HOMOGENEOUS POISSON STRUCTURES ON SYMMETRIC SPACES

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Abstract. We calculate, in a relatively explicit way, the Hamiltonian systems which arise from the Evens-Lu construction of homogeneous Poisson structures on both compact and noncompact type symmetric spaces. A corollary is that the Hamiltonian system arising in the noncompact case is isomorphic to the generic Hamiltonian system arising in the compact case. In the group case these systems are also isomorphic to those arising from the Bruhat Poisson structure on the flag space, and hence, by results of Lu, can be completely factored.

0. Introduction

Suppose that X is a simply connected compact symmetric space with a fixed basepoint. From this we obtain: a diagram of groups,

(0.1)
$$\begin{tikzcd}
 & G & \\
G_0 \arrow[ur] & & U \arrow[ul] \\
 & K \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where U is the universal covering of the identity component of the isometry group of X, $X \simeq U/K$, G is the complexification of U, and $X_0 = G_0/K$ is the noncompact type symmetric space dual to X; a diagram of equivariant totally geodesic (Cartan) embeddings of symmetric spaces:

(0.2)
$$\begin{tikzcd}
U/K \arrow[r, "\phi"] \arrow[d] & U \arrow[d] & \\
G/G_0 \arrow[r, "\phi"] & G & G/U \arrow[l, "\psi"'] \\
 & G_0 \arrow[u] & G_0/K \arrow[l, "\psi"'] \arrow[u]
\end{tikzcd}$$

Let θ denote the involution corresponding to the pair (U, K). We consider one additional ingredient: a triangular decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+,$$

which is θ-stable and for which $\mathfrak{t}_0 = \mathfrak{h} \cap \mathfrak{k}$ is maximal abelian in $\mathfrak{k}$.

This data determines standard Poisson Lie group structures, denoted Π_U and Π_{G_0}, for the groups U and G_0, respectively. By a general construction of Evens and Lu ([EL]), the symmetric spaces X and X_0 acquire Poisson structures Π_X and Π_{X_0}, respectively, which are homogeneous for the respective actions of the Poisson Lie groups (U, Π_U) and (G_0, Π_{G_0}). The compact case was considered in [C] and [FL], and the noncompact case in [FO].

In the noncompact case there is just one type of symplectic leaf, and this leaf is naturally Hamiltonian with respect to the maximal torus $T_0 = \exp(\mathfrak{t}_0) \subset K$. In the compact case the types of symplectic leaves are indexed by representatives $\mathbf{w}$ of the Weyl group of U which lie in the image of the Cartan embedding $\phi \colon U/K \to U$. Each such leaf is naturally Hamiltonian with respect to a torus T_w depending on the corresponding Weyl group element w. In a reasonably natural way, these leaves are parameterized by double cosets $R \backslash G_0/K$, where R depends upon $\mathbf{w}$ and choice of basepoint. We will refer to these as Evens-Lu Hamiltonian systems.

The plan of this paper is the following. In section 1 we recall standard notation and introduce an important operator, which is used throughout the paper.

In section 2 we exhibit a family of closed two-forms on G_0/K depending on a parameter $\mathbf{w}_1 \in U$. For special values of the parameter $\mathbf{w}_1$, these forms descend to the double coset spaces $R \backslash G_0/K$ and explicitly describe the Evens-Lu Hamiltonian systems. The results in this section for special values of $\mathbf{w}_1$ also follow from the calculations in sections 3 and 4, and general facts about Poisson geometry. However this direct approach is suggestive.

In section 3, we prove that the Hamiltonian system with $\mathbf{w} = 1$ is equal to the Hamiltonian system arising from the Evens-Lu construction in the noncompact case.

In section 4 we prove that these Hamiltonian systems are naturally isomorphic to the Hamiltonian systems arising from the Evens-Lu construction in the compact case. Our proof of this involves a brutal calculation, which is lacking conceptual insight.

All of the results of sections 2-4 generalize in a perfunctory way to loop spaces, using the now commonplace insight that finite dimensional complex semisimple Lie algebras and (centrally extended) loop algebras fit into the common framework of Kac-Moody Lie algebras.

In section 5 we specialize to the group case, $X = K$, where K is a simply connected compact Lie group. There are two main points in this section. The first is that (X, Π_X) is Poisson isomorphic to the standard Poisson Lie group structure on K, where the isomorphism is essentially translation by a representative for the longest Weyl group element. This translation interchanges the Birkhoff decomposition (intersected with K), the isotypic symplectic components for Π_X, with the Bruhat decomposition, the isotypic symplectic decomposition for the standard Poisson structure. This equivalence is a special finite dimensional feature.

The second main point is that the Hamiltonian systems which arise in this case can all be viewed as torus-invariant symplectic submanifolds of the generic Hamiltonian system. This is a corollary of work of Lu ([Lu]), who has completely factored these systems, in finite dimensions. It is natural to wonder whether this factorization is possible for more general symmetric spaces, or in infinite dimensions.

In the context of loop groups, one can think of these small (finite dimensional) systems as natural cutoffs for the enveloping (infinite dimensional) system, the primary object of interest for the authors of this paper.

1. Background and Notation

Throughout the remainder of this paper, U will denote a simply connected compact Lie group, and θ will denote an involution of U. This involution admits a unique holomorphic extension to G and determines involutive automorphisms of the Lie algebras of U and G, respectively. Slightly abusing notation, we will also write θ for the extension to G and the corresponding maps of algebras. The identity component of the fixed point set of θ in U will be denoted by K, and X will denote the quotient, U/K. We will also assume that X is irreducible, in the sense of symmetric space theory.

Corresponding to the diagram of groups in (0.1), there is a Lie algebra diagram

$$\begin{tikzcd}
 & \mathfrak{g} = \mathfrak{u} + i\mathfrak{u} & \\
\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p} \arrow[ur] & & \mathfrak{u} = \mathfrak{k} + i\mathfrak{p} \arrow[ul] \\
 & \mathfrak{k} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where θ, acting on the Lie algebra level is $+1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. It will be convenient to write the action of θ as a superscript, i.e., $\theta(g) = g^{\theta}$. We let $(\cdot)^{-*}$ denote the Cartan involution for the pair (G, U). The Cartan involution for the pair (G, G_0) is then given by $\sigma(g) = g^{\sigma} = g^{-*\theta}$. Since θ, $(\cdot)^{-*}$, and σ all commute, our practice of writing these involutions as superscripts should not cause confusion.

There are totally geodesic embeddings of symmetric spaces

$$\begin{tikzcd}
U/K \arrow[r, "\phi"] \arrow[d] & U \arrow[d] & : & uK \arrow[r] & uu^{-\theta} \\
G/G_0 \arrow[r, "\phi"] & G & : & gG_0 \arrow[r] & gg^{-\sigma}
\end{tikzcd}$$

where the symmetric space structures are derived from the Killing form (the embeddings ψ in (0.2) are defined in a similar way, but will not play a role in this paper).

Fix a maximal abelian subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}$. By computing the centralizer $\mathfrak{h}_0$ of $\mathfrak{t}_0$ in $\mathfrak{g}_0$, we obtain θ-stable Cartan subalgebras

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0, \quad \mathfrak{t} = \mathfrak{t}_0 + i\mathfrak{a}_0, \text{ and } \mathfrak{h} = \mathfrak{h}_0^{\mathbb{C}}$$

for $\mathfrak{g}_0$, $\mathfrak{u}$, and $\mathfrak{g}$, respectively, where $\mathfrak{a}_0 \subset \mathfrak{p}$. We write

$$\mathfrak{a} = \mathfrak{h}_{\mathbb{R}} = i\mathfrak{t} = i\mathfrak{t}_0 + \mathfrak{a}_0,$$

$A = \exp(\mathfrak{a})$, and we let T_0 and T denote the maximal tori in K and U corresponding to $\mathfrak{t}_0$ and $\mathfrak{t}$, respectively. We also fix a θ-stable triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^{-} + \mathfrak{h} + \mathfrak{n}^{+},$$

so that $\sigma(\mathfrak{n}^{\pm}) = \mathfrak{n}^{\mp}$. Let $N^{\pm} = \exp(\mathfrak{n}^{\pm})$, $H = \exp(\mathfrak{h})$, $B^{\pm} = HN^{\pm}$, and Wdenote the Weyl group $W(G, H)$. Note that $W = N_U(T)/T \simeq N_G(H)/H$.

We will write $x = x_- + x_{\mathfrak{h}} + x_+$ for the triangular decomposition of $x \in \mathfrak{g}$, and $x = x_{\mathfrak{k}} + x_{\mathfrak{p}}$ for the Cartan decomposition of $x \in \mathfrak{g}_0$, and $y = y_{\mathfrak{k}} + y_{i\mathfrak{p}}$ for the Cartan decomposition of $y \in \mathfrak{u}$. To do calculations we will frequently need to make use of the $\mathbb{R}$-linear orthogonal projections to the $\mathbb{R}$-subspaces $\mathfrak{u}, i\mathfrak{u}, \mathfrak{p}$, etc. In keeping with the above notation scheme, we will write $\{Z\}_{\mathfrak{u}}$ for the orthogonal projection to $\mathfrak{u}$ of $Z \in \mathfrak{g}$, and similarly $\{Z\}_{i\mathfrak{u}}$, $\{Z\}_{\mathfrak{p}}$, for the orthogonal projection to $i\mathfrak{u}$, $\mathfrak{p}$, etc.

There are two Iwasawa decompositions of $\mathfrak{g}$ determined by the above data:

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{a} + \mathfrak{u} \text{ and } \mathfrak{g} = \mathfrak{n}^- + i\mathfrak{h}_0 + \mathfrak{g}_0. \tag{1.1}$$

The former leads to a global decomposition of the group $G = N^-AU$. The standard Poisson Lie group structures on U (resp. G_0) that we consider are those associated to the Iwaswa decompositions in (1.1). Given $g \in G$, we write

$$g = \mathbf{l}(g)\mathbf{a}(g)\mathbf{u}(g) \tag{1.2}$$

relative to the Iwasawa decomposition $G = N^-AU$. We will write $\mathrm{pr}_{\mathfrak{u}}$ for the projection $\mathfrak{g} \to \mathfrak{u}$ with kernel $\mathfrak{n}^- + \mathfrak{a}$, and $\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}$ for the projection $\mathfrak{g} \to \mathfrak{n}^- + \mathfrak{a}$ with kernel $\mathfrak{u}$. Similarly, $\mathrm{pr}_{\mathfrak{g}_0}$ will denote the projection to $\mathfrak{g}_0$ with kernel $\mathfrak{n}^- + i\mathfrak{h}_0$, and $\mathrm{pr}_{\mathfrak{n}^-+i\mathfrak{h}_0}$ will denote the projection to $\mathfrak{n}^- + i\mathfrak{h}_0$ with kernel $\mathfrak{g}_0$. Note that the Iwasawa projections $\mathrm{pr}_{\mathfrak{u}}$ (resp. $\mathrm{pr}_{\mathfrak{g}_0}$) and the orthogonal projections $\{\cdot\}_{\mathfrak{u}}$ (resp. $\{\cdot\}_{\mathfrak{g}_0}$) do not agree, as the fomer has kernel $\mathfrak{n}^- + \mathfrak{a}$ (resp. $\mathfrak{n}^- + i\mathfrak{h}_0$) whereas the latter has kernel $i\mathfrak{u}$ (resp. $i\mathfrak{g}_0$).

We identify the dual of $\mathfrak{p}$ (resp. $i\mathfrak{p}$) with $\mathfrak{p}$ (resp. $i\mathfrak{p}$) using the Killing form. To do calculations, we use the induced isomorphisms

$$G_0 \times_K \mathfrak{p} \to T(G_0/K) = T^*(G_0/K), \tag{1.3}$$

$$U \times_K i\mathfrak{p} \to T(U/K) = T^*(U/K). \tag{1.4}$$

To keep track of functoriality, we will write $[g_0, x]$, $[g_0, y]$, and so on, for tangent vectors, and $[g_0, \phi]$, $[g_0, \psi]$, and so on, for cotangent vectors.

A key player throughout this paper is the "Hilbert transform" $\mathcal{H} \colon \mathfrak{g} \to \mathfrak{g}$ associated to the triangular decomposition of $\mathfrak{g}$.

$$x_- + x_0 + x_+ = x \mapsto \mathcal{H}(x) = -ix_- + ix_+ \tag{1.5}$$

The real subspaces $\mathfrak{g}_0$, $i\mathfrak{g}_0$, $\mathfrak{u}$, and $i\mathfrak{u}$ are all stabilized by $\mathcal{H}$, and $\mathcal{H}$ is skew-symmetric with respect to the Killing form. This operator also stabilizes $\mathfrak{n}^- + \mathfrak{n}^+$ and squares to -1 there. Thus, $\mathcal{H}$ is similar to a complex structure on the vector space $\mathfrak{g}$, except that it has a kernel, namely $\mathfrak{h}$. However, we still have the following result which will be important in the proof of the first theorem of section 3.

Proposition 1.1. *The Nijenhuis torsion for $\mathcal{H}$ on $\mathfrak{g}$,*

$$\mathcal{N}(A, B) = [A, B] + \mathcal{H}([\mathcal{H}(A), B] + [A, \mathcal{H}(B)]) - [\mathcal{H}(A), \mathcal{H}(B)], \quad A, B \in \mathfrak{g}, \tag{1.6}$$

is identically zero.

Proof. Since $\mathcal{H}$ is defined in terms of the triangular decomposition, we will show that each component of the triangular decomposition of $\mathcal{N}(A, B)$ vanishes. Let $A = A_- + A_0 + A_+$ and $B = B_- + B_0 + B_+$ denote the triangular decompositions of A and B. Since $[A_0, B_0] = 0$ we may write

$$[A, B] = [A_- + A_0, B_- + B_0] + ([A_-, B_+] + [A_+, B_-]) + [A_0 + A_+, B_0 + B_+]. \tag{1.7}$$

The first of the three terms on the right hand side of (1.7) is in $\mathfrak{n}^-$ since $[\mathfrak{b}^-, \mathfrak{b}^-] \subset \mathfrak{n}^-$. Similarly, the last term is in $\mathfrak{n}^+$. Hence, the diagonal part of $[A, B]$ is the same as the diagonal part of the middle term in (1.7). But, $\mathcal{H}$ leaves that term invariant, so we have $([\mathcal{H}(A), \mathcal{H}(B)])_0 = ([A, B])_0$.

We can now see that the diagonal of $\mathcal{N}(A, B)$ vanishes from the formula in (1.6). The second of the three terms on the right hand side of (1.6) is in the image of $\mathcal{H}$ and hence has no diagonal part, whereas we have just established the equality of the diagonal parts of the first and last terms.

