$  is an inner automorphism, each Cartan subalgebra of  $\mathfrak{k}$  is, in fact, a Cartan subalgebra of  $\mathfrak{u}$ . Thus  $\mathfrak{h} \cap \mathfrak{u} = \mathfrak{t}$  and  $\mathfrak{h} \cap \mathfrak{k} = \mathfrak{t}_0$  are equal. Write  $T_0$  for  $\exp(\mathfrak{t}_0)$ . Many statements simplify dramatically in the inner case. For example, the torus  $T_w = \exp(\ker\{\text{Ad}(w)\theta|_t - 1\})$  which acts on the layer of the Birkhoff decomposition of  $X$  corresponding to  $w$  admits a much simpler description in the inner case as  $T_0 \cap (T_0)^w$ .

**Theorem 4.3.** *In symmetric spaces for which  $\theta$  is an inner automorphism, each connected component of the layer of the Birkhoff decomposition of  $X$  corresponding to the trivial element of the Weyl group is an open symplectic leaf. The components are indexed by the elements of order two in  $T_0$ .*

*Proof.* From corollary 3.3: the symplectic leaves of maximal dimension foliate the connected components of the layer corresponding to  $w = T_0 \in W = N_U(T_0)/T_0$ . For brevity, this layer will be referred to as the *top layer*. The leaves are indexed by the elements  $\mathbf{w} \in \phi(U/K) \cap T_0$ . Such an element satisfies the equation  $\mathbf{w}^{-1} = \mathbf{w}^\theta$ , but

$T_0$  is fixed by  $\theta$ , so  $\mathbf{w}$  must be an element of order 2. Theorem 3 in [Pic06] shows that for each such  $\mathbf{w}$  there exists an element  $\mathbf{w}_1 \in N_U(T_0)$  such that  $\phi(\mathbf{w}_1 K) = \mathbf{w}$ .

In theorem 2.3 it was shown that with the presentation of  $T^*(U/K) \simeq U \times_K i\mathfrak{p}$  the Evens-Lu Poisson bivector can be expressed relative to the Killing form as

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y \rangle$$

where  $X, Y \in i\mathfrak{p}$ , and  $\Omega_u: i\mathfrak{p} \rightarrow i\mathfrak{p}$  is given by

$$\Omega_u(X) = \{\text{Ad}(u^{-1}) \circ \mathcal{H} \circ \text{Ad}(u)(X)\}_{i\mathfrak{p}}.$$

The kernel of  $\Omega_{\mathbf{w}_1}(X)$  is thus equal to  $\mathfrak{t}_0^{\mathbf{w}_1^{-1}} \cap i\mathfrak{p} = \mathfrak{t}_0 \cap i\mathfrak{p} = 0$ . Hence, the leaves in  $X$  whose  $\phi$ -images lie in the top layer of the Birkhoff decomposition are open.  $\square$

The elements of order two in  $T_0$  are precisely the  $\phi$ -images of the preferred basepoints in  $X$  whose existence was established in lemma 3.1.

From the classification of symmetric spaces in [Hel78], the list of irreducible compact symmetric spaces for which the involution is an inner automorphism includes, but is not limited to, the compact Hermitian symmetric spaces. Complex variables will be used in the following chapter to exhibit locally the Evens-Lu Poisson structure on some spaces of this type.

## CHAPTER 5

### EXAMPLES

In this chapter, local expressions are computed for the Evens-Lu Poisson structure in a number of explicit examples. The first example is the complex Grassmannian, i.e., the space of  $m$ -planes in  $\mathbb{C}^{m+n}$ . As a symmetric space, this may be presented as the quotient of the compact group  $U = \mathrm{SU}(m+n)$  by the closed subgroup  $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ . This presentation arises from the natural action of  $\mathrm{SU}(m+n)$  on  $\mathbb{C}^{m+n}$  by linear isometries of the standard Hermitian inner product, which further descends to a transitive action on the set of complex  $m$ -planes through the origin.

The complexification of  $\mathfrak{u} = \mathfrak{su}(m+n)$  is  $\mathfrak{g} = \mathfrak{sl}(m+n, \mathbb{C})$ . Let  $\mathfrak{h}$  be the diagonal matrices in  $\mathfrak{sl}(m+n, \mathbb{C})$  and the triangular decomposition be the usual one where  $\mathfrak{n}_+$  consists of the strictly upper triangular matrices and  $\mathfrak{n}_-$  the strictly lower triangular matrices. This  $\mathfrak{u}$ -compatible triangular decomposition generates a standard Poisson Lie group structure  $\pi_U$  on  $U$ . The corresponding Birkhoff decomposition of

$$G = \mathrm{SL}(m+n, \mathbb{C})$$

corresponds to the factorization produced in linear algebra through Gaussian elimination.

Choose a basis for  $\mathbb{C}^{m+n}$ . The point in the Grassmannian corresponding to the plane spanned by the first  $m$  basis vectors is an example of a preferred basepoint in the sense of lemma 3.1. Denote this plane by  $\mathbb{C}^m$ . The stability subgroup of this basepoint consists of the special unitary transformations preserving this plane (and by necessity its Hermitian orthogonal complement), i.e., special unitary matrices of

the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where  $A$  (resp.  $D$ ) is an  $m \times m$  (resp.  $n \times n$ ) unitary matrix. This is the subgroup  $S(U(m) \times U(n))$ . From this presentation one can readily see that the involution,  $\theta$ , fixing this subgroup, is the automorphism which negates the off-diagonal blocks. This involution is an inner automorphism, given by conjugation by a scalar multiple of the block diagonal matrix with an  $m \times m$  identity matrix in the upper left block and an  $n \times n$  diagonal matrix with negative ones on the diagonal in the lower right block. When  $n$  is even, the scalar multiple is one. When  $n$  is odd, the multiple is a primitive  $2(m+n)$ -th root of unity.

The complex Grassmannian is, additionally, a Hermitian symmetric space. It is diffeomorphic to the quotient of  $SL(m+n, \mathbb{C})$  by the parabolic subgroup of the upper block triangular matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

It is through an identification such as this that the Grassmannian inherits a complex structure. Holomorphic coordinates can thus be used to present local formulas.

## 5.1 Graph Coordinates

The graph of a  $\mathbb{C}$ -linear transformation  $Z \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$  in  $\mathbb{C}^{m+n}$ , written

$$\{(X, ZX) : X \in \mathbb{C}^m\},$$

uniquely determines a point in the Grassmannian. In fact, every complex  $m$ -dimensional subspace of  $\mathbb{C}^{m+n}$  which is transverse to  $(\mathbb{C}^m)^\perp = \mathbb{C}^n$  can be realized in this way. In this fashion  $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ , provides an affine coordinate chart for the Grassmannian with each point in a Zariski open subset described by an  $n \times m$  matrix  $Z$  of complex numbers.

Each coset of  $S(U(m) \times U(n))$  corresponding to the graph of a linear transformation contains a unique element  $u$  which has positive definite diagonal blocks. This can be seen as follows. Apply polar decomposition to the diagonal blocks  $A$  and  $D$  of a special unitary matrix  $u$ , writing  $A = |A|P_A$  and  $D = |D|P_D$  and then factor  $u$  to find

$$u = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} |A| & BP_D^* \\ CP_A^* & |D| \end{pmatrix} \begin{pmatrix} P_A & 0 \\ 0 & P_D \end{pmatrix}. \quad (5.1)$$

The diagonal blocks of a special unitary matrix corresponding to the graph of a linear transformation are invertible, so their polar decomposition is unique. Furthermore, the diagonal blocks of a special unitary matrix have conjugate determinants (see proposition 1.3 of [For92]). Thus,  $P_A$  and  $P_D$  have conjugate determinants, and the factorization in (5.1) produces a unique representative for  $u$  modulo  $S(U(m) \times U(n))$  with the desired properties.

**Proposition 5.1.** *As a function of  $Z$ , this preferred coset representative is given by*

$$u(Z) = \begin{pmatrix} (1 + Z^*Z)^{-1/2} & -(1 + Z^*Z)^{-1/2}Z^* \\ Z(1 + Z^*Z)^{-1/2} & (1 + ZZ^*)^{-1/2} \end{pmatrix}. \quad (5.2)$$

*Proof.* The conditions for unitarity of  $u$ , i.e.,  $uu^* = 1$  and  $u^*u = 1$  impose eight conditions on the blocks  $A$ ,  $B$ ,  $C$ , and  $D$ , two of which are redundant. After removing these, the remaining conditions are

$$AA^* + BB^* = 1 \quad (5.3)$$

$$AC^* + BD^* = 0 \quad (5.4)$$

$$CC^* + DD^* = 1 \quad (5.5)$$

$$A^*A + C^*C = 1$$

$$A^*B + C^*D = 0 \quad (5.6)$$

$$B^*B + D^*D = 1$$

giving six equations in four non-commuting variables, of which only  $A$  and  $D$  are invertible. Since it is assumed that  $u$  carries  $\mathbb{C}^m$  to the graph of  $Z$ , the relation that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X' \\ ZX' \end{pmatrix}$$

implies that  $AX = X'$  and  $CX = ZX'$ . Thus  $CA^{-1} = Z$ . The conjugate transpose of equation (5.6) can then be used to show that

$$B^*A + D^*C = 0 \Rightarrow CA^{-1} = -(BD^{-1})^*.$$

Whence,  $Z^* = -BD^{-1}$ . From equation (5.5), one derives that

$$DD^* = 1 - CC^* = 1 - CA^{-1}AC^* = 1 - ZAC^*.$$

Then, equation (5.4) can be used to show that

$$DD^* = 1 - ZAC^* = 1 - Z(-BD^*) = 1 - Z(-BD^{-1})DD^* = 1 - ZZ^*DD^*$$

from which it follows that  $DD^* = (1 + ZZ^*)^{-1}$ . Taking  $D$  to be positive definite allows one to write  $D = (1 + ZZ^*)^{-1/2}$ . Similarly, using equation (5.3) and the conjugate transpose of equation (5.4), one is able to show that  $AA^*(1 + Z^*Z) = 1$ . Taking  $A$  to be positive definite,  $A = (1 + Z^*Z)^{-1/2}$ .

To complete the proof of the proposition, one uses the equations  $Z = CA^{-1}$  and  $Z^* = -BD^{-1}$  to find formulas for the blocks  $B$  and  $C$ , obtaining  $C = ZA = Z(1 + Z^*Z)^{-1/2}$  and  $B = -Z^*D = -Z^*(1 + ZZ^*)^{-1/2}$ . The final form for  $B$  is obtained by noting that  $(1 + Z^*Z)$  and  $(1 + ZZ^*)$  have the same spectrum, and that  $Z$  and  $Z^*$  intertwine these two operators. Thus,  $B = -Z^*(1 + ZZ^*)^{-1/2} = -(1 + Z^*Z)^{-1/2}Z^*$ .  $\square$

The formula for  $u(Z)$  in (5.2) will be referred to as the *canonical representative* for the coset corresponding to  $uK$  depending on  $Z$ .