Let us now turn to the $\mathfrak{n}^+$-part of $\mathcal{N}(A, B)$. With the previous observations we have that the $\mathfrak{n}^+$ part of the sum of the first and third terms in the Nijenhuis torsion (1.6) is

$$([A, B] - [\mathcal{H}(A), \mathcal{H}(B)])_+ = [A_0 + A_+, B_0 + B_+] + [A_+, B_+]. \tag{1.8}$$

Making further use of (1.7), we compute that

$$([\mathcal{H}(A), B])_+ = (-i[A_-, B_+] + i[A_+, B_-])_+ + i[A_+, B_0 + B_+] \tag{1.9}$$

and likewise

$$([A, \mathcal{H}(B)])_+ = (i[A_-, B_+] - i[A_+, B_-])_+ + i[A_0 + A_+, B_+]. \tag{1.10}$$

Summing the right hand sides of (1.9) and (1.10) and then applying $\mathcal{H}$ gives that

$$(\mathcal{H}([\mathcal{H}(A), B] + [A, \mathcal{H}(B)]))_+ = -[A_+, B_0 + B_+] - [A_0 + A_+, B_+] \tag{1.11}$$

which is the $\mathfrak{n}^+$-part of the second term in (1.6). The sum of right hand sides of (1.8) and (1.11) gives the $\mathfrak{n}^+$-part of $\mathcal{N}(A, B)$.

$$\begin{aligned} (\mathcal{N}(A, B))_+ &= [A_0 + A_+, B_0 + B_+] + [A_+, B_+] \\ &\quad -[A_+, B_0 + B_+] - [A_0 + A_+, B_+] && (1.12) \\ &= 0. && (1.13) \end{aligned}$$

The vanishing of the sum on the right hand side of (1.12) is readily apparent after one expands the terms using bilinearity of the bracket. A completely analogous calculation shows that the $\mathfrak{n}^-$-part of $\mathcal{N}(A, B)$ is also zero and thus completes the proof of the proposition. □

Two additional properties of $\mathcal{H}$ which will be important in this paper concern its relationship with the Iwasawa projections to $\mathfrak{u}$ and to $\mathfrak{g}_0$. Given $Z \in \mathfrak{g}$, we will write $Z \mapsto iZ$ for the complex structure on $\mathfrak{g}$ and denote the corresponding map of $\mathfrak{g}$ by i.

Proposition 1.2. *The following diagrams commute.*

(1.14)

$$\begin{tikzcd} \mathfrak{u} \arrow[r, "i"] \arrow[dr, "\mathcal{H}"'] & i\mathfrak{u} \arrow[d, "\mathrm{pr}_{\mathfrak{u}}"] \\ & \mathfrak{u} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g}_0 \arrow[r, "i"] \arrow[dr, "\mathcal{H}"'] & i\mathfrak{g}_0 \arrow[d, "\mathrm{pr}_{\mathfrak{g}_0}"] \\ & \mathfrak{g}_0 \end{tikzcd}$$

Proof. To see that the first diagram commutes, observe that the triangular decomposition of element $Z \in \mathfrak{u}$ has the form $-(Z_+)^* + Z_0 + Z_+$ where $Z_0 \in \mathfrak{t}$. Hence, the Iwasawa projection to $\mathfrak{u}$ of iZ is

$$\mathrm{pr}_{\mathfrak{u}}(iZ) = -(iZ_+)^* + iZ_+ = -i(-(Z_+)^*) + iZ_+ = \mathcal{H}(Z)$$

since $i\mathfrak{t} = \mathfrak{a}$ is contained in the kernel of $\mathrm{pr}_{\mathfrak{u}}$ and the involution $-(\cdot)^*$ is complex anti-linear.

Similarly, $Z \in \mathfrak{g}_0$ has triangular decomposition $(Z_+)^\sigma + Z_0 + Z_+$ where $Z_0 \in \mathfrak{h}_0$. Hence, the projection to $\mathfrak{g}_0$ of iZ is

$$\mathrm{pr}_{\mathfrak{g}_0}(iZ) = (iZ_+)^\sigma + iZ_+ = -i(Z_+)^\sigma + iZ_+ = \mathcal{H}(Z)$$

as $i\mathfrak{h}_0$ is contained in the kernel of $\mathrm{pr}_{\mathfrak{g}_0}$ and the involution σ is complex anti-linear. □

2. Evens-Lu Hamiltonian Systems

In this section we introduce symplectic structures on certain double coset spaces of G_0. The double coset spaces and their symplectic structures depend on a parameter $\mathbf{w}_1 \in U$. For certain values of this parameter, these spaces admit Hamiltonian torus actions for which we compute the momentum maps.

Our definition of the relevant two-forms involves the Hilbert transform $\mathcal{H} \colon \mathfrak{g} \to \mathfrak{g}$ associated to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ (cf. (1.5)) and the map $\mathbf{u} \colon G \to U$ arising the Iwasawa factorization (c.f. (1.2)).

Definition. For each $\mathbf{w}_1 \in U$ we define a two-form $\omega_{\mathbf{w}_1}$ on G_0/K by the formula

$$\omega_{\mathbf{w}_1}([g_0, x] \wedge [g_0, y]) = \langle Ad(\mathbf{u}(\mathbf{w}_1 g_0)^{-1}) \circ \mathcal{H} \circ Ad(\mathbf{u}(\mathbf{w}_1 g_0))(x), y \rangle. \tag{2.1}$$

Theorem 2.1. *For each $\mathbf{w}_1 \in U$, the two-form $\omega_{\mathbf{w}_1}$ on G_0/K is closed.*

Proof. It will be convenient to abbreviate $\mathbf{u}(\mathbf{w}_1 g_0)$ by simply $\mathbf{u}$, and just keep in mind that $\mathbf{w}_1 g_0 = \mathbf{lau}$. Let us define an $i\mathfrak{u}$-valued one-form α on G_0/K by

$$\alpha([g_0, x]) = x^{\mathbf{u}}.$$

Then $\omega = \langle \mathcal{H}(\alpha) \wedge \alpha \rangle$ and

$$d\omega = \langle \mathcal{H}(d\alpha) \wedge \alpha \rangle - \langle \mathcal{H}(\alpha) \wedge d\alpha \rangle. \tag{2.2}$$

Given $X \in \mathfrak{g}_0$, let $\kappa(X) = [g_0, \{X^{g_0^{-1}}\}_{\mathfrak{p}}]$ denote the corresponding vector field on G_0/K. Note that $\{X^{g_0^{-1}}\}_{\mathfrak{p}} = \{X^{g_0^{-1}}\}_{i\mathfrak{u}}$ because $X^{g_0^{-1}} \in \mathfrak{g}_0$. Thus,

$$\alpha(\kappa(X)) = (\{X^{g_0^{-1}}\}_{i\mathfrak{u}})^{\mathbf{u}} = \{X^{\mathbf{u} g_0^{-1}}\}_{i\mathfrak{u}} = \{(X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}\}_{i\mathfrak{u}} \tag{2.3}$$

where we have used the factorization $\mathbf{w}_1 g_0 = \mathbf{lau}$ and the fact that $\mathrm{Ad}(\mathbf{u})$ commutes with the othogonal projection to $i\mathfrak{u}$.

Now suppose that $X, Y \in \mathfrak{g}_0$. Then

$$\begin{aligned} d\alpha(\kappa(X) \wedge \kappa(Y)) &= \kappa(X)\alpha(\kappa(Y)) - \kappa(Y)\alpha(\kappa(X)) - \alpha([\kappa(X), \kappa(Y)]) \\ &= \kappa(X)\alpha(\kappa(Y)) - \kappa(Y)\alpha(\kappa(X)) + \alpha(\kappa[X, Y]) \end{aligned}$$

where in the last equation we used the fact that $\kappa \colon \mathfrak{g}_0 \to \Gamma(T(G_0/K))$ is an anti-homomorphism of Lie algebras.

To compute the derivative of α we must first compute the derivative of function $\mathbf{la}$. This derivative takes values in $T(N^-A)$, which we may identify with $N^-A \times (\mathfrak{n}^- + \mathfrak{a})$ by left translation, so we can consider $(\mathbf{la})^{-1} d(\mathbf{la})$ as an $(\mathfrak{n}^- + \mathfrak{a})$-valued 1-form representing the derivative.

We assert that $(\mathbf{la})^{-1} d(\mathbf{la})(\kappa(X)) = \mathrm{pr}_{\mathfrak{n}^- + \mathfrak{a}}((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})$. Recall from section 1 that $\mathrm{pr}_{\mathfrak{n}^- + \mathfrak{a}}(Z) = Z_- + (Z_+)^* + Z_{\mathfrak{a}}$ for each $Z \in \mathfrak{g}$..

To see this, let $Y \in \mathfrak{g}_0$ and let ε denote a small real parameter. Then $\mathbf{w}_1 g_0 e^{\varepsilon Y} = \mathbf{lau} e^{\varepsilon Y} = \mathbf{la} e^{\varepsilon Y^{\mathbf{u}}} \mathbf{u}$. After translation, the linearization of $(\mathbf{la})$ at $\varepsilon = 0$ is $Y \mapsto \mathrm{pr}_{\mathfrak{n}^- + \mathfrak{a}}(Y^{\mathbf{u}})$.

Note that $\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(Z) = \mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(\{Z\}_{\mathfrak{u}} + \{Z\}_{i\mathfrak{u}}) = \mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(\{Z\}_{i\mathfrak{u}})$. Thus, after evaluating the linearization of $(\mathbf{la})$ on $Y = \kappa(X) = \{X^{g_0^{-1}}\}_{\mathfrak{p}} = \{X^{g_0^{-1}}\}_{i\mathfrak{u}}$, we obtain

$$\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(\{X^{g_0^{-1}}\}^{\mathbf{u}}_{i\mathfrak{u}}) = \mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(\{X^{\mathbf{u}g_0^{-1}}\}_{i\mathfrak{u}}) = \mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})$$

as desired.

Next, we need to compute the derivative of the $\mathfrak{g}$-valued function

$$g_0K \mapsto \mathrm{Ad}((\mathbf{la})^{-1})(Y)$$

in the direction of $\kappa(X)$ for each $Y \in \mathfrak{g}$. Using the chain rule, and the preceeding assertion, we obtain the expression

$$-\mathrm{ad}(\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}))(\mathrm{Ad}(\mathbf{la})^{-1}(Y)) = -[\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}), Y^{(\mathbf{la})^{-1}}]$$

for this derivative. Thus, $d\alpha(\kappa(X) \wedge \kappa(Y)) = \{W\}_{i\mathfrak{u}} = \frac{1}{2}(W + W^*)$ where

$$\begin{aligned} W(X,Y) &= -[\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}), (Y^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}] \\ &\quad +[\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}((Y^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}), (X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}] + [X^{\mathbf{w}_1}, Y^{\mathbf{w}_1}]^{(\mathbf{la})^{-1}}. \end{aligned}$$

To obtain a formula for the value of $d\alpha$ on cotangent vectors represented by $[g_0, x]$ and $[g_0, y]$ we make the substitutions $X = x^{g_0}$ and $Y = y^{g_0}$. Then, $X^{\mathbf{w}_1} = x^{\mathbf{w}_1 g_0} = x^{\mathbf{lau}}$ and

$$d\alpha([g_0, x] \wedge [g_0, y]) = \frac{1}{2}(W + W^*) \tag{2.4}$$

where

$$\begin{aligned} W &= -[\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(x^{\mathbf{u}}), y^{\mathbf{u}}] + [\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(y^{\mathbf{u}}), x^{\mathbf{u}}] + [x,y]^{\mathbf{u}} && (2.5)\\ &= [\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})] - [\mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(x^{\mathbf{u}}), \mathrm{pr}_{\mathfrak{n}^-+\mathfrak{a}}(y^{\mathbf{u}})]. && (2.6)\\ &= [\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), y^{\mathbf{u}}] + [x^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})] - [x^{\mathbf{u}}, y^{\mathbf{u}}]. && (2.7) \end{aligned}$$

In order to obtain (2.6) from (2.5) in the above, one: uses that $\mathrm{Ad}(\mathbf{u})$ is a Lie algebra homomorphism to write $[x,y]^{\mathbf{u}}$ as $[x^{\mathbf{u}}, y^{\mathbf{u}}]$; then decomposes $x^{\mathbf{u}}$ and $y^{\mathbf{u}}$ under the Iwaswa decompostion; then expands to observe cancellations. Similarly, one can use the Iwasawa decomposition to verify that (2.7) is equivalent to (2.6). The expression in (2.7) is what will be used as it contains a minimal number of projections.

Note that $[\mathfrak{u}, i\mathfrak{u}] \subset i\mathfrak{u}$, which is fixed by $(\cdot)^*$ whereas $[i\mathfrak{u}, i\mathfrak{u}] \subset \mathfrak{u}$ is negated by $(\cdot)^*$. Hence

$$\begin{aligned} W^* &= [\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), y^{\mathbf{u}}]^* + [x^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})]^* - [x^{\mathbf{u}}, y^{\mathbf{u}}]^* \\ &= [\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), y^{\mathbf{u}}] + [x^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})] + [x^{\mathbf{u}}, y^{\mathbf{u}}] \\ &= W + 2[x^{\mathbf{u}}, y^{\mathbf{u}}]. \end{aligned}$$

With this expression for W^*, (2.4) becomes

$$\begin{aligned} \frac{1}{2}(W + W^*) &= \frac{1}{2}(W + W + 2[x^{\mathbf{u}}, y^{\mathbf{u}}]) \\ &= W + [x^{\mathbf{u}}, y^{\mathbf{u}}] \\ &= [\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), y^{\mathbf{u}}] + [x^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})]. && (2.8) \end{aligned}$$

Using (2.2) we have that $d\omega([g_0, x] \wedge [g_0, y] \wedge [g_0, z])$ is equal to

$$\langle \mathcal{H}([\mathrm{pr}_{\mathfrak{u}}(x^{\mathbf{u}}), y^{\mathbf{u}}] + [x^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}})]), z^{\mathbf{u}}\rangle - \langle \mathcal{H}(x^{\mathbf{u}}), [\mathrm{pr}_{\mathfrak{u}}(y^{\mathbf{u}}), z^{\mathbf{u}}] + [y^{\mathbf{u}}, \mathrm{pr}_{\mathfrak{u}}(z^{\mathbf{u}})]\rangle$$

$$+ \text{ cyclic permutations of } x,\ y, \text{ and } z.$$

This can be further rewritten as

$$\begin{aligned}-2(&\langle[\mathrm{pr}_{\mathfrak{u}}(a),b]+[a,\mathrm{pr}_{\mathfrak{u}}(b)],\mathcal{H}(c)\rangle\\ &+\langle[\mathrm{pr}_{\mathfrak{u}}(c),a]+[c,\mathrm{pr}_{\mathfrak{u}}(a)],\mathcal{H}(b)\rangle\\ &+\langle[\mathrm{pr}_{\mathfrak{u}}(b),c]+[b,\mathrm{pr}_{\mathfrak{u}}(c)],\mathcal{H}(a)\rangle)\end{aligned} \tag{2.9}$$

using skew-symmetry of $\mathcal{H}$ with respect to the Killing form on $i\mathfrak{u}$ and abbreviating $x^{\mathbf{u}}$, $y^{\mathbf{u}}$, and $z^{\mathbf{u}}$, by a, b, and c, repsectively. Using the commutativity of the left diagram in (1.14), we can replace $\mathrm{pr}_{\mathfrak{u}}(\cdot)$ with $\mathcal{H}(-i\cdot)=-i\mathcal{H}(\cdot)$ in line (2.9) obtaining

$$2i(\langle[\mathcal{H}(a),b],\mathcal{H}(c)]\rangle+\langle\mathcal{H}(a),[b,\mathcal{H}(c)]\rangle \tag{2.10}$$

$$+\langle[\mathcal{H}(c),a],\mathcal{H}(b)]\rangle+\langle\mathcal{H}(c),[a,\mathcal{H}(b)]\rangle \tag{2.11}$$

$$+\langle[\mathcal{H}(b),c],\mathcal{H}(a)]\rangle+\langle\mathcal{H}(b),[c,\mathcal{H}(a)]\rangle). \tag{2.12}$$

The Killing form satisfies the identities

$$\langle X,[Y,Z]\rangle=\langle Y,[Z,X]\rangle=\langle Z,[X,Y]\rangle \tag{2.13}$$

for each $X,Y,Z\in\mathfrak{g}$. Using the symmetry of the Killing form and applying these identities to the terms in (2.10), (2.11), and (2.12), and simplifying gives

$$4i(\langle a,[\mathcal{H}(b),\mathcal{H}(c)]\rangle+\langle b,[\mathcal{H}(c),\mathcal{H}(a)]\rangle+\langle c,[\mathcal{H}(a),\mathcal{H}(b)]\rangle). \tag{2.14}$$

Let us note that by making further use of the Killing form identities (2.13), the skew-symmetry of $\mathcal{H}$, and omitting the multiple of $4i$, we can rewrite (2.14) as

$$\begin{aligned}&\langle a,[\mathcal{H}(b),\mathcal{H}(c)]-\mathcal{H}([\mathcal{H}(b),c]+[b,\mathcal{H}(c)])\rangle\\ =\ &\langle a,[b,c]-\mathcal{N}(b,c)\rangle\end{aligned} \tag{2.15}$$

where $\mathcal{N}(b,c)$ denotes the Nijenhuis torsion for $\mathcal{H}$ on $\mathfrak{g}$ (cf. 1.6). In Proposition 1.1, the Nijenhuis torsion for $\mathcal{H}$ was shown to vanish. Thus,

$$d\omega([g_0,x]\wedge[g_0,y]\wedge[g_0,z])=4i\langle x^{\mathbf{u}},[y^{\mathbf{u}},z^{\mathbf{u}}]\rangle=4i\langle x,[y,z]\rangle \tag{2.16}$$

as $\mathrm{Ad}(\mathbf{u})$ is a homomorphism of $\mathfrak{g}$ and the Killing form is Ad-invariant. The final expression in (2.16) is zero because $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$ which is othogonal to $\mathfrak{p}$. The proof is complete. □

The identification of U with $N^-A\backslash G$ gives rise to a right action of G_0 on U.

$$\begin{array}{ccc}U\times G_0 & \to & U\\ (u,g_0) & \mapsto & \mathbf{u}(ug_0)\end{array}$$

Given $\mathbf{w}_1\in U$, we can compute the stabilizer using the uniqueness of the Iwasawa decomposition. Since $\mathbf{w}_1 g_0=g_0^{\mathbf{w}_1}\mathbf{w}_1$, it follows that $\mathbf{u}(\mathbf{w}_1 g_0)=\mathbf{w}_1$ if and only if $g_0^{\mathbf{w}_1}\in N^-A$.

Notation. We write $R(\mathbf{w}_1)$ for the G_0-subgroup $(N^-A)^{\mathbf{w}_1^{-1}}\cap G_0$, i.e., the stabilizer of $\mathbf{w}_1$ under the G_0-action on U.

Theorem 2.2. *The closed two-form $\omega_{\mathbf{w}_1}$ on G_0/K descends to a symplectic form on the double coset space $R(\mathbf{w}_1)\backslash G_0/K$.*

Proof. Fix $\mathbf{w}_1 \in U$. To simplify notation, we will suppress the dependence on $\mathbf{w}_1$ by writing simply R for $R(\mathbf{w}_1)$, and ω for $\omega_{\mathbf{w}_1}$. To show that ω descends to $R\backslash G_0/K$, we must show that ω is left R-translation invariant, and that $\kappa(\mathfrak{r}) \subset \ker(\omega)$, where $\mathfrak{r}$ denotes the Lie algebra of R, and κ denotes the action by vector fields on G_0/K.

If $r \in R$, then $\mathbf{w}_1 r \mathbf{w}_1^{-1} \in N^- A$ and

$$\mathbf{w}_1 r g_0 = \mathbf{w}_1 r \mathbf{w}_1^{-1}(\mathbf{w}_1 g_0), \tag{2.17}$$

hence $\mathbf{u}(\mathbf{w}_1 r g_0) = \mathbf{u}(\mathbf{w}_1 g_0)$. From this and the formula for ω, it follows that ω is preserved by left translation by r.

As in the proof of theorem 2.1, we abbreviate $\mathbf{u}(\mathbf{w}_1 g_0)$ by $\mathbf{u}$ and make use of the $i\mathfrak{u}$-valued one-form α on G_0/K defined by $\alpha([g_0, x]) = x^{\mathbf{u}}$. Recall that $\omega = \langle \mathcal{H}(\alpha) \wedge \alpha \rangle$and $\alpha(\kappa(X)) = \{(X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}\}_{i\mathfrak{u}}$ for each $X \in \mathfrak{g}_0$. If $X \in \mathfrak{r}$, then $X^{\mathbf{w}_1} \in \mathfrak{n}^- + \mathfrak{a}$, as is $Z = (X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}}$. Thus, $\{Z\}_{i\mathfrak{u}} = \frac{1}{2}(Z_- + 2Z + (Z_-)^*)$, and $2\mathcal{H}(\{Z\}_{i\mathfrak{u}}) = -iZ_- + i(Z_-)^* = -iZ_- + (-iZ_-)^*$. With this, we have

$$\begin{aligned}
\omega(\kappa(X) \wedge [g_0, y]) &= \langle \mathcal{H}(\alpha(\kappa(X))) \wedge \alpha([g_0, y]) \rangle \\
&= \tfrac{1}{2}\langle -iZ_- + (-iZ_-)^*, y^{\mathbf{u}} \rangle \\
&= \tfrac{1}{2}\left(\langle -iZ_-, y^{\mathbf{u}} \rangle + \langle (-iZ_-)^*, (y^{\mathbf{u}})^* \rangle \right) \\
&= \mathrm{Re}\langle -iZ_-, y^{\mathbf{u}} \rangle \\
&= \mathrm{Im}\langle Z_-, y^{\mathbf{u}} \rangle
\end{aligned}$$

where we have used that $y^{\mathbf{u}} \in i\mathfrak{u}$ is fixed by $(\cdot)^*$ which is an complex anti-linear isometry of the Killing form.

Since $Z = (X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}} \in \mathfrak{n}^- + \mathfrak{a}$, we can write $Z_- = Z - Z_0$ and

$$\omega(\kappa(X) \wedge [g_0, y]) = \mathrm{Im}\langle Z, y^{\mathbf{u}} \rangle - \mathrm{Im}\langle Z_0, y^{\mathbf{u}} \rangle. \tag{2.18}$$

The second term on the right hand side of (2.18) vanishes since $Z_0 \in \mathfrak{a} \subset i\mathfrak{u}$ and the Killing form is real on $i\mathfrak{u} \times i\mathfrak{u}$. Substituting back $Z = X^{(\mathbf{la})^{-1}\mathbf{w}_1}$ into the first term on the right hand side of (2.18), we obtain

$$\mathrm{Im}\langle Z, y^{\mathbf{u}} \rangle = \mathrm{Im}\langle X^{(\mathbf{la})^{-1}\mathbf{w}_1}, y^{\mathbf{u}} \rangle = \mathrm{Im}\langle X, y^{g_0} \rangle$$

using the factorization $\mathbf{w}_1 g_0 = \mathbf{lau}$. This term also vanishes because the Killing form is real on $\mathfrak{g}_0$. Hence ω descends to the quotient $R\backslash G_0/K$.

The next claim to be established is the non-degeneracy of ω on $R\backslash G_0/K$. Let us write $\Omega(g_0)$ for the linear transformation

$$\mathfrak{p} \in x \mapsto \Omega(g_0)(x) = \{\mathrm{Ad}(\mathbf{u})^{-1} \circ \mathcal{H} \circ \mathrm{Ad}(\mathbf{u})(x)\}_{\mathfrak{p}} \tag{2.19}$$

so that $\omega([g_0, x], [g_0, y]) = \langle \Omega(g_0)(x), y \rangle$. We must show that $\kappa(\mathfrak{r})|_{g_0 K} = \ker(\Omega(g_0))$. We established one containment to show descent. For the other, it is convenient to introduce an extension $\tilde{\Omega}(g_0)$ of $\Omega(g_0)$ to all of $\mathfrak{g}_0$. Specifically, we set

$$\tilde{\Omega}(g_0) = I \circ \mathrm{Ad}(\mathbf{u})^{-1} \circ \mathrm{pr}_{\mathfrak{u}} \circ \mathrm{Ad}(\mathbf{u}) \tag{2.20}$$

where the linear transformation $I \colon \mathfrak{u} \to \mathfrak{g}_0$ is the identity on $\mathfrak{k}$ and multiplication by i on $i\mathfrak{p}$. Relative to the decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$, the extension is represented by the block 2×2 matrix

$$\tilde{\Omega}(g_0) = \begin{pmatrix} 1 & b \\ 0 & \Omega \end{pmatrix}$$

for some linear transformation $b\colon \mathfrak{p} \to \mathfrak{k}$. It's follows from (2.20) that $\tilde{\Omega}$ is the identity on $\mathfrak{k}$. For its value on $\mathfrak{p}$ one should note that if $x \in \mathfrak{p}$, then $\mathrm{Ad}(\mathbf{u})(x) \in i\mathfrak{u}$, and $\mathrm{pr}_{\mathfrak{u}} = \mathcal{H}(-i\cdot) = -i\mathcal{H}(\cdot)$ on $i\mathfrak{u}$. Thus, for $x \in \mathfrak{p}$,

$$\tilde{\Omega}(g_0)(x) = I(-i\mathrm{Ad}(\mathbf{u})^{-1} \circ \mathcal{H} \circ \mathrm{Ad}(\mathbf{u})(x)) = b(x) + \Omega(g_0)(x)$$

for some $b(x) \in \mathfrak{k}$. This shows that the compression to $\mathfrak{p}$ of $\tilde{\Omega}(g_0)$ agrees with $\Omega(g_0)$. We introduce this extension because there is an isomorphism $x \in \ker(\Omega) \mapsto (-bx, x) \in \ker(\tilde{\Omega})$ and it is possible to compute the kernel of $\tilde{\Omega}(g_0)$.

The extension $\tilde{\Omega}$ can be factored as a composition of four operators:

$$\mathfrak{g}_0 \xrightarrow{\mathrm{Ad}(g_0)} \mathfrak{g}_0 \xrightarrow{T} \mathfrak{u} \xrightarrow{\mathrm{Ad}(\mathbf{u})^{-1}} \mathfrak{u} \xrightarrow{I} \mathfrak{g}_0 \tag{2.21}$$

where $T = \mathrm{pr}_{\mathfrak{u}} \circ \mathrm{Ad}(\mathbf{u} g_0^{-1}) = \mathrm{pr}_{\mathfrak{u}} \circ \mathrm{Ad}((\mathbf{la})^{-1}\mathbf{w}_1)$. The kernel of $\tilde{\Omega}$ is thus determined by the kernel of T as the other three arrows in (2.21) are isomorphisms of vector spaces. The operator T maps $X \in \mathfrak{g}_0$ to

$$-(((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})_+)^* + (X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})_{\mathfrak{t}} + (X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})_+$$

which can be rewritten as

$$-((((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_+)^* + ((X^{\mathbf{w}_1} + (X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_{\mathfrak{t}} + ((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_+ \tag{2.22}$$

using the identity $((X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}})_+ = (((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_+$. Since (2.22) gives the triangular decomposition of $T(X)$, we can immediately see that $T(X) = 0$ only if $(X^{\mathbf{w}_1})^{(\mathbf{la})^{-1}} \in \mathfrak{b}^-$ or, equivalently, $X^{\mathbf{w}_1} \in \mathfrak{b}^-$. After examining the diagonal part of (2.22), it follows that $(X^{\mathbf{w}_1})_{\mathfrak{t}} = 0$ if $T(X) = 0$. Thus, $X^{\mathbf{w}_1} \in \mathfrak{n}^- + \mathfrak{a}$, or equivalently, $X \in \mathfrak{r} = (\mathfrak{n}^- + \mathfrak{a})^{\mathbf{w}_1^{-1}} \cap \mathfrak{g}_0$, and we have shown the desired containment $\ker(\tilde{\Omega}(g_0)) \subset \kappa(\mathfrak{r})|_{g_0 K}$, completing the proof of the theorem. $\square$

The sub-torus $T_0 \subset T$ acts from the left on G_0/K in a natural way. In what follows, we introduce other sub-tori of T which will act on the double coset space $R(\mathbf{w}_1)\backslash G_0/K$ for certain values of the parameter $\mathbf{w}_1 \in U$.

Notation. For $w \in W$, we write

$$\mathfrak{t}_w = \{x \in \mathfrak{t}\colon \mathrm{Ad}(w) \circ \theta(x) = x\}$$

and

$$T_w = \{t \in T\colon w t^\theta w^{-1} = t\}.$$

Lemma 2.3. *Denote by* $\mathbf{w}$ *the* $\mathbf{w}_1 K$ *Cartan image,* $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_1^{-\theta}$.

a) $Ad(\mathbf{w}_1) \circ \theta \circ Ad(\mathbf{w}_1)^{-1} = Ad(\mathbf{w}) \circ \theta$.

b) If $\mathbf{w} \in N_U(T)$ *and* $w = \mathbf{w}T \in W$, *then:*

  i. $\mathfrak{h}$, $\mathfrak{a}$ *and* $\mathfrak{t}$ *are* $\theta^{Ad(\mathbf{w}_1)}$*-stable,*

  ii. $\mathfrak{t}_w = \{x \in \mathfrak{t} : \theta^{Ad(\mathbf{w}_1)}(x) = x\} = \mathfrak{t} \cap \mathfrak{g}_0^{\mathbf{w}_1} = \mathfrak{t} \cap \mathfrak{k}^{\mathbf{w}_1}$, *and*

  iii. $T_w = T \cap G_0^{\mathbf{w}_1} = T \cap K^{\mathbf{w}_1} = exp(\mathfrak{t}_w)$.