Cotangent vectors at  $Z$  will be represented by  $m \times n$  complex matrices using the identification of the real cotangent space to the Grassmannian with the holomorphic cotangent space, which is further identified with  $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$  at each point.

The identification that will be used here fits with the presentation of the tangent bundle to  $U/K$  as  $U \times_K \mathfrak{ip}$  using right translation used in previous chapters. If  $X$  denotes an element of  $\mathfrak{ip}$  representing a tangent vector at the basepoint, then the map  $t \mapsto ue^{tX}K$  represents a curve passing through the point  $uK$  at time  $t = 0$ . The identification of this model of the tangent space with  $n \times m$  complex matrices is given by

$$X = \begin{pmatrix} 0 & -P^* \\ P & 0 \end{pmatrix}$$

where  $P$  is a such a complex matrix. A short computation shows that

$$ue^{tX} = \begin{pmatrix} A + tBP & -tAP^* + B \\ C + tDP & -tCP^* + D \end{pmatrix} + O(t^2).$$

Thus, the map  $t \mapsto (C + tDP)(A + tBP)^{-1}$  represents the curve  $ue^{tX}K$  to first order in  $t$  in the  $Z$  coordinate, where  $A$ ,  $B$ ,  $C$ , and  $D$  are as in the proof of proposition 5.1. The derivative at zero gives a formula for the push-forward under translation by  $u$

$$\left. \frac{d}{dt} \right|_{t=0} (C + tDP)(A + tBP)^{-1} = (D - CA^{-1}B)PA^{-1}. \quad (5.7)$$

Formula (5.7) can be reduced to a much more convenient form by using the relations exposed in the proof of proposition 5.1. Specifically,

$$D - CA^{-1}B = (1 + CA^{-1}(-BD^{-1}))D = (1 + ZZ^*)D = D^{-2}D = D^{-1}.$$

Substituting the result of this calculation into the result of (5.7) yields that the push-forward of the tangent vector  $P$  at the basepoint 0 to a tangent vector at  $Z$  is given by the linear map

$$P \mapsto D^{-1}PA^{-1}. \quad (5.8)$$

What will be more important in what follows is the formula for the induced map of a cotangent vector at  $Z$  to a cotangent vector at 0. The transpose of the linear transformation in (5.8) is given by

$$V \mapsto A^{-1}VD^{-1} \quad (5.9)$$

where  $V$  is an  $m \times n$  complex matrix representing a cotangent vector at  $Z$ .

## 5.2 The Complex Grassmannian

**Theorem 5.2.** *With respect to the coordinates constructed in the previous section, the action of the Evens–Lu bivector  $\pi$  on two cotangent vectors represented by complex  $m \times n$  matrices  $V$  and  $W$  may be computed by*

$$\pi(V, W) = i[\operatorname{tr}(L_Z V)^* W) - \operatorname{tr}((L_Z V)W^*)] \quad (5.10)$$

where  $L_Z$  is the  $\mathbb{R}$ -linear transformation  $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^n) \rightarrow \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$  given by

$$\begin{aligned} L_Z V &= V - Z^* Z V Z Z^* \\ &\quad + Z^*((ZV - V^* Z^*)_+ + c.t.) \\ &\quad - ((Z^* V^* - VZ)_+ + c.t.)Z^*. \end{aligned} \quad (5.11)$$

In the above expression,  $(\cdot)_+$  denotes the upper triangular part as before,  $V^*$  denotes the conjugate transpose of the matrix  $V$ , and *c.t.* denotes the conjugate transpose of the preceding term.

*Proof.* This local formula is obtained by direct calculation from the equivariant formula for  $\pi$  in theorem 2.3. Recall that the Killing form was used to identify the dual of  $\mathfrak{ip}$  with  $\mathfrak{ip}$  itself. The cotangent bundle of  $U/K$  was presented as  $U \times_K \mathfrak{ip}$  and, with respect to this presentation, the Evens–Lu bivector was given by

$$\pi([u, X], [u, Y]) = \langle \Omega_u X, Y \rangle$$

where  $X, Y \in i\mathfrak{p}$  and  $\Omega_u: i\mathfrak{p} \rightarrow i\mathfrak{p}$

$$\Omega_u(X) = \{\text{Ad}(u^{-1}) \circ \mathcal{H} \circ \text{Ad}(u)(X)\}_{i\mathfrak{p}}$$

and  $\{\cdot\}_{i\mathfrak{p}}$  denotes the projection to  $i\mathfrak{p}$  along the decomposition  $\mathfrak{g} = i\mathfrak{u} + \mathfrak{k} + i\mathfrak{p}$ .

Using the identifications discussed in the previous section and, in particular formula (5.9), we see that the cotangent vector at  $Z$  represented by an  $m \times n$  complex matrix  $V$  corresponds to the element of  $[u(Z), X] \in U \times_K i\mathfrak{p}$  with

$$X = \begin{pmatrix} 0 & A^{-1}VD^{-1} \\ -D^{-1}V^*A^{-1} & 0 \end{pmatrix}. \quad (5.12)$$

Similarly, the same formula as in (5.12) with  $X$  replaced by  $Y$  and  $V$  replaced by  $W$  will be used to represent the class  $[u(Z), Y]$  corresponding to the cotangent vector  $W$  at  $Z$ .