Proof. Part a) follows from the fact that $\theta \circ \mathrm{Ad}(\mathbf{w}_1) = \mathrm{Ad}(\mathbf{w}_1^\theta) \circ \theta$. Given the validty of a) and the θ-stability of $\mathfrak{h}$, $\mathfrak{t}$, and $\mathfrak{a}$, it follows that each of these is $\theta^{\mathrm{Ad}(\mathbf{w}_1)}$-stable when $\mathbf{w}_1 \mathbf{w}_1^{-\theta} \in N_U(T)$. For b), part ii) the set theoretic description of $\mathfrak{t}_w$ follows from a). Since $\mathfrak{g}_0$ is fixed by σ, $\mathfrak{g}_0^{\mathbf{w}_1}$ is fixed by $\mathrm{Ad}(\mathbf{w}) \circ \sigma$. Thus, by intersection $\mathfrak{g}_0^{\mathbf{w}_1}$ with $\mathfrak{t}$ we obtain $\mathfrak{t}_w$ which is the fixed point set of in $\mathfrak{t}$ of $\mathrm{Ad}(\mathbf{w}) \circ \sigma$. For the same reasons, we have that $\mathfrak{t}_w = \mathfrak{t} \cap \mathfrak{k}^{\mathbf{w}_1}$ as σ and θ agree and are equal to the identity on $\mathfrak{k}$. The equalities in iii) follow routinely from those in ii). $\square$

Theorem 2.4. *Suppose that* $\mathbf{w}_1 \in U$ *is such that* $\mathbf{w} = \mathbf{w}_1\mathbf{w}_1^{-\theta} \in N_U(T)$ *and let* $w = \mathbf{w}T$ *denote the element of the Weyl group represented by* $\mathbf{w}$*.*

a) The double coset space $R(\mathbf{w}_1)\backslash G_0/K$ *is contractible.*

b) The torus T_w *acts on* $R(\mathbf{w}_1)\backslash G_0/K$ *as follows. Consider* $\mathbf{w}_1$ *as fixed and abbreviate* $R(\mathbf{w}_1)$ *by* R*.*

$$\begin{array}{ccc} T_w \times R\backslash G_0/K & \rightarrow & R\backslash G_0/K \\ (t, Rg_0K) & \mapsto & R\mathbf{w}_1^{-1}t\mathbf{w}_1g_0K \end{array} \tag{2.23}$$

Moreover, this action is preserves the symplectic form $\omega_{\mathbf{w}_1}$*.*

c) Let $\mathfrak{t}_w^\vee$ *denote the dual space of* $\mathfrak{t}_w$*. The action of* T_w *on* $R(\mathbf{w}_1)\backslash G_0/K$ *is Hamiltonian with momemtum map*

$$\begin{array}{rccc} \Phi^{\mathbf{w}_1}: & R\backslash G_0/K & \rightarrow & \mathfrak{t}_w^\vee \\ & Rg_0K & \mapsto & \langle i\log \mathbf{a}(\mathbf{w}_1g_0), \cdot\rangle. \end{array} \tag{2.24}$$

Proof. Suppose that $\mathbf{w}_1$ and $\mathbf{w}$ are as in the statement of the theorem and regard these as fixed. As usual, we will abbreviate $R(\mathbf{w}_1)$ by R and $\omega_{\mathbf{w}_1}$ by ω to simplify notation. For part a), we refer to the proof of Theorem 4 a) in [Pi1] which makes use of the assumption $\mathbf{w} \in N_U(T)$.

Let $t \in T_w$. From Lemma 2.3, we know that $T_w = T \cap G_0^{\mathbf{w}_1}$. Therefore, $\mathbf{w}_1^{-1}t\mathbf{w}_1 \in G_0$ and T_w acts from the left on G_0/K by $(t, g_0K) \mapsto t^{\mathbf{w}_1^{-1}}g_0K$. The fact that $\mathrm{Ad}(t^{\mathbf{w}_1^{-1}}) = \mathrm{Ad}(\mathbf{w}_1^{-1}) \circ \mathrm{Ad}(t) \circ \mathrm{Ad}(\mathbf{w}_1)$ preserves R implies that left T_w-action on G_0/K descends to the quotient $R\backslash G_0/K$ as in (2.23). Note that

$$\mathbf{u}(\mathbf{w}_1(\mathbf{w}_1^{-1}t\mathbf{w}_1)g_0) = t\mathbf{u}(\mathbf{w}_1g_0)$$

and $\mathrm{Ad}(t)$ commutes with $\mathcal{H}$. From these observations, together with the formula in (2.1), it follows that ω is T_w-invariant. This proves b).

Now let $X \in \mathfrak{t}_w$, then $\kappa(X^{\mathbf{w}_1^{-1}})$ is vector field on G_0/K representing the infintesimal action of X. We must show that contraction of ω in the direction of $\kappa(X^{\mathbf{w}_1^{-1}})$ is equal to the one-form $d\Phi_X$ where Φ_X is the function

$$g_0K \mapsto \Phi_X(g_0K) = \langle iX, \log \mathbf{a}(\mathbf{w}_1g_0)\rangle.$$

First we compute $d\Phi_X$. Let ε denote a small real parameter, let $y \in \mathfrak{p}$, and consider Φ_X evaluated along the curve $\varepsilon \mapsto g_0e^{\varepsilon Y}K$. Observe that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \log \mathbf{a}(\mathbf{w}_1g_0e^{\varepsilon y}) = [g_0, \{y^{\mathbf{u}}\}_{\mathfrak{a}}]$$

since the orthogonal projection to $\mathfrak{a}$ and the Iwasawa projection to $\mathfrak{a}$ give the same result. Thus

$$d\Phi_X([g_0, y]) = \langle iX, \{y^{\mathbf{u}}\}_{\mathfrak{a}}\rangle = \langle iX, Y^{\mathbf{u}}\rangle = \langle iX^{\mathbf{u}^{-1}}, y\rangle$$

since $X \in \mathfrak{t}_w \subset \mathfrak{t}$. This shows that $d\Phi_X$ is represented by the class $[g_0, \{X^{\mathbf{u}^{-1}}\}_{\mathfrak{p}}]$. We must show that this agrees with the class representing the contraction of ω in the direction of $\kappa(X^{\mathbf{w}_1^{-1}}) = [g_0, \{X^{(\mathbf{w}_1g_0)^{-1}}\}_{\mathfrak{p}}]$. Let us write $\mathbf{w}_1g_0 = \mathbf{lau}$ to simplify notation. Computing $\omega(\kappa(X^{\mathbf{w}_1^{-1}}) \wedge [g_0, y])$, one obtains

$$\begin{aligned} & \langle \mathrm{Ad}(\mathbf{u}^{-1}) \circ \mathcal{H} \circ \mathrm{Ad}(\mathbf{u})(\{\mathrm{Ad}((\mathbf{w}_1g_0)^{-1})(X)\}_{\mathfrak{p}}), y\rangle \\ = \ & \tfrac{1}{2}\langle \mathcal{H} \circ \mathrm{Ad}(\mathbf{u})(X^{(\mathbf{w}_1g_0)^{-1}} + (X^{(\mathbf{w}_1g_0)^{-1}})^*), y^{\mathbf{u}}\rangle \\ = \ & \tfrac{1}{2}\langle \mathcal{H}(X^{(\mathbf{la})^{-1}} + (X^{(\mathbf{la})^{-1}})^*), y^{\mathbf{u}}\rangle \end{aligned} \tag{2.25}$$

making use of the facts that: 1) $X^{(\mathbf{w}_1 g_0)^{-1}} \in \mathfrak{g}_0$, since $X \in \mathfrak{t}_w$; 2) given $Z \in \mathfrak{g}_0$, the projection $\{Z\}_{\mathfrak{p}} = \frac{1}{2}(Z + Z^*)$ since θ agrees with $-(\cdot)^*$ on $\mathfrak{g}_0$; and 3) $\mathrm{Ad}(\mathbf{u})$ commutes with $(\cdot)^*$. To continue, note that $X^{(\mathbf{la})^{-1}} \in \mathfrak{b}^-$ as $X \in \mathfrak{t}_w \subset \mathfrak{t}$. Then, using (2.25), $2\omega(\kappa(X^{\mathbf{w}_1^{-1}}) \wedge [g_0, y])$ equals

$$\begin{align} & \langle \mathcal{H}(X^{(\mathbf{la})^{-1}}), y^{\mathbf{u}} \rangle + \langle \mathcal{H}((X^{(\mathbf{la})^{-1}})^*), y^{\mathbf{u}} \rangle \notag \\ = \;& \langle -i(X^{(\mathbf{la})^{-1}})_-, y^{\mathbf{u}} \rangle + \langle i((X^{(\mathbf{la})^{-1}})^*)_+, y^{\mathbf{u}} \rangle \tag{2.26} \\ = \;& -\langle i(X^{(\mathbf{la})^{-1}})_-, y^{\mathbf{u}} \rangle - \langle ((iX^{(\mathbf{la})^{-1}})_-)^*, (y^{\mathbf{u}})^* \rangle \tag{2.27} \\ = \;& -2\mathrm{Re}\langle i(X^{(\mathbf{la})^{-1}})_-, y^{\mathbf{u}} \rangle. \tag{2.28} \end{align}$$

In the above, (2.27) is obtained from (2.26) using that complex anti-linear map $(\cdot)^*$ intertwines the projections to $\mathfrak{n}^+$ and $\mathfrak{n}^-$, and that $y^{\mathbf{u}} \in i\mathfrak{u}$, which is fixed by $(\cdot)^*$. Observe that $(X^{(\mathbf{la})^{-1}})_- = X^{(\mathbf{la})^{-1}} - X$ since $X^{(\mathbf{la})^{-1}} \in \mathfrak{b}^-$ and $(X^{(\mathbf{la})^{-1}})_0 = X$ since $X \in \mathfrak{t}_w \subset \mathfrak{t}$. With this, we continue from (2.28), finding that

$$\begin{align} \omega(\kappa(X^{\mathbf{w}_1^{-1}}) \wedge [g_0, y]) &= -\mathrm{Re}\langle i(X^{(\mathbf{la})^{-1}})_-, y^{\mathbf{u}} \rangle \notag \\ &= \mathrm{Re}\langle iX, y^{\mathbf{u}} \rangle - \mathrm{Re}\langle X^{(\mathbf{la})^{-1}}, y^{\mathbf{u}} \rangle \notag \\ &= \mathrm{Re}\langle iX, y^{\mathbf{u}} \rangle - \mathrm{Re}\langle iX, y^{\mathbf{w}_1 g_0} \rangle \tag{2.29} \\ &= \mathrm{Re}\langle iX, y^{\mathbf{u}} \rangle - \mathrm{Re}\langle X^{\mathbf{w}_1^{-1}}, iy^{g_0} \rangle \tag{2.30} \end{align}$$

making use in (2.29) and (2.30) of the factorization $\mathbf{w}_1 g_0 = \mathbf{lau}$. By ii. of part b) in Lemma 2.3, $X^{\mathbf{w}_1^{-1}} \in \mathfrak{g}_0$. Thus, the second term in (2.30) vanishes since iy^{g_0} is in $i\mathfrak{g}_0$, and the Killing form is purely imaginary on $\mathfrak{g}_0 \times i\mathfrak{g}_0$. Furthermore, the first Killing form pairing on the right hand side of (2.30) is real since iX and $y^{\mathbf{u}}$ are in $i\mathfrak{u}$. Thus, we have shown that

$$\omega(\kappa(X^{\mathbf{w}_1^{-1}}) \wedge [g_0, y]) = \langle iX, y^{\mathbf{u}} \rangle = \langle iX^{\mathbf{u}^{-1}}, y \rangle = d\Phi_X([g_0, y])$$

completing the proof of the theorem. □

3. The Noncompact Case

We will write X_0 for the non-compact symmetric space G_0/K. The Evens-Lu Poisson structure on X_0 is given by the formula

$$\Pi_{X_0}([g_0, \phi] \wedge [g_0, \psi]) = \langle \Omega(g_0)(\phi), \psi \rangle, \tag{3.1}$$

where

$$\Omega(g_0)(\phi) = \{Ad(g_0)^{-1} \circ \mathcal{H} \circ Ad(g_0)(\phi)\}_{\mathfrak{p}}. \tag{3.2}$$

Given the commutativity of the right diagram in (1.14), the transformation $\Omega(g_0)$ can be rewritten as

$$\mathfrak{p} \ni \phi \mapsto \Omega(g_0)(\phi) = \{(\mathrm{pr}_{\mathfrak{g}_0}(i\phi^{g_0}))^{g_0^{-1}}\}_{\mathfrak{p}} \tag{3.3}$$

which is how this formula first appeared in [Pi1]. (See section 5 of that paper for a derivation of this formula from the general construction in [EL].) In this section, we explicitly describe the geometry of the symplectic foliation for Π_{X_0}. This structure is regular and we can compute a Casimir. Lastly, we show that along the symplectic leaves in G_0/K, the two-form $\Pi_{X_0}^{-1}$ agrees with the restriction of the global two-form $\omega_{\mathbf{w}_1}$ (introduced in section 2) with the parameter $\mathbf{w}_1 = 1 \in U$.

We will write $g_0 = \mathbf{lau} = \mathbf{l}\mathbf{a}_0\mathbf{a}_1\mathbf{u}$ for the Iwasawa factorization of g_0 together with the further factorization $\mathbf{a} = \mathbf{a}_0\mathbf{a}_1$ where $\mathbf{a}_0 \in A_0$. It will be convenient to introduce an extension $\tilde{\Omega}(g_0)$ of (3.2) as in (34) of [Pi1]. Specifically,

$$\tilde{\Omega}(g_0) = \mathrm{Ad}(g_0)^{-1} \circ \mathrm{pr}_{\mathfrak{g}_0} \circ \mathrm{Ad}(g_0) \circ I \tag{3.4}$$

where the linear transformation $I\colon \mathfrak{g}_0 \to \mathfrak{u}$ is the identity on $\mathfrak{k}$ and multiplication by i on $i\mathfrak{p}$. Relative to the decomposition $\mathfrak{k} + \mathfrak{p}$,

$$\tilde{\Omega}(g_0) = \begin{pmatrix} 1 & b \\ 0 & \Omega \end{pmatrix} \tag{3.5}$$

for some linear transformation $b\colon \mathfrak{p} \to \mathfrak{k}$ depending on g_0. This extension can be factored as a composition of four operators

$$\begin{tikzcd} \mathfrak{g}_0 \arrow[r, "I"] & \mathfrak{u} \arrow[r, "\mathrm{Ad}(\mathbf{u})"] & \mathfrak{u} \arrow[r, "T(g_0)"] & \mathfrak{g}_0 \arrow[r, "\mathrm{Ad}((\mathbf{a}_0^{-1} g_0))^{-1}"] & \mathfrak{g}_0, \end{tikzcd} \tag{3.6}$$

where $T(g_0) = \mathrm{Ad}(\mathbf{a}_0^{-1}) \circ \mathrm{pr}_{\mathfrak{g}_0} \circ \mathrm{Ad}(\mathbf{la})$. Because $\mathbf{a}_0 \in G_0 \cap H$ the operator $\mathrm{Ad}(\mathbf{a}_0)$ commutes with σ and stabilizes the triangular decomposition. Hence, it also commutes with the Iwasawa projection to $\mathfrak{g}_0$, which means that $T(g_0) = \mathrm{pr}_{\mathfrak{g}_0} \circ \mathrm{Ad}(\mathbf{L})(x)$ where $\mathbf{L} = \mathbf{a}_0^{-1}\mathbf{l}\mathbf{a}_0\mathbf{a}_1$. The value of $T(g_0)$ on $X \in \mathfrak{u}$ can be written in two equivalent ways:

$$T(g_0)(X) = ((X^{\mathbf{L}})_+)^\sigma + (X^{\mathbf{L}})_{\mathfrak{h}_0} + (X^{\mathbf{L}})_+ \tag{3.7}$$

or (as in [Pi1])

$$T(g_0)(X) = (((X_+)^{\mathbf{L}})_+)^\sigma + X_{\mathfrak{t}_0} + ((X_+)^{\mathbf{L}})_{\mathfrak{h}_0} + ((X_+)^{\mathbf{L}})_+ \tag{3.8}$$

both of which will be of use in what follows.

Remark. In [Pi1], displayed line (37), $\mathbf{l}'$ should have been set equal to $\mathbf{l}' = \mathbf{a}_0^{-1}\mathbf{l}\mathbf{a}_0$ rather than $\mathbf{a}_0\mathbf{l}\mathbf{a}_0^{-1}$. However, this has no effect on the remaining results in that paper.

For later purposes, we now establish a number of facts about the operator $T(g_0)$.

Lemma 3.1. *For each $g_0 \in G_0$:*

a) $\ker(T(g_0)) = i\mathfrak{a}_0$.

b) The adjoint of $T(g_0)$, relative to the Killing forms on $\mathfrak{u}$ and $\mathfrak{g}_0$,

$$T^*(g_0)\colon \mathfrak{g}_0 \to \mathfrak{u}$$

is given by the formula

$$T^*(g_0)(y) = \frac{1}{2}\left(((y_{\mathfrak{h}_0} + 2y_-)^{\mathbf{L}^{-1}})_- + 2y_{\mathfrak{t}_0} - ((y_{\mathfrak{h}_0} + 2y_-)^{\mathbf{L}^{-1}})_-)^*\right). \tag{3.9}$$

c) The cokernel of $T(g_0)$ is

$$\ker(T^*(g_0)) = \left\{(y^{\mathbf{L}} - y) + 2y + (y^{\mathbf{L}} - y)^\sigma : y \in \mathfrak{a}_0\right\}.$$

d) The image of $T(g_0)$ consists of $y \in \mathfrak{g}_0$ such that the $\mathfrak{h}_0$ part of y is in the image of the following map.

$$\begin{array}{ccc} \mathfrak{t}_0 + \mathfrak{n}^+ & \to & \mathfrak{h}_0 \\ x_{\mathfrak{t}_0} + x_+ & \mapsto & x_{\mathfrak{t}_0} + \{(x_+)^{\mathbf{L}}\}_{\mathfrak{h}_0} \end{array} \tag{3.10}$$

Given such a $y \in \mathfrak{g}_0$, the solution of $T(x \in \mathfrak{u} \ominus i\mathfrak{a}_0) = y$ is solved in stages by

$$x_+ = ((y_+)^{\mathbf{L}^{-1}})_+,\ x_- = (x_+)^*$$

and

$$x_{\mathfrak{t}_0}=y_{\mathfrak{t}_0}-\{(x_+)^{\mathbf{L}}\}_{\mathfrak{t}_0}.$$

The moral is that it is easy to solve for $T(g_0)^{-1}$ *if one knows that there is a solution.*

Proof. For part (a), note that the containment $i\mathfrak{a}_0\subset\ker(T(g_0))$ is clear. For the reverse, we make use of the formula for $T(g_0)(x)$ in (3.8). From that formula one can see that if $T(x)=0$, then $(x_+)^{\mathbf{L}}=0$ from which it follows that $x_+=0$ and $x_{\mathfrak{t}_0}=0$. Since $x_-=-(x_+)^*$, it must be the case that $x\in i\mathfrak{a}_0$.

Let $x\in\mathfrak{u}$ and $y\in\mathfrak{g}_0$. Then $x_{\mathfrak{t}_0}=x_{\mathfrak{h}_0}$, and so the diagonal part of $T(g_0)(x)$ can written $x_{\mathfrak{t}_0}+((x_+)^{\mathbf{L}})_{\mathfrak{h}_0}=(x+(x_+)^{\mathbf{L}})_{\mathfrak{h}_0}$. Using (3.7) we compute

$$T^*\langle y,x\rangle=\langle Tx,y\rangle=\langle((x^{\mathbf{L}})_+)^\sigma,y_+\rangle+\langle(x^{\mathbf{L}})_{\mathfrak{h}_0},y_{\mathfrak{h}_0}\rangle+\langle(x^{\mathbf{L}})_+,y_-\rangle. \tag{3.11}$$

Since $y\in\mathfrak{g}_0$, $y_+=(y_-)^\sigma$ and thus the first term on the right hand side of (3.11) can be rewritten as $\langle((x_+)^{\mathbf{L}})^\sigma,(y_-)^\sigma\rangle$ which is equal to the conjugate of the third term on the right hand side of (3.11). Hence,

$$\langle T^*y,x\rangle=2\mathrm{Re}\langle(x^{\mathbf{L}})_+,y_-\rangle+\langle(x^{\mathbf{L}})_{\mathfrak{h}_0},y_{\mathfrak{h}_0}\rangle. \tag{3.12}$$

To compute the $\mathfrak{h}_0$-part of $x^{\mathbf{L}}$ we first take the orthogonal projection to $\mathfrak{g}_0$ and then compute the diagonal part. Since this term is paired with $y_{\mathfrak{h}_0}$ in the Killing form, this diagonal projection may be dropped. Therefore

$$2\langle(x^{\mathbf{L}})_{\mathfrak{h}_0},y_{\mathfrak{h}_0}\rangle=2\langle\{x^{\mathbf{L}}\}_{\mathfrak{g}_0},y_{\mathfrak{h}_0}\rangle=\langle x^{\mathbf{L}},y_{\mathfrak{h}_0}\rangle+\langle(x^{\mathbf{L}})^\sigma,(y_{\mathfrak{h}_0})^\sigma\rangle \tag{3.13}$$

using that $y_{\mathfrak{h}_0}$ is fixed by σ. It follows from (3.13) that

$$\langle(x^{\mathbf{L}})_{\mathfrak{h}_0},y_{\mathfrak{h}_0}\rangle=\mathrm{Re}\langle x^{\mathbf{L}},y_{\mathfrak{h}_0}\rangle=\mathrm{Re}\langle x,(y_{\mathfrak{h}_0})^{\mathbf{L}^{-1}}\rangle. \tag{3.14}$$

Now, because $L\in B^-$ and $y_{\mathfrak{h}_0}$ is diagonal, $(y_{\mathfrak{h}_0})^{\mathbf{L}^{-1}}=((y_{\mathfrak{h}_0})^{\mathbf{L}^{-1}})_-+y_{\mathfrak{h}_0}$. Continuing from (3.14) we have that

$$\langle(x^{\mathbf{L}})_{\mathfrak{h}_0},y_{\mathfrak{h}_0}\rangle=\mathrm{Re}\langle x_+,((y_{\mathfrak{h}_0})^{\mathbf{L}^{-1}})_-\rangle+\langle x_{\mathfrak{t}_0},y_{\mathfrak{t}_0}\rangle \tag{3.15}$$

since the real part of the pairing of x and $y_{\mathfrak{h}_0}$ is the pairing of their $\mathfrak{t}_0$-parts.

Next, we rewrite the first term of (3.11) as $2\mathrm{Re}\langle x_+,((y_-)^{\mathbf{L}^{-1}})\rangle$. Combining this with the result in (3.15), gives that $\langle T^*x,y\rangle$ equals

$$\frac{1}{2}\langle-(x_+)^*+x_{\mathfrak{t}_0}+x_+,((y_{\mathfrak{h}_0}+2y_-)^{\mathbf{L}^{-1}})_-+2y_{\mathfrak{t}_0}-((y_{\mathfrak{h}_0}+2y_-)^{\mathbf{L}^{-1}})_-)^*\rangle.$$

This calculation, together with part (a) establishes (b).

Now suppose $y\in\mathfrak{g}_0$ and $T^*(y)=0$. Then from part (b) we know that $y_{\mathfrak{t}_0}=0$, so $y_{\mathfrak{a}_0}=y_{\mathfrak{h}_0}$. Since the lower triangular part of $T^*(g_0)(y)$ must also vanish, we have that

$$\begin{aligned}0&=((y_{\mathfrak{h}_0}+2y_-)^{\mathbf{L}^{-1}})_-\\&=(y_{\mathfrak{a}_0}+2y_-)^{\mathbf{L}^{-1}}-y_{\mathfrak{a}_0}\\&=((y_{\mathfrak{a}_0})^{\mathbf{L}^{-1}})_-+2(y_-)^{\mathbf{L}^{-1}}+y_{\mathfrak{a}_0}-y_{\mathfrak{a}_0}.\end{aligned}$$

by the usual manipulations involving the action of $\mathbf{L}^{-1}$ on elements of $\mathfrak{h}$ and $\mathfrak{n}^-$. It follows that $-2y_-=(((y_{\mathfrak{a}_0})^{\mathbf{L}^{-1}})_-)^{\mathbf{L}}$. This can be further simplified:

$$-2y_-=(((y_{\mathfrak{a}_0})^{\mathbf{L}^{-1}})_-)^{\mathbf{L}}=((y_{\mathfrak{a}_0})^{\mathbf{L}^{-1}}-y_{\mathfrak{a}_0})^{\mathbf{L}})=-((y_{\mathfrak{a}_0})^{\mathbf{L}}-y_{\mathfrak{a}_0})$$

yeilding that y_- is determined by $y_{\mathfrak{a}_0}$. Specifically,

$$y_- = \tfrac{1}{2}((y_{\mathfrak{a}_0})^{\mathbf{L}} - y_{\mathfrak{a}_0}).$$

After rescaling y to $2y$, the description of the cokernel in (c) follows immediately.

The first part of (d) concerning the image of $T(g_0)$ is follows easily after examining the formula for $T(g_0)$ in (3.8). Let $x \in \mathfrak{u} \ominus i\mathfrak{a}_0$ and y be in the image of $T(g_0)$. Then

$$y_+ = ((x_+)^{\mathbf{L}})_+ = (x_+)^{\mathbf{L}} + z \tag{3.16}$$

where $z \in \mathfrak{b}^-$. Using that $\mathbf{L}^{-1} \in B^-$ we obtain that $((y_+)^{\mathbf{L}^{-1}})_+ = x_+$. Once x_+ is determined, we know that we can find $x_{\mathfrak{t}_0}$ by the equation

$$x_{\mathfrak{t}_0} = y_{\mathfrak{h}_0} - \{(x_+)^{\mathbf{L}}\}_{\mathfrak{h}_0} = y_{\mathfrak{t}_0} - \{(x_+)^{\mathbf{L}}\}_{\mathfrak{t}_0}. \tag{3.17}$$

This completes the proof of the lemma. □

Lemma 3.2. *The tangent vector* $[g_0, x]$ *is tangent to the symplectic leaf through* g_0K *if and only if* $\mathrm{Ad}(\mathbf{u})(x)$ *is perpendicular to* $\mathfrak{a}_0$ *relative to the Killing form on* $i\mathfrak{u}$.

Proof. As usual, write $g_0 = \mathbf{lau}$ for the Iwasawa factorization of g_0. The subspace tangent to the symplectic leaf through g_0K is the image of the anchor map $\Pi^{\#}_{X_0}\colon T^*(G_0/K) \to T(G_0/K)$ at g_0K. In terms of our working identifications

$$\Pi^{\#}_{X_0}([g_0, \phi]) = [g_0, \Omega(g_0)(\phi)].$$

Note that $\Omega(g_0) \in \mathfrak{so}(\mathfrak{p})$ for each $g_0 \in G_0$. Thus, its image is equal to the orthogonal complement of its kernel. It follows from (3.5) that there is an isomorphism $\ker(\Omega(g_0)) \ni \phi \mapsto (-b\phi, \phi) \in \ker(\tilde{\Omega}(g_0))$. The factorization of $\tilde{\Omega}(g_0)$ in (3.6) together with part (a) of Lemma 3.1 shows that

$$\ker(\tilde{\Omega}(g_0)) = I^{-1} \circ \mathrm{Ad}(\mathbf{u})^{-1}(\ker(T(g_0))) = I^{-1}(i\mathfrak{a}_0^{\mathbf{u}^{-1}}).$$

This lemma follows. □

Proposition 3.3.

a) The Poisson structure Π_{X_0} *is regular.*

b) The image of the anchor map $\Pi^{\#}_{X_0}\colon T^*(G_0/K) \to T(G_0/K)$ *defines a flat connection for the principal bundle*

$$A_0 \longrightarrow G_0/K \longrightarrow A_0\backslash G_0/K.$$

c) The symplectic leaves are the level sets of the function $\mathbf{a}_0$.

d) The horizontal parameterization for the symplectic leaf through the basepoint is given by the map $s\colon A_0\backslash G_0/K \to G_0/K$

$$A_0 g_0 K \to s(A_0 g_0 K) = \mathbf{a}_0^{-1} g_0 K \tag{3.18}$$

where $g_0 = \mathbf{l}\mathbf{a}_0\mathbf{a}_1\mathbf{u}$.