To proceed with the calculation of the action of  $\pi$  in a somewhat efficient manner, we will record the result that

$$\text{Ad}(u)(X) = \begin{pmatrix} VD^{-1}B^* - BD^{-1}V^* & V - BD^{-1}V^*A^{-1}C^* \\ CA^{-1}VD^{-1}B^* - V^* & CA^{-1}V - V^*A^{-1}C^* \end{pmatrix}$$

which can be reduced to

$$\text{Ad}(u)(X) = \begin{pmatrix} -VZ + Z^*V^* & V + Z^*V^*Z^* \\ ZVZ - V^* & ZV - V^*Z^* \end{pmatrix} \quad (5.13)$$

using the relations  $Z = CA^{-1}$  and  $Z^* = -BD^{-1}$  from proposition 5.1.

Let  $M = -VZ + Z^*V^*$ , the upper left block of (5.13), and  $N = ZV - V^*Z^*$ , the lower right block. Then, recalling the definition of  $\mathcal{H}$  from (2.1), one may write

$$\mathcal{H} \circ \text{Ad}(u)(X) = \begin{pmatrix} -iM_- + iM_+ & i(V + Z^*V^*Z^*) \\ -i(-ZVZ - V^*) & -iN_- + iN_+ \end{pmatrix}.$$

Rather than determining the result of  $\text{Ad}(u^{-1}) \circ \mathcal{H} \circ \text{Ad}(u)(X)$  completely, note that what is most relevant is the resulting projection to  $\mathfrak{ip}$  which is determined by the upper right block of this matrix. This block is given by

$$\begin{aligned} P &= A(-iM_- + iM_+)B + iC^*(V^* + ZVZ)B \\ &\quad + iA(V + Z^*V^*Z^*)D + C^*(-iN_- + iN_+)D. \end{aligned} \quad (5.14)$$

Since the matrices under consideration are elements of  $\mathfrak{sl}(n, \mathbb{C})$ , the Killing form applied to  $\Omega_u(X)$  and  $Y$  is a constant multiple of the trace of the product of  $\Omega_u(X)$  and  $Y$ . Suppressing this normalization, one can write

$$\begin{aligned} \langle \Omega_u(X), Y \rangle &= \text{tr} \left( \begin{pmatrix} 0 & P \\ -P^* & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{-1}WD^{-1} \\ -D^{-1}W^*A^{-1} & 0 \end{pmatrix} \right) \\ &= -\text{tr}(PD^{-1}W^*A^{-1}) - \text{tr}(P^*A^{-1}WD^{-1}) \\ &= -\text{tr}(A^{-1}PD^{-1}W^*) - \text{tr}((A^{-1}PD^{-1})^*W). \end{aligned} \quad (5.15)$$

It remains then to compute the quantity  $A^{-1}PD^{-1}$ . It is from (5.14) that one obtains equation (5.16).

$$A^{-1}PD^{-1} = (iM_- + iM_+)BD^{-1} + iA^{-1}C^*(V^* + ZVZ)BD^{-1} \quad (5.16)$$

$$+ i(V + Z^*V^*Z^*) + A^{-1}C^*(-iN_- + iN_+) \quad (5.17)$$

$$\begin{aligned} &= i[(-M_- + M_+)(-Z^*) + Z^*(V^* + ZVZ)(-Z^*) \\ &\quad (V + Z^*V^*Z^*) + Z^*(-N_- + N_+)] \\ &= i[V - Z^*ZVZZ^* + Z^*(-N_- + N_+) - (-M_- + M_+)Z^*] \end{aligned} \quad (5.18)$$

The right hand side of (5.18) shows that  $A^{-1}PD^{-1}$  is of the form  $iL_ZV$  where  $L_Z$  is a  $\mathbb{R}$ -linear automorphism of the vector space of  $m \times n$  complex matrices. Substituting this result into (5.15) establishes that

$$\pi(V, W) = i(\text{tr}((L_ZV)^*W) - \text{tr}((L_ZV)W^*))$$

as in the statement of the theorem. To complete the proof, one need only observe that  $M = Z^*V^* - VZ$  and  $N = ZV - V^*Z^*$  are skew-adjoint. Thus

$$\begin{aligned} -N_- + N_+ &= ((ZV - V^*Z^*)_+ + c.t.) \\ \text{and } -M_- + M_+ &= ((Z^*V^* - VZ)_+ + c.t.) \end{aligned}$$

which corresponds with the definition of  $L_Z$  given in the statement of the theorem.  $\square$

### 5.3 Complex Projective Space

Complex projective space of dimension  $n$ , denoted  $\mathbb{C}P^n$ , is the space of complex lines through the origin in  $\mathbb{C}^{n+1}$  and is the Grassmannian with  $m = 1$ . In this case, the coordinate  $Z$  is a column vector and the matrices representing cotangent vectors are row vectors. The quantity  $Z^*Z$  is a scalar which, for brevity, we write as

$$Z^*Z = \|Z\|^2 = |z_1|^2 + \cdots + |z_n|^2.$$

From the formula in theorem 5.2 one can derive a more familiar coordinate expression in this case.