Proof. We continue to write $g_0 = \mathbf{lau}$ for the Iwaswa factorization of g_0. Essentially, parts (a) and (b) were established in [FO]. We supply alternative arguments here. For part (a) it sufficient to show that the map

$$\mathfrak{a}_0 \to \{\mathfrak{a}_0^{\mathbf{u}^{-1}}\}_{\mathfrak{p}} \tag{3.19}$$

is injective for each $g_0 \in G_0$. Let $x_0 \in \mathfrak{a}_0$, the $x_0^{\mathbf{u}^{-1}} \in i\mathfrak{u}$ and hence its projection to $\mathfrak{p}$ agrees with its projection to $\mathfrak{g}_0$. Thus,

$$\{x_0^{\mathbf{u}^{-1}}\}_{\mathfrak{p}} = (\{x_0^{\mathbf{la}}\}_{\mathfrak{g}_0})^{g_0^{-1}} = (\{(x_0^{\mathbf{la}})_- + x_0\}_{\mathfrak{g}_0})^{g_0^{-1}} \tag{3.20}$$

because $\mathbf{u}^{-1} = g_0^{-1}\mathbf{la}$ and $\mathrm{Ad}(g_0^{-1})$ commutes with the projection to $\mathfrak{g}_0$. Since

$$\{(x_0^{\mathbf{la}})_- + x_0\}_{\mathfrak{g}_0} = \tfrac{1}{2}((x_0^{\mathbf{la}})_- + 2x_0 + ((x_0^{\mathbf{la}})_-)^{\sigma}$$

it is clear that the right hand side of (3.20) is zero if and only if $x_0 = 0$. This proves (a).

Given part (a) it suffices to show infinitesimally that the A_0-orbits have trivial intersection with the symplectic leaves. Flatness of the connection then follows because the symplectic leaf distribution is integrable. The tangent space to the A_0-orbit through g_0K is

$$\{[g_0, \{\mathrm{Ad}(g_0^{-1})(y_0)\}_{\mathfrak{p}}] \colon y_0 \in \mathfrak{a}_0\}.$$

It is clear that at the basepoint this subspace intersects the subspace tangent to the symplectic leaf only at zero. In general, let $y_0 \in \mathfrak{a}_0$ and suppose that there exists $x \in \mathfrak{p}$ with $x^{\mathbf{u}} \perp \mathfrak{a}_0$ such that $\{y_0^{g_0^{-1}}\}_{\mathfrak{p}} = x$. Let $\kappa \in \mathfrak{k}$ be such that $\mathrm{Ad}(g_0^{-1})(y_0) = \kappa + x$. Given $z_0 \in \mathfrak{a}_0$, the pairing $\langle \mathrm{Ad}(\mathbf{u}g_0^{-1})(y_0), z_0\rangle$ can be written in two equivalent ways. On the one hand $\mathrm{Ad}(\mathbf{u}g_0^{-1})(y_0) = \mathrm{Ad}((\mathbf{la})^{-1})(y_0)$ so

$$\langle \mathrm{Ad}(\mathbf{u}g_0^{-1})(y_0), z_0\rangle = \langle \mathrm{Ad}((\mathbf{la})^{-1})(y_0), z_0\rangle = \langle y_0, z_0\rangle \tag{3.21}$$

since $y_0 \in \mathfrak{h}$ and $(\mathbf{la}) \in N^-$. On the other hand

$$\langle \mathrm{Ad}(\mathbf{u}g_0^{-1})(y_0), z_0\rangle = \langle \kappa^{\mathbf{u}} + x^{\mathbf{u}}, z_0\rangle = \langle \kappa^{\mathbf{u}}, z_0\rangle \tag{3.22}$$

since $x^{\mathbf{u}} \perp \mathfrak{a}_0$. The right hand side of (3.21) is real whereas the right hand side of (3.22) is purely imaginary since $\kappa^{\mathbf{u}} \in \mathfrak{u}$, so they must both be zero. This implies that x must be zero, proving (b).

For part (c), identify the tangent bundle to A_0 with $A_0 \times \mathfrak{a}_0$ using left translation. Then $d\mathbf{a}_0$ is identified with the $\mathfrak{a}_0$-valued one-form $[g_0, x] \mapsto \{x^{\mathbf{u}}\}_{\mathfrak{a}_0}$. From the preceeding lemma, it follows that the symplectic leaves are the level sets of $\mathbf{a}_0$.

We now turn to part (d). Let $a_0 \in A_0$.

$$a_0 g_0 = a_0 \mathbf{l} \mathbf{a}_0 \mathbf{a}_1 \mathbf{u} = \mathbf{l}^{a_0} a_0 \mathbf{a}_0 \mathbf{a}_1 \mathbf{u}$$

Since $\mathrm{Ad}(a_0)$ stabilizes N^-, it follows from the uniqueness of the Iwasawa decomposition that the A_0 factor of $a_0 g_0$ is $a_0\mathbf{a}_0$. This shows that the cross section (3.18) is well-defined. It remains to show that the image of s is horizontal.

Let ε be a small real parameter so that given $x \in \mathfrak{p}$, the map $\varepsilon \mapsto g_0 e^{\varepsilon x} K$ is a smooth curve passing through g_0K at $\varepsilon = 0$. Then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} s(g_0 e^{\varepsilon x} K) = [\mathbf{a}_0^{-1} g_0, x - \{(\{x^{\mathbf{u}}\}_{\mathfrak{a}_0})^{g_0^{-1}}\}_{\mathfrak{p}}].$$

To show that this is horizontal, we must check that the pairing

$$\langle (x - \{(\{x^{\mathbf{u}}\}_{\mathfrak{a}_0})^{g_0^{-1}}\}_{\mathfrak{p}})^{\mathbf{u}}, y_0\rangle \tag{3.23}$$

vanishes for $y_0 \in \mathfrak{a}_0$. Again, the projection to $\mathfrak{p}$ in (3.23) may be replaced with the projection to $i\mathfrak{u}$ as it is being applied to an element of $\mathfrak{g}_0$. Thus, (3.23) becomes

$$\langle x^{\mathbf{u}}, y_0\rangle - \langle \{(\{x^{\mathbf{u}}\}_{\mathfrak{a}_0})^{(\mathbf{la})^{-1}}\}_{i\mathfrak{u}}, y_0\rangle \tag{3.24}$$

using the factorization $g_0 = \mathbf{lau}$ and the fact that $\mathrm{Ad}(\mathbf{u})$ commutes with the projection to $i\mathfrak{u}$. Note that if $Z \in i\mathfrak{t} \subset i\mathfrak{u}$, then the diagonal part of $\{\mathrm{Ad}((\mathbf{la})^{-1})(Z)\}_{i\mathfrak{u}}$ is Z since $\mathbf{la} \in N^-$. Apply this observation to $Z = \{x^{\mathbf{u}}\}_{\mathfrak{a}_0}$ we have that the second term in 3.24) is

$$\langle \{(\{x^{\mathbf{u}}\}_{\mathfrak{a}_0})^{(\mathbf{la})^{-1}}\}_{i\mathfrak{u}}, y_0\rangle = \langle \{x^{\mathbf{u}}\}_{\mathfrak{a}_0}, y_0\rangle = \langle x^{\mathbf{u}}, y_0\rangle.$$

Therefore (3.23) vanishes. The proof is complete. □

Theorem 3.4. *Along the symplectic leaves,* $\Pi_{X_0}^{-1}$ *agrees with the restriction of the closed two-form* $\omega_{\mathbf{w}_1}$ *from (2.1) with* $\mathbf{w}_1 = 1$*.*

Proof. To compute the inverse of Π_{X_0} we make use of the extened operator (3.2) and the factorization (3.6). Suppose that $[g_0, x]$ and $[g_0, y]$ represent vectors tangent to the symplectic leaf of Π_{X_0} passing through g_0K. By lemma 3.2, we know that $\chi = \mathrm{Ad}(\mathbf{u})(x)$ is perpendicular to $\mathfrak{a}_0$. Thus, the $\mathfrak{h}_0$ part of $\mathrm{Ad}(\mathbf{a}_0^{-1}g_0)(x) = \mathrm{Ad}(\mathbf{a}_0^{-1}\mathbf{la})(\chi)$, which equals

$$\chi_0^{\mathbf{a}_1} + ((\chi_+)^{\mathbf{a}_0^{-1}\mathbf{la}})_0 \tag{3.25}$$

is in the image of the map (3.10). By Lemma 3.1, part (d), there exists a solution X to the equation $T(g_0)(X) = \mathrm{Ad}(\mathbf{a}_0^{-1}\mathbf{la})(\chi)$ with

$$X_+ = \chi_+,\ X_- = -(\chi_+)^* \tag{3.26}$$

and

$$X_{\mathfrak{t}_0} = \{\chi_0^{\mathbf{a}_1} + ((\chi_+)^{\mathbf{a}_0^{-1}\mathbf{la}}\}_{\mathfrak{t}_0} - \{(\chi_+)^{\mathbf{a}_0^{-1}\mathbf{la}}\}_{\mathfrak{t}_0} = \chi_{\mathfrak{t}_0}^{\mathbf{a}_1}. \tag{3.27}$$

Thus $X_{\mathfrak{t}_0} = \chi_{\mathfrak{t}_0}^{\mathbf{a}_1} = 0$ because $\chi \in i\mathfrak{u}$. From the factorization in (3.6) we have that $(\tilde{\Omega}(g_0))^{-1}(x) = I^{-1} \circ \mathrm{Ad}(\mathbf{u})(-(x^{\mathbf{u}})_- + (x^{\mathbf{u}})_+)$ and hence

$$\begin{aligned} \Pi_{X_0}^{-1}([g_0, x] \wedge [g_0, y]) &= \langle I^{-1} \circ \mathrm{Ad}(\mathbf{u})(-(x^{\mathbf{u}})_- + (x^{\mathbf{u}})_+), y\rangle \\ &= -\langle \mathrm{Ad}(\mathbf{u})^{-1} \circ \mathcal{H} \circ \mathrm{Ad}(\mathbf{u})(ix), iy\rangle \\ &= \omega_{\mathbf{w}_1}([g_0, x] \wedge [g_0, y]) \end{aligned}$$

with $\mathbf{w}_1 = 1 \in U$. □

4. The Compact Case

The Evens-Lu Poisson structure on $X = U/K$ is given by the formula

$$\Pi_X([u, \phi] \wedge [u, \psi]) = \langle \Omega(u)(\phi), \psi\rangle \tag{4.1}$$

where the linear transformation $\Omega(u) \colon i\mathfrak{p} \to i\mathfrak{p}$ is given by

$$\Omega(u) = \{\mathrm{Ad}(u)^{-1} \circ \mathcal{H} \circ \mathrm{Ad}(u)(\phi)\}_{i\mathfrak{p}}. \tag{4.2}$$

See [C] for a derivation of this formula for the Evens-Lu construction. Recall that G_0 acts from the right on U through the Iwasawa decomposition,

$$\begin{array}{ccc} U \times G_0 & \to & U \\ (u, g_0) & \mapsto & \mathbf{u}(ug_0) \end{array}$$

It was shown in [FL] that the symplectic leaves of Π_X are the projections of the G_0-orbits in U to U/K. Building on this work and that of [Pi1], a finer description was given in [C] using the connection with the Birkhoff decomposition.

Corresponding to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ there is a decomposition of the group

$$G = \coprod_{w \in W} \Sigma_w, \text{ where } \Sigma_w = N^- w H N^+$$

into submanifolds Σ_w whose codimension increases with the length of the indexing Weyl group element. The symmetric space X inherits a decomposition into the pre-images of the Σ_w under the composition

$$\begin{tikzcd} X = U/K \arrow[r] & U \arrow[r] & G \end{tikzcd}$$

where the first arrow is the Cartan embedding and the second is inclusion. As a variety in U, the image of the Cartan embedding is connected component containing the identity of $\{u^{-1} = u^\theta\} \subset U$. As in [C], the pre-image of Σ_w will be referred to as the layer of the Birkhoff decomposition indexed by w. Literally viewing the Weyl group $W = N_U(T)/T$ as the set of connected components of the normalizer of T in U one obtains that the layers of the Birkhoff decomposition of X are indexed by those elements $w \in W$ such that $w \cap \{u^{-1} = u^\theta\}_0 \neq \emptyset$. Each layer may consist of multiple connected components.

With the assumption that the triangular decomposition of $\mathfrak{g}$ is θ-stable, (as taken in this paper) the symplectic foliation of Π_X aligns with the Birkhoff decomposition. Each connected component of a given layer is foliated by contractible symplectic leaves. When restricted to a given layer, Π_X is regular. The torus $T_w = \{t \in T\colon w t^\theta w^{-1} = t\}$ (cf. section 2) acts on the layer indexed by w preserving the symplectic leaves. The action on each leaf is Hamiltonian and has a unique fixed point. The images of the T_w-fixed points under the Cartan embedding are the elements of the intersection of the image of the Cartan embedding with $w \subset U$. We thus label the symplectic leaves of (X, Π_X) by the representatives $\mathbf{w} \in w \cap \{u^{-1} = u^\theta\}_0$.

Notation. We will denote by $S(\mathbf{w})$ the symplectic leaf of (X, Π_X) corresponding to $\mathbf{w}$. When we write, "Let $S(\mathbf{w})$ be a symplectic leaf," we implicitly declare that $\mathbf{w}$ is in $N_U(T)$ and in the image of the Cartan embedding. By $\Pi_{\mathbf{w}}$ we denote the restriction of the Poisson tensor Π_X to the symplectic leaf $S(\mathbf{w})$.

Let $S(\mathbf{w})$ be a symplectic leaf. Fix a choice of $\mathbf{w}_1 \in U$ such that $\mathbf{w}_1 \mathbf{w}_1^{-\theta} = \mathbf{w}$. The map $\tilde{\mathbf{u}}\colon G_0 \to U$

$$g_0 \mapsto \mathbf{u}(\mathbf{w}_1 g_0)$$

is equivariant for the right actions of K on G_0 and U, invariant under the left action of $R(\mathbf{w}_1) = (N^- A)^{\mathbf{w}_1^{-1}} \cap G_0$ on G_0 and descends to a T_w-equivariant diffeomorphism

$$\tilde{\mathbf{u}}\colon R(\mathbf{w}_1)\backslash G_0/K \to S(\mathbf{w}). \tag{4.3}$$

The main result of this section is the following theorem.