**Corollary 5.3** (to Theorem 5.2). *In terms of the holomorphic coordinates  $z_1, \dots, z_n$  the Evens-Lu bivector on  $\mathbb{C}P^n$  is given by*

$$\begin{aligned} \pi &= -i \left\{ \sum_{k=1}^n S_k \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial \bar{z}_k} \right. \\ &\quad \left. + \left( \sum_{k=1}^n \sum_{j=1}^{k-1} z_j z_k \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k} - \sum_{k=1}^n \sum_{j=1}^{k-1} z_j \bar{z}_k \|Z\|^2 \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_k} \right) - c.c. \right\} \end{aligned} \quad (5.19)$$

where

$$S_j = 1 + \sum_{j=1}^{k-1} |z_j|^2 - |z_k|^2 \|Z\|^2 - \sum_{j=k+1}^n |z_j|^2. \quad (5.20)$$

*Proof.* Write  $V = (v_1, \dots, v_n)$  and  $W = (w_1, \dots, w_n)$ . The final term of  $L_Z V$  in (5.11) vanishes as the matrix  $Z^* V^* - V Z$  is one-by-one and thus has no upper triangular part. Translating (5.11) into components, one finds that

$$(L_Z V)_k = v_k - \|Z\|^2 \left( \sum_{j=1}^n v_j z_j \bar{z}_k \right) + \sum_{j=1}^n \bar{z}_j A_{jk} \quad (5.21)$$

where the matrix  $A$  is shorthand for  $((ZV - V^* Z^*)_+ + c.t.)$ . This matrix has components

$$A_{jk} = \begin{cases} \bar{z}_k \bar{v}_j - v_k z_j & j > k \\ 0 & j = k \\ z_j v_k - \bar{v}_j \bar{z}_k & j < k \end{cases} .$$

With this, one then computes that

$$\sum_{j=1}^n \bar{z}_j A_{jk} = \sum_{j=k+1}^n (\bar{z}_j \bar{z}_k \bar{v}_j - |z_j|^2 v_k) + \sum_{j=1}^{k-1} (|z_j|^2 v_k - \bar{z}_j \bar{z}_k \bar{v}_j) . \quad (5.22)$$

Breaking apart the sum in the second term on the right hand side of (5.21), and substituting in (5.22) one finds the formula

$$\begin{aligned} (L_Z V)_k &= v_k - |z_k|^2 \|Z\|^2 v_k - \|Z\|^2 \sum_{j=1}^{k-1} z_j \bar{z}_k v_j - \|Z\|^2 \sum_{j=k+1}^n z_j \bar{z}_k v_j \\ &\quad + \sum_{j=1}^{k-1} |z_j|^2 v_k - \sum_{j=k+1}^n |z_j|^2 v_k + \sum_{j=k+1}^n \bar{z}_j \bar{z}_k \bar{v}_j - \sum_{j=1}^{k-1} \bar{z}_j \bar{z}_k \bar{v}_j \end{aligned}$$

which can be recombined as

$$\begin{aligned} (L_Z V)_k &= \left( 1 + \sum_{j=1}^{k-1} |z_j|^2 - |z_k|^2 \|Z\|^2 - \sum_{j=k+1}^n |z_j|^2 \right) v_k - \sum_{j=1}^{k-1} \bar{z}_j \bar{z}_k \bar{v}_j \\ &\quad + \sum_{j=k+1}^n \bar{z}_j \bar{z}_k \bar{v}_j - \sum_{j=1}^{k-1} z_j \bar{z}_k \|Z\|^2 v_j - \sum_{j=k+1}^n z_j \bar{z}_k \|Z\|^2 v_j \end{aligned}$$

by collecting the coefficients of  $v_k$ .

For brevity, write

$$S_k = 1 + \sum_{j=1}^{k-1} |z_j|^2 - |z_k|^2 \|Z\|^2 - \sum_{j=k+1}^n |z_j|^2$$

as in (5.20) for these coefficients. Note that these are real quantities, and are thus unchanged under complex conjugation. With this short hand,

$$(L_Z V) W^* = \sum_{k=1}^n S_k v_k \bar{w}_k - \sum_{k=1}^n \sum_{j=1}^{k-1} \bar{z}_j \bar{z}_k \bar{v}_j \bar{w}_k + \sum_{k=1}^n \sum_{j=k+1}^n \bar{z}_j \bar{z}_k \bar{v}_j \bar{w}_k \quad (5.23)$$

$$- \sum_{k=1}^n \sum_{j=1}^{k-1} z_j \bar{z}_k \|Z\|^2 v_j \bar{w}_k - \sum_{k=1}^n \sum_{j=k+1}^n z_j \bar{z}_k \|Z\|^2 v_j \bar{w}_k \quad (5.24)$$

and

$$\text{tr}((L_Z V)^* W) = \sum_{k=1}^n S_k \bar{v}_k w_k - \sum_{k=1}^n \sum_{j=1}^{k-1} z_j z_k v_j w_k + \sum_{k=1}^n \sum_{j=k+1}^n z_j z_k v_j w_k \quad (5.25)$$

$$- \sum_{k=1}^n \sum_{j=1}^{k-1} \bar{z}_j z_k \|Z\|^2 \bar{v}_j w_k - \sum_{k=1}^n \sum_{j=k+1}^n \bar{z}_j z_k \|Z\|^2 \bar{v}_j w_k. \quad (5.26)$$

The next step is to substitute these two formulas into (5.10) and then rearrange the terms to express the result as a linear combination of the elements of the coordinate basis of  $\bigwedge^2 T\mathbb{C}P^n$ . Before doing that, it will be convenient to interchange the order of summation in the third sum on the right hand side of (5.23) and, after relabeling the indices, obtain a double sum over terms of with  $j$  less than  $k$ , i.e.,

$$\sum_{k=1}^n \sum_{j=k+1}^n \bar{z}_j \bar{z}_k \bar{v}_j \bar{w}_k = \sum_{k=1}^n \sum_{j=1}^{k-1} \bar{z}_j \bar{z}_k \bar{v}_k \bar{w}_j. \quad (5.27)$$

By performing the same procedure on the second term of (5.24), the third sum on the right hand side of (5.25) and the second term of (5.26), all of the double sums can be converted to double sums where  $j$  is less than  $k$  as in (5.27). With these

notational conversions, it is then possible to organize the result of (5.10) to obtain

$$\begin{aligned}
\pi(V, W) &= -i [(L_Z V)W^* - \text{tr}((L_Z V)W^*)] \\
&= -i \left[ \sum_{k=1}^n S_k (v_k \bar{w}_k - \bar{v}_k w_k) \right. \\
&\quad + \sum_{k=1}^n \sum_{j=1}^{k-1} z_j z_k (v_j w_k - v_k w_j) + \sum_{k=1}^n \sum_{j=1}^{k-1} z_j \bar{z}_k \|Z\|^2 (v_j \bar{w}_k - \bar{v}_k w_j) \\
&\quad \left. - \sum_{k=1}^n \sum_{j=1}^{k-1} \bar{z}_j \bar{z}_k (\bar{v}_j \bar{w}_k - \bar{v}_k \bar{w}_j) - \sum_{k=1}^n \sum_{j=1}^{k-1} \bar{z}_j z_k \|Z\|^2 (\bar{v}_j w_k - v_k \bar{w}_j) \right].
\end{aligned}$$

From this expression for  $\pi(V, W)$ , one need only replace the bilinear forms in the components of  $V$  and  $W$  with their bivector counterparts to obtain the formula (5.19). The dictionary is given by

$$\begin{aligned}
v_k \bar{w}_k - \bar{v}_k w_k &\longleftrightarrow \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial \bar{z}_k} \\
v_j w_k - v_k w_j &\longleftrightarrow \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k} \\
v_j \bar{w}_k - \bar{v}_k w_j &\longleftrightarrow \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_k}
\end{aligned}$$

and the conjugates. This completes the proof of the corollary.  $\square$

#### 5.4 Birkhoff Decomposition of $\mathbb{C}\mathbb{P}^2$

It is interesting to consider  $\mathbb{C}\mathbb{P}^2$  and  $\mathbb{C}\mathbb{P}^1$  in further detail. It appears that the coefficients  $S_j$  are reducible polynomials in the variables  $|z_j|^2$  only in these cases. It is not clear what the significance of this is (if any). For  $\mathbb{C}\mathbb{P}^2$  the bivector can be expressed locally as

$$\begin{aligned}
\pi &= -i \left\{ S_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} + S_2 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial \bar{z}_2} \right. \\
&\quad \left. + \left( z_1 z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} - z_1 \bar{z}_2 \|Z\|^2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_2} \right) - c.c. \right\}
\end{aligned}$$

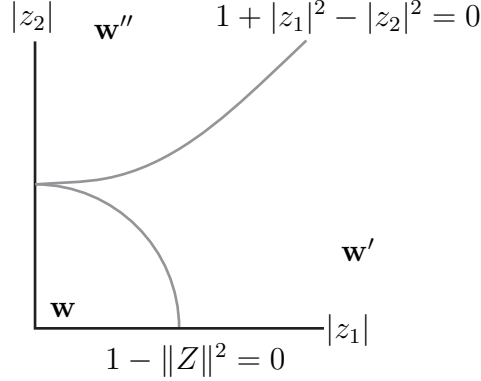


Figure 5.1: The degeneracy locus for  $\mathbb{C}P^2$ . The three elements  $\mathbf{w} = \text{diag}(1, 1, 1)$ ,  $\mathbf{w}' = \text{diag}(-1, -1, 1)$ ,  $\mathbf{w}'' = \text{diag}(-1, 1, -1)$  in  $T_0^{(2)} \cap \phi(U/K)$  are depicted with each component of the top stratum they determine.

where the coefficients  $S_1$  and  $S_2$  factor as  $S_1 = (1 + |z_1|^2)(1 - \|Z\|^2)$  and  $S_2 = (1 - |z_2|^2)(1 + \|Z\|^2)$ . Since the symplectic leaves are open, one can write down the induced symplectic structure in these coordinates by inverting the Poisson tensor. Explicitly,

$$\omega = \frac{i}{p(Z, Z^*)} \left\{ -S_2 dz_1 \wedge d\bar{z}_1 - S_1 dz_2 \wedge d\bar{z}_2 + (z_1 z_2 d\bar{z}_1 \wedge d\bar{z}_2 + z_1 \bar{z}_2 \|Z\|^2 d\bar{z}_1 \wedge dz_2) - c.c \right\}$$

where  $p(Z, Z^*) = (1 + |z_1|^2 - |z_2|^2)(1 - \|Z\|^2)(1 + \|Z\|^2)$ . As one would expect, the symplectic structure is singular along the degeneracy locus of  $\pi$ , i.e., the variety  $p(Z, Z^*) = 0$ .