Theorem 4.1. *Let $S(\mathbf{w})$ be a symplectic leaf. Fix a choice of $\mathbf{w}_1 \in U$ such that $\mathbf{w}_1 \mathbf{w}_1^{-\theta} = \mathbf{w}$. Then the map $\tilde{\mathbf{u}}$ induces an isomorphism of T_w-Hamiltonian spaces*

$$(R(\mathbf{w}_1)\backslash G_0/K, \omega_{\mathbf{w}_1}) \to (S(\mathbf{w}), \Pi_{\mathbf{w}}^{-1}) \tag{4.4}$$

where $\omega_{\mathbf{w}_1}$ is as in (2.1).

We remark here that there is a sense in which this result does not depend upon the choice of $\mathbf{w}_1$. Let $k \in K$. Note that conjugation by k^{-1} maps $R(\mathbf{w}_1)$ to $R(\mathbf{w}_1 k)$. The following diagram of isomorphisms commutes.

$$\begin{tikzcd}
(R(\mathbf{w}_1)\backslash G_0/K, \omega_{\mathbf{w}_1}) \arrow[r, "\mathbf{u}(\mathbf{w}_1(\cdot))"] \arrow[d, "conj(k^{-1})"'] & (S(\mathbf{w}), \Pi_{\mathbf{w}}^{-1}) \\
(R(\mathbf{w}_1 k)\backslash G_0/K, \omega_{\mathbf{w}_1 k}) \arrow[ur, "\mathbf{u}(\mathbf{w}_1 k(\cdot))"'] &
\end{tikzcd}$$

In order to prove theorem 4.1, we first compute an explicit formula for the symplectic structure $\Pi_{\mathbf{w}}^{-1}$ on $S(\mathbf{w})$. To do this, as in the noncompact case (cf. section 3), it will be convenient to introduce an extension $\tilde{\Omega}(u)$ of $\Omega(u)$ to all of $\mathfrak{u}$. Specifically,

$$\tilde{\Omega}(u) = \mathrm{Ad}(u)^{-1} \circ \mathrm{pr}_{\mathfrak{u}} \circ \mathrm{Ad}(u) \circ I,$$

where the linear transformation I is the identity on $\mathfrak{k}$ and multiplication by i on $i\mathfrak{p}$. This extension is the identity on $\mathfrak{k}$ and the commutativity of the left diagram in (1.14) implies that its compression to $i\mathfrak{p}$ agrees with $\Omega(u)$. With respect to the decomposition $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, $\tilde{\Omega}(u)$ has the form

$$\tilde{\Omega}(u) = \begin{pmatrix} 1 & b \\ 0 & \Omega(u) \end{pmatrix} \tag{4.5}$$

for some linear transformation $b \colon i\mathfrak{p} \to \mathfrak{k}$ depending on u. Furthermore,

$$\Pi_X([u,\phi] \wedge [u,\psi]) = \langle \tilde{\Omega}(u)(\phi), \psi \rangle. \tag{4.6}$$

Let $S(\mathbf{w})$ be a symplectic leaf and $uK \in S(\mathbf{w})$. Since $S(\mathbf{w})$ is the projection modulo K of a G_0-orbit in U, it is the case that $u = \mathbf{u}(\mathbf{w}_1 g_0)$ for some $g_0 \in G_0$ and some $\mathbf{w}_1 \in U$ such that $\mathbf{w}_1 \mathbf{w}_1^{-\theta} = \mathbf{w}$. Write $\mathbf{w}_1 g_0 = \mathbf{la}u$ for the Iwasawa factorization of $\mathbf{w}_1 g_0$. Given these choices, the operator $\tilde{\Omega}(u)$ admits a factorization as a composition of four operators

$$\mathfrak{u} \xrightarrow{I} \mathfrak{g}_0 \xrightarrow{\mathrm{Ad}(g_0)} \mathfrak{g}_0 \xrightarrow{T_{\mathbf{w}_1}(u)} \mathfrak{u} \xrightarrow{\mathrm{Ad}(u)^{-1}} \mathfrak{u} \tag{4.7}$$

where

$$T_{\mathbf{w}_1}(u) = \mathrm{pr}_{\mathfrak{u}} \circ \mathrm{Ad}((\mathbf{la})^{-1}\mathbf{w}_1). \tag{4.8}$$

The factorization in (4.7) implies that the kernel of $\tilde{\Omega}(u)$ is the inverse image under $\mathrm{Ad}(g_0) \circ I$ of $\ker(T_{\mathbf{w}_1}(u))$. As an operator on $\mathfrak{g}$, the right hand side of (4.8) has kernel

$$\mathrm{Ad}((\mathbf{la})^{-1}\mathbf{w}_1)^{-1}(\mathfrak{n}^- + \mathfrak{a}) = (\mathfrak{n}^- + \mathfrak{a})^{\mathbf{w}_1^{-1}}.$$

Thus, $\ker(T_{\mathbf{w}_1}(u)) = (\mathfrak{n}^- + \mathfrak{a})^{\mathbf{w}_1^{-1}} \cap \mathfrak{g}_0 = \mathfrak{r}(\mathbf{w}_1)$. Notice that $\mathfrak{r}(\mathbf{w}_1)$ is precisely the Lie algebra of the subgroup $R(\mathbf{w}_1)$ appearing in the statement of theorem (4.1).

Let $X \in \mathfrak{g}_0$. Using the identity $(X^{(\mathbf{la})^{-1}\mathbf{w}_1})_+ = (((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_+)$, it follows that

$$(T_{\mathbf{w}_1}(u)(X))_+ = (((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_+ \tag{4.9}$$

and

$$(T_{\mathbf{w}_1}(u)(X))_{\mathfrak{t}} = (X^{\mathbf{w}_1} + ((X^{\mathbf{w}_1})_+)^{(\mathbf{la})^{-1}})_{\mathfrak{t}}. \tag{4.10}$$

Since $T_{\mathbf{w}_1}(u)(X)$ is in $\mathfrak{u}$ its $\mathfrak{n}^-$-part is determined by its $\mathfrak{n}^+$-part using the involution $-(\cdot)^*$.

Lemma 4.2. *Assume the hypotheses of theorem 4.1. Given u representing $uK \in S(\mathbf{w})$ find $g_0 \in G_0$ such $u = \tilde{\mathbf{u}}(g_0) = \mathbf{u}(\mathbf{w}_1 g_0)$.*

a) The derivative of the $\tilde{\mathbf{u}}$-map is given by

$$\begin{array}{ccc} G_0 \times_K \mathfrak{p} & \to & U \times_K i\mathfrak{p} \\ [g_0, y] & \to & [u, \{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(-iy)\}_{i\mathfrak{p}}]. \end{array} \tag{4.11}$$

b) The adjoint of the derivative map in a) is given by

$$\begin{array}{ccc} U \times_K i\mathfrak{p} & \to & G_0 \times_K \mathfrak{p} \\ [u, \phi] & \to & [g_0, \{Ad(u^{-1}) \circ i\mathcal{H} \circ Ad(u)(\phi)\}_{\mathfrak{p}}]. \end{array} \tag{4.12}$$

c) The tangent space to $S(\mathbf{w})$ at uK is

$$\{[u, x] : ix^{g_0} \perp \mathfrak{r}(\mathbf{w}_1) \text{ in } \mathfrak{g}_0\}. \tag{4.13}$$

Proof. Let ε denote a small real parameter. A curve representing $[g_0, y]$ is $\varepsilon \mapsto g_0 e^{\varepsilon y} K$. A curve representing the image of $[g_0, y]$ under the derivative of $\tilde{\mathbf{u}}$ is

$$\varepsilon \mapsto \mathbf{u}(\mathbf{w}_1 g_0 e^{\varepsilon y}) K. \tag{4.14}$$

Now, $\mathbf{w}_1 g_0 e^{\varepsilon y} = \mathbf{la} u e^{\varepsilon y} = \mathbf{la} e^{\varepsilon y^u} u$. Therefore, the linearization at of (4.14) at $\varepsilon = 0$ is given by

$$[u, \{(\text{pr}_{\mathfrak{u}}(y^u))^{u^{-1}}\}_{i\mathfrak{p}}]$$

since $\mathbf{u}(\mathbf{w}_1 g_0 e^{\varepsilon y})$ equals $uu^{-1} \exp(\varepsilon \text{pr}_{\mathfrak{u}}(y^u))u$ to first order. The expression in (4.11) follows using the commutativity of the left diagram in (1.14).

Part b) follows from part a). For part c), we first observe that

$$T_{uK}(S(\mathbf{w})) = \{[u, \Omega(u)(x)] \colon x \in i\mathfrak{p}\}.$$

The range of $\Omega(u)$ agrees with the orthogonal complement of its kernel because $\Omega(u)$ is skew-symmetric on $i\mathfrak{p}$. In light of (4.5),

$$\ker(\Omega(u)) = \{\ker \tilde{\Omega}(u)\}_{i\mathfrak{p}} = i\{\text{Ad}(g_0^{-1})(\mathfrak{r}(\mathbf{w}_1))\}_{\mathfrak{p}}.$$

Let $x \in i\mathfrak{p}$ and let $Y \in \mathfrak{r}(\mathbf{w}_1)$. Then

$$0 = \langle i\{Y^{g_0^{-1}}\}_{\mathfrak{p}}, x\rangle = \langle Y^{g_0^{-1}}, ix\rangle = \langle Y, ix^{g_0}\rangle$$

which completes the proof. □

Throughout this section, we will write

$$\text{Ad}(\mathbf{w}) = \begin{pmatrix} A & 0 & B \\ 0 & w & 0 \\ C & 0 & D \end{pmatrix} \tag{4.15}$$

relative to $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$.

Lemma 4.3. *Assume the hypotheses of theorem 4.1. Given u representing $uK \in S(\mathbf{w})$ find $g_0 \in G_0$ such that $u = \tilde{\mathbf{u}}(g_0) = \mathbf{u}(\mathbf{w}_1 g_0)$. Let $[u, x]$ and $[u, y]$ represent tangent vectors to $S(\mathbf{w})$.*

If $\mathbf{w} = \mathbf{w}_1 = 1 \in U$ then

$$\Pi_1^{-1}([u, x] \wedge [u, y]) = \langle Ad(g_0)^{-1} \circ \mathcal{H} \circ \text{Ad}(g_0)(x), y\rangle. \tag{4.16}$$

In general

$$\Pi_{\mathbf{w}}^{-1}([u, x] \wedge [u, y]) = \langle \text{Ad}(\mathbf{w}_1 g_0)^{-1} \circ \mathcal{H}_{\mathbf{w}} \circ Ad(\mathbf{w}_1 g_0)(x), y\rangle \tag{4.17}$$

where $\mathcal{H}_{\mathbf{w}} : (Range(1 - C\sigma) \subset \mathfrak{g}_0^{\mathbf{w}_1}) \rightarrow \mathfrak{g}_0^{\mathbf{w}_1}$ *is given by*

$$\chi \mapsto -i\frac{1+C\sigma}{1-C\sigma}\chi_- + i\chi_+. \tag{4.18}$$

Proof. We will prove the general case, as the specific case follows from the fact that $C = 0$ when $\mathbf{w} = 1$. To compute $\Pi_{\mathbf{w}}^{-1}$ we invert $\tilde{\Omega}(u)$ on the complement of its kernel as (4.5) shows that the compression to $i\mathfrak{p}$ of $\tilde{\Omega}^{-1}(u)$ agrees with $\Omega^{-1}(u)$. From the factorization of $\tilde{\Omega}(u)$ in (4.7), it follows that

$$\begin{aligned} \Pi_{\mathbf{w}}^{-1}([u,x] \wedge [u,y]) &= \langle I^{-1} \circ \mathrm{Ad}(g_0)^{-1} \circ T_{\mathbf{w}_1}^{-1}(u) \circ \mathrm{Ad}(u)(x), y\rangle \\ &= \langle T_{\mathbf{w}_1}^{-1}(u)(x^u), iy^{g_0}\rangle \end{aligned} \tag{4.19}$$

where “$T_{\mathbf{w}_1}^{-1}(u)(x^u)$” denotes a solution $X \in \mathfrak{g}_0$ to the equation $T_{\mathbf{w}_1}(u)(X) = x^u$. Such a solution is unique modulo $\mathfrak{r}(\mathbf{w}_1)$.

Write $\chi = x^{\mathbf{w}_1 g_0} \in i\mathfrak{g}_0^{\mathbf{w}_1}$ and note that $x^u = \chi^{(\mathbf{la})^{-1}} \in \mathfrak{u}$. We seek $X \in \mathfrak{g}_0$ such that $T_{\mathbf{w}_1}(u)(x^u) = \chi^{(\mathbf{la})^{-1}}$. Using the formulas in (4.9) and (4.10), the equality of the $\mathfrak{n}^+$-components gives $(X^{\mathbf{w}_1})_+ = \chi_+$, and the equality of the $\mathfrak{h}$-components gives that $\{X^{\mathbf{w}_1}\}_{\mathfrak{t}} = \{\chi\}_{\mathfrak{t}}$. By c) of lemma 4.2, $ix^{g_0} \perp \mathfrak{r}(\mathbf{w}_1)$, thus $i\chi \perp ((\mathfrak{n}^- + \mathfrak{a}) \cap g_0^{\mathbf{w}_1})$. In particular, $\{i\chi\}_{\mathfrak{a}} = i\{\chi\}_{\mathfrak{t}} = 0$, and hence $\{X^{\mathbf{w}_1}\}_{\mathfrak{t}} = 0$.