To observe the connection with triangular factorization, the canonical representatives can be used to compute the complement of the top layer of the Birkhoff decomposition in these coordinates. By introducing the real analytic function  $\varphi = (\sqrt{1 + \|Z\|^2} - 1)/\|Z\|^2$ , one can compute the matrix  $(1 + ZZ^*)^{-1/2}$ . Then,

the canonical representative from (5.2) is given by

$$u = u(z_1, z_2) = \frac{1}{\sqrt{1 + \|Z\|^2}} \begin{pmatrix} 1 & -\bar{z}_1 & -\bar{z}_2 \\ z_1 & 1 + |z_2|^2\varphi & -z_1\bar{z}_2\varphi \\ z_2 & -\bar{z}_1z_2\varphi & 1 + |z_1|^2\varphi \end{pmatrix}$$

and

$$uu^{-\theta} = \frac{1}{1 + \|Z\|^2} \begin{pmatrix} 1 - \|Z\|^2 & -2\bar{z}_1 & -2\bar{z}_2 \\ 2z_1 & 1 - |z_1|^2 + |z_2|^2 & -2z_1\bar{z}_2 \\ 2z_2 & -2\bar{z}_1z_2 & 1 + |z_1|^2 - |z_2|^2 \end{pmatrix}. \quad (5.28)$$

The matrix  $uu^{-\theta}$  in (5.28) is in  $\phi(U/K) \cap \Sigma_1^G$  provided the principal minors are non-vanishing. Thus, the complement of the top layer is given in coordinates by the vanishing locus of the product of the principal minors of (5.28). One can check that, in this case, this product is given by the smooth rational function  $p(Z, Z^*)/(1 + \|Z\|^2)^3$ .

In general, the degeneracy locus for  $\pi$  on the complex Grassmannian is given by the vanishing locus of a reducible polynomial whose factors are given by the explicit formulas of Habermas ([Hab06]).

## 5.5 $SU(2)$ and $\mathbb{CP}^1$

For  $X = \mathbb{CP}^1$  the group  $U$  is  $SU(2)$  and  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ . As is typical with this example, everything can be computed. The Lu-Weinstein Poisson Lie group structure, the Evens-Lu Poisson structure that  $SU(2)$  inherits as a symmetric space, and the Evens-Lu Poisson structure on  $\mathbb{CP}^1$  will all be displayed. The end of this subsection returns to a topic discussed in the introduction, relating the structure produced by the Evens-Lu construction to the one produced by the Foth-Lu construction ([FL04]). To conclude, the Evens-Lu Poisson structure is exhibited as an element of the one parameter family from [KRR93].

Consider the standard triangular decomposition of  $\mathfrak{g}$  as for the Grassmannian:  $\mathfrak{h}$  is the set of traceless diagonal matrices and  $\mathfrak{n}_{\pm}$  are the strictly upper triangular (resp. lower triangular) matrices. Temporarily, set

$$E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{and } E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then  $\mathfrak{h} = \text{span}_{\mathbb{C}}\{H\}$ ,  $\mathfrak{n}_{\pm} = \text{span}_{\mathbb{C}}\{E_{\pm}\}$ , and  $\mathfrak{u} = \text{span}_{\mathbb{R}}\{H, X, Y\}$  where  $X = E_+ - E_-$ , and  $Y = i(E_+ + E_-)$ . The triangular decomposition is stable with respect to the involution selecting the stability subgroup  $S(U(1) \times U(1))$ .

Using right translation, the tangent bundle to  $U$  can be identified with  $U \times \mathfrak{u}$ . Thus a bivector field on  $U$  can be identified with a smooth map  $U \rightarrow \bigwedge^2 \mathfrak{u}$ . In this presentation the value of the Lu-Weinstein Poisson Lie group structure  $\pi_U$  at an element

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(2)$$

is given by

$$\pi_U = (1 - |a|^4 + |b|^4)X \wedge Y + 2\text{Im}(\bar{a}b)Y \wedge H - 2\text{Re}(a\bar{b})H \wedge X \quad (5.29)$$

whereas the Evens-Lu Poisson structure that  $\text{SU}(2)$  inherits as a symmetric space is given by

$$\pi^{\text{EL}} = (1 + |a|^4 - |b|^4)X \wedge Y + 2\text{Im}(ab)Y \wedge H - 2\text{Re}(ab)H \wedge X.$$

These expressions are computed directly from the formulas (4.2) and (4.4) from chapter 4. One can see that the Evens-Lu Poisson structure  $\pi^{\text{EL}}$  vanishes precisely when the principal minor, i.e.,  $a$ , is zero. This shows that the symplectic foliation of  $\pi^{\text{EL}}$  respects the Birkhoff decomposition of  $\text{SU}(2) \subset \text{SL}(2, \mathbb{C})$ .

With the presentation of  $\mathbb{C}\text{P}^1$  as  $\text{SU}(2)/S(U(1) \times U(1))$ , the Evens-Lu construction gives the Poisson structure

$$\pi = -i(1 - |z|^4) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}. \quad (5.30)$$

The top layer of the Birkhoff decomposition has two connected components, the upper and lower hemispheres. The degeneracy locus of  $\pi$  is the equator.

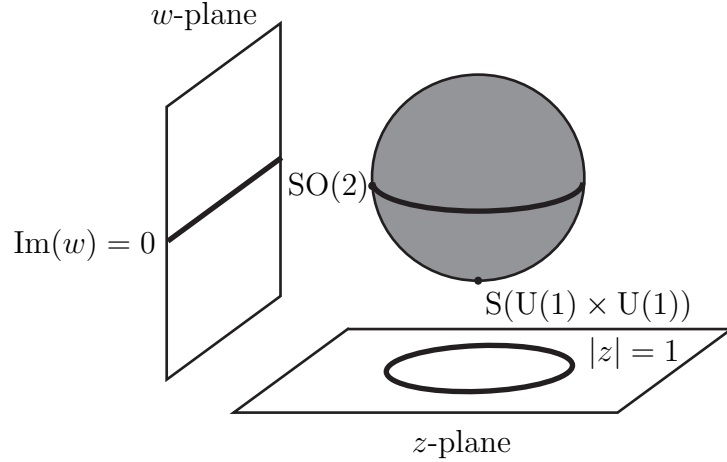


Figure 5.2: The symplectic foliation for the  $(U, \pi_U)$ -homogeneous Poisson structure on the  $\mathbb{C}\mathbb{P}^1$ . The degeneracy locus is pictured in the  $z$ -coordinate when the stability subgroup of the basepoint is  $S(U(1) \times U(1))$  and the  $w$ -coordinate when the stability subgroup is  $SO(2)$ .

Alternatively, Foth and Lu choose the basepoint of this symmetric space to be any of the points along the equator  $z = e^{i\phi}$ . For these basepoints, the triangular decomposition is  $\sigma$ -stable. Recall that  $\sigma$  denotes the anti-holomorphic involution of  $G$  obtained by composing the Cartan involution with  $\theta$ , i.e.,  $g \mapsto g^\sigma = g^{-*\theta}$ .

Let  $\theta^z$  denote the involution corresponding to the basepoint  $z$ . Let  $\mathfrak{k}^z$  denote the subalgebra of  $\mathfrak{su}(2)$  fixed by  $\theta^z$ , and  $\mathfrak{g}_0^z$  denote the corresponding non-compact real form of  $\mathfrak{g}$  fixed by  $g \mapsto g^{\sigma^z} = g^{-*\theta^z}$ . Write  $\mathfrak{a}_0^z$  for the intersection  $\mathfrak{g}_0^z \cap \mathfrak{h}$  and  $\mathfrak{n}_0^z$  for the intersection  $\mathfrak{g}_0^z \cap \mathfrak{n}_-$ .

The involutions corresponding to the basepoints  $z = e^{i\phi}$  are

$$\theta^{e^{i\phi}} = \text{conjugation by } \begin{pmatrix} 0 & ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}.$$

The corresponding non-compact real forms of  $\mathfrak{g}$ ,

$$\mathfrak{g}_0^{z=e^{i\phi}} = \left\{ \begin{pmatrix} a & ibe^{-i\phi} \\ ice^{i\phi} & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\},$$

are precisely those for which  $\mathfrak{g}_0^z = \mathfrak{n}_0^z + \mathfrak{a}_0^z + \mathfrak{k}^z$  is an Iwasawa decomposition for  $\mathfrak{g}_0^z$ , i.e., for which the Borel subalgebra  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}_+$  is Iwasawa relative to  $\mathfrak{g}_0^z$ .

With these choices of basepoint the quotient map  $U \rightarrow U/K$  projects  $\pi_U$  to a  $(U, \pi_U)$ -homogeneous Poisson structure on  $U/K$ . In particular, consider case  $z = i$ . The stability subgroup of this point is  $\mathrm{SO}(2)$  and the corresponding non-compact real form is  $\mathfrak{sl}(2, \mathbb{R})$ . The map

$$w \mapsto \frac{1}{\sqrt{1+|w|^2}} \begin{pmatrix} 1 + i\mathrm{Re}(w) & i\mathrm{Im}(w) \\ i\mathrm{Im}(w) & 1 - i\mathrm{Re}(w) \end{pmatrix}$$

gives a local cross section of the projection  $\mathrm{SU}(2) \rightarrow \mathrm{SU}(2)/\mathrm{SO}(2)$ , and a holomorphic coordinate for  $\mathbb{CP}^1$  with  $w = 0$  corresponding to  $z = i$ . In this coordinate, the projection of  $\pi_U$  (5.29) is given by

$$\pi = -2i\mathrm{Im}(w)(1+|w|^2) \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}}.$$

These two points of view are illustrated in figure 3. There, one can see that the Foth-Lu construction and the Evens-Lu construction produce the same Poisson structure on  $\mathbb{CP}^1$ .

In [KRR93] the existence of a parabolic subgroup of  $G$  such that  $U \cap P$  is a Poisson Lie subgroup was established under the assumption that  $X$  was a Hermitian symmetric space. The projection of  $\pi_U$  under the quotient map  $U \mapsto U/(U \cap P)$  then defines a  $(U, \pi_U)$ -homogeneous Poisson structure on  $X$  which is compatible with the invariant Poisson structure  $\pi_{KKS}$  that a symmetric space carries because it is a coadjoint orbit.

In this example, the parabolic subgroup  $P$  is actually the Borel subgroup  $HN^+ \subset \mathrm{SL}(2, \mathbb{C})$  and  $\mathrm{SU}(2) \cap P$  is the diagonal torus  $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ . Pushing forward  $\pi_U$

from (5.29) under the quotient map gives the Poisson structure

$$\pi_{PL} = 2i|z|^2(1 + |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}$$

which is degenerate only at the basepoint. Furthermore, the invariant Poisson structure of Kostant-Kirillov-Souriau is given in the  $z$ -coordinate by

$$\pi_{KKS} = i(1 + |z|^2)^2 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

The reader is invited to check that the Evens-Lu Poisson structure  $\pi$  in (5.30) is equal to  $\pi_{PL} + \lambda\pi_{KKS}$  with  $\lambda = -1$ .

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