We now know that $X^{\mathbf{w}_1} = L + d + \chi_+$ for some $L \in \mathfrak{n}_-$ and $d \in \mathfrak{g}_0^{\mathbf{w}_1} \cap \mathfrak{a}$. The fixed point set of $\mathrm{Ad}(\mathbf{w}) \circ \sigma$ is $\mathfrak{g}_0^{\mathbf{w}_1}$ (this follows from part a) of lemma 2.3). Thus, $\mathrm{Ad}(\mathbf{w}) \circ \sigma(X^{\mathbf{w}_1}) = X^{\mathbf{w}_1}$ because $X^{\mathbf{w}_1} \in \mathfrak{g}_0^{\mathbf{w}_1}$. Using the triangular decomposition of $X^{\mathbf{w}_1}$ and the matrix representation of $\mathrm{Ad}(\mathbf{w})$ in 4.15, this equation implies the following two equations for the $\mathfrak{n}^-$ and $\mathfrak{n}^+$-components of $X^{\mathbf{w}_1}$.

$$\mathrm{Ad}(L) = (1 - B\sigma)(\chi_+) \tag{4.20}$$

$$(1 - C\sigma)(L) = D\sigma(\chi_+) \tag{4.21}$$

The minus one eigenspace of $\mathrm{Ad}(\mathbf{w}) \circ \sigma$ on $\mathfrak{g}$ is $i\mathfrak{g}_0^{\mathbf{w}_1}$ which contains χ. The equation $\mathrm{Ad}(\mathbf{w}) \circ \sigma(\chi) = -\chi$ implies the following two equations for the $\mathfrak{n}^-$ and $\mathfrak{n}^+$-components of χ.

$$B\sigma\chi_+ = -\chi_+ - A\sigma\chi_- \tag{4.22}$$

$$D\sigma\chi_+ = -(1 + C\sigma)\chi_- \tag{4.23}$$

Equations (4.21) and (4.23) together imply that

$$(1 - C\sigma)(L) = -(1 + C\sigma)\chi_- \tag{4.24}$$

whereas (4.20) and (4.22) together give

$$A\sigma(L) = 2\chi_+ + A\sigma\chi_-.$$

The condition that $(1 - C\sigma)(L)$ and $A\sigma(L)$ both vanish is equivalent to the the statement that $L \in \mathfrak{n}^-$ belongs to $\mathfrak{g}_0^{\mathbf{w}_1}$. Thus (4.24) implies that

$$L = -\frac{1+C\sigma}{1-C\sigma}\chi_- \text{ modulo } \mathfrak{n}^- \cap \mathfrak{g}_0^{\mathbf{w}_1}.$$

Therefore $X = \mathrm{Ad}(\mathbf{w}_1)^{-1}(L + \chi_+) + d^{\mathbf{w}_1^{-1}}$ modulo $\mathfrak{n}^{\mathbf{w}_1^{-1}} \cap \mathfrak{g}_0$. Note that $d \in \mathfrak{a} \cap \mathfrak{g}_0^{\mathbf{w}_1}$, and thus $d^{\mathbf{w}_1^{-1}} \in \mathfrak{a}^{\mathbf{w}_1^{-1}} \cap \mathfrak{g}_0$. So, in fact,

$$X = \mathrm{Ad}(\mathbf{w}_1^{-1})(L + \chi_+) = \mathrm{Ad}(\mathbf{w}_1^{-1}) \circ \mathcal{H}_{\mathbf{w}}(-i\chi) \text{ modulo } \mathfrak{r}(\mathbf{w}_1)$$

where $\mathcal{H}_{\mathbf{w}}$ is as in (4.18). Substituting this for $T^{-1}(u)(x^u)$ in (4.19) completes the proof of the lemma. □

Lemma 4.4. $\mathbf{w}_1 g_0 \mathbf{u}(\mathbf{w}_1 g_0)^{-\theta} = \mathbf{w}(\mathbf{la})^{\sigma}$.

Proof. The left hand side equals

$$\mathbf{lauu}^{-\theta} = (\mathbf{la})((\mathbf{la})^{-1}\mathbf{w}(\mathbf{la})^{\sigma}).$$

□

First assume $\mathbf{w}_1 = 1$.

Lemma 4.5. *Suppose that* $x \in \mathfrak{p}$. *Then*

$$\begin{aligned}
& Ad(g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(x)\}_{\mathfrak{p}}). \\
= \quad & -i(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-) + \\
& +\sigma(-i(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-)) \\
= \quad & -i(\frac{1}{2}(Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)) - ((ix^{\mathbf{u}})_{\mathfrak{t}_0}) \\
& +\sigma(-i(\frac{1}{2}(Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)
\end{aligned}$$

The proof is a long calculation:

$$\begin{aligned}
& Ad(g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(x)\}_{\mathfrak{p}}) \\
= \quad & \frac{1}{2}Ad(g_0)(Ad(u)^{-1} \circ \mathcal{H} \circ Ad(u)(x) - Ad(u^{-\theta}) \circ \mathcal{H} \circ Ad(u^{\theta})(x^{\theta}))
\end{aligned}$$

(Note: $x^{\theta} = -x$, and use the Lemma (4.4))

$$= \quad \frac{1}{2}(Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{g_0}) + Ad((\mathbf{la})^{\sigma}) \circ \mathcal{H} \circ Ad((\mathbf{la})^{-\sigma})(x^{g_0}))$$

$$= \quad \frac{1}{2}(Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{g_0}) + \sigma \circ Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{g_0})) \tag{4.25}$$

(Note: $\sigma(x^{g_0}) = x^{g_0}$, and $\sigma \circ \mathcal{H} = \mathcal{H} \circ \sigma$)

$$\begin{aligned}
= \quad & \frac{1}{2}((Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H})(x^{g_0}) + \\
& \sigma \circ (Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H})(x^{g_0})) + \mathcal{H}(x^{g_0}).
\end{aligned}$$

Lemma 4.6. *Let* p_-, p_0, *and* p_+ *denote the projections corresponding to the triangular decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. *Then*

a)

$$Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H} = i\mathcal{Z}$$

where $\mathcal{Z} : \mathfrak{g} \to \mathfrak{n}^- + \mathfrak{h}$ *equals*

$$\begin{aligned}
& -p_0 \circ Ad(\mathbf{la})^{-1} \circ p_+ + 2p_- \circ Ad(\mathbf{la}) \circ p_+ \circ Ad(\mathbf{la})^{-1} \circ p_+ + \\
& p_- Ad(\mathbf{la}) \circ p_0 \circ Ad(\mathbf{la})^{-1} \circ p_+ + p_- \circ Ad(\mathbf{la}) \circ p_0.
\end{aligned}$$

b)

$$\mathcal{Z}(x^{g_0}) \quad = \quad -((x^{\mathbf{u}})_0 - (x^{g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+)_- + \tag{4.26}$$

$$(Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-. \tag{4.27}$$

Proof of Lemma 4.6. Relative to the triangular decomposition, written in the order $\mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$, $\mathrm{Ad}(\mathbf{la})$, $\mathcal{H}$, and $\mathrm{Ad}(\mathbf{la})^{-1}$ are represented as 3×3 matrices,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \alpha' & 1 & 0 \\ \gamma & \delta' & \delta \end{pmatrix}, \quad \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} A & 0 & 0 \\ A' & 1 & 0 \\ C & D' & D \end{pmatrix},$$

respectively, where $\alpha = p_+ \circ Ad(\mathbf{la}) \circ p_+$, etc. Note that

$$\alpha' A + A', \quad \gamma A + \delta' A' + \delta C, \quad \textit{and} \quad \delta' + \delta D'$$

all vanish. Then

$$\begin{aligned} \mathrm{Ad}(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H} \quad &= \begin{pmatrix} 0 & 0 & 0 \\ i\alpha' A & 0 & 0 \\ i(\gamma A - \delta C) & -i\delta D' & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 & 0 \\ i\alpha' A & 0 & 0 \\ i(2\gamma A + \delta' A') & -i\delta D' & 0 \end{pmatrix} \quad &= \begin{pmatrix} 0 & 0 & 0 \\ -iA' & 0 & 0 \\ i(2\gamma A + \delta' A') & i\delta' & 0 \end{pmatrix} \end{aligned}$$

This last expression is equivalent to (a).

For part (b),

$$\begin{aligned} & \mathcal{Z}(x^{g_0}) \\ = \quad & -p_0 \circ Ad(\mathbf{la})^{-1}((x^{g_0})_+) + 2p_- \circ Ad(\mathbf{la}) \circ p_+ \circ Ad(\mathbf{la})^{-1}((x^{g_0})_+) + \\ & p_- Ad(\mathbf{la}) \circ p_0 \circ Ad(\mathbf{la})^{-1}((x^{g_0})_+) + p_- \circ Ad(\mathbf{la})((x^{g_0})_0) \\ = \quad & -((x^{\mathbf{u}})_0 - (x^{g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+)_- + \\ & (Ad(\mathbf{la})((x^{\mathbf{u}})_0 - (x^{g_0})_0))_- + (Ad(\mathbf{la})((x^{g_0})_0)_- \\ = \quad & -((x^{\mathbf{u}})_0 - (x^{g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + \\ & (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-. \end{aligned}$$

□

Proof of Lemma 4.5. Lemma 4.6 implies that (4.25)

$$\begin{aligned} = \quad & \frac{1}{2}(i\mathcal{Z}(x^{g_0}) + \sigma(i\mathcal{Z}(x^{g_0})) + \mathcal{H}(x^{g_0}) \\ = \quad & \frac{i}{2}(-((x^{\mathbf{u}})_0 - (x^{g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- \\ & -\sigma(-((x^{\mathbf{u}})_0 - (x^{g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-)) \\ & +\mathcal{H}(x^{g_0}) \\ = \quad & \frac{i}{2}(-(x^{\mathbf{u}})_0 + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-) \\ & -\sigma(-(x^{\mathbf{u}})_0 + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-)) \\ & +\mathcal{H}(x^{g_0}). \end{aligned}$$

We now use

$$(\mathcal{H}(x^{g_0}))_- = -iAd(\mathbf{la})((x^{\mathbf{u}})_-) - i(Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- - i(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_-$$

and the fact that $\mathcal{H}(x^{g_0}) \in \mathfrak{g}_0$. When this latter expression is plugged into our calculation, there are cancellations, and we obtain

$$\begin{aligned} &= \frac{i}{2}(-(x^{\mathbf{u}})_0 - (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-) - iAd(\mathbf{la})((x^{\mathbf{u}})_-) \\ &\quad +\sigma(\frac{i}{2}(-(x^{\mathbf{u}})_0 - (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-) - iAd(\mathbf{la})((x^{\mathbf{u}})_-) \\ &= \frac{-i}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) - iAd(\mathbf{la})((x^{\mathbf{u}})_-) \\ &\quad +\sigma(\frac{-i}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) - iAd(\mathbf{la})((x^{\mathbf{u}})_-) \end{aligned}$$

This proves the Lemma. □

Proof of Theorem. We must now plug the first formula in Lemma (4.5) this into the first formula in Lemma (), our formula for the symplectic form when $\mathbf{w}_1 = 1$. The operator $\mathcal{H}$ kills zero modes. Also $\langle \sigma(v), w\rangle = \langle v, \sigma(w)\rangle^*$. Thus

$$\langle \mathcal{H} \circ Ad(g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(x)\}_{\mathfrak{p}}), Ad(g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(y)\}_{\mathfrak{p}})\rangle =$$

$$\langle \mathcal{H}(\frac{i}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + iAd(\mathbf{la})((x^{\mathbf{u}})_-)), \sigma(\frac{i}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0) + iAd(\mathbf{la})((y^{\mathbf{u}})_-)\rangle \\ +c.c. =$$

$$-i\langle \frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) - \frac{1}{2}(x^{\mathbf{u}})_0 + Ad(\mathbf{la})((x^{\mathbf{u}})_-), \frac{1}{2}Ad((\mathbf{la})^{\sigma})((y^{\mathbf{u}^{\theta}})_0) + Ad((\mathbf{la})^{\sigma})((y^{\mathbf{u}^{\theta}})_+)\rangle \\ +c.c. =$$

$$i\langle \frac{1}{2}(x^{\mathbf{u}})_0, \frac{1}{2}(y^{\mathbf{u}^{\theta}})_0\rangle - \\ i\langle \frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-), \frac{1}{2}Ad((\mathbf{la})^{\sigma})((y^{\mathbf{u}^{\theta}})_0) + Ad((\mathbf{la})^{\sigma})((y^{\mathbf{u}^{\theta}})_+)\rangle \\ +c.c. =$$

(use $x^{\mathbf{u}}, y^{\mathbf{u}^{\theta}} \in i\mathfrak{k} + \mathfrak{p}$, implying $(x^{\mathbf{u}})_0, (y^{\mathbf{u}^{\theta}})_0 \in \mathfrak{a}$ and the first inner product is real, hence cancels when added with its complex conjugate)

$$i\langle \frac{1}{2}(x^{\mathbf{u}})_0 + (x^{\mathbf{u}})_-, \frac{1}{2}Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_0) + Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_+)\rangle \\ +c.c. \tag{4.28}$$

First observe:

$$\begin{aligned} &i\langle \frac{1}{2}(x^{\mathbf{u}})_0, \frac{1}{2}Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_0) + Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_+)\rangle \\ &-i\langle \frac{1}{2}(x^{\mathbf{u}})_0, \frac{1}{2}Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_0) + Ad(\mathbf{uu}^{-\theta})((y^{\mathbf{u}^{\theta}})_-)\rangle \\ &\qquad = \langle \frac{1}{2}(x^{\mathbf{u}})_0, Ad(\mathbf{uu}^{-\theta})(\mathcal{H}(y^{\mathbf{u}^{\theta}}))\rangle \end{aligned}$$

This implies (4.28)

$$= \langle \frac{1}{2}(x^{\mathbf{u}})_0, \mathbf{uu}^{-\theta}\mathcal{H}(\mathbf{u}^{\theta}y)\rangle +$$

$$\begin{gathered}
\langle (x^{\mathbf{u}})_-, Ad(\mathbf{u}\mathbf{u}^{-\theta})(\frac{i}{2}(y^{\mathbf{u}^\theta})_0 + i(y^{\mathbf{u}^\theta})_+)\rangle + \\
\langle (x^{\mathbf{u}})_+, Ad(\mathbf{u}\mathbf{u}^{-\theta})(\frac{-i}{2}(y^{\mathbf{u}^\theta})_0 - i(y^{\mathbf{u}^\theta})_-)\rangle
\end{gathered}$$

$$\begin{gathered}
= \langle \frac{1}{2}(x^{\mathbf{u}})_0, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(\mathbf{u}^\theta y)\rangle + \\
-\frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), \mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_0\rangle + \\
\langle (x^{\mathbf{u}})_-, i\mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_+)\rangle + \\
\langle (x^{\mathbf{u}})_+, Ad(\mathbf{u}\mathbf{u}^{-\theta})(-i(y^{\mathbf{u}^\theta})_-)\rangle
\end{gathered}$$

$$\begin{gathered}
= \langle \frac{1}{2}(x^{\mathbf{u}})_0, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(\mathbf{u}^\theta y)\rangle + \\
-\frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), \mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_0\rangle + \\
\langle (x^{\mathbf{u}})_-, i\mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_+)\rangle - \langle (x^{\mathbf{u}})_-, i\mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_-\rangle \\
\langle (x^{\mathbf{u}})_+, Ad(\mathbf{u}\mathbf{u}^{-\theta})(-i(y^{\mathbf{u}^\theta})_-)\rangle + \langle (x^{\mathbf{u}})_-, i\mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_-\rangle
\end{gathered}$$

$$\begin{gathered}
= \langle \frac{1}{2}(x^{\mathbf{u}})_0, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(\mathbf{u}^\theta y)\rangle + \\
-\frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), \mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_0\rangle + \\
\langle (x^{\mathbf{u}})_-, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(y^{\mathbf{u}^\theta}))\rangle \\
-\langle \mathcal{H}(x^{\mathbf{u}}), Ad(\mathbf{u}\mathbf{u}^{-\theta})((y^{\mathbf{u}^\theta})_-)\rangle
\end{gathered}$$

$$\begin{gathered}
= \langle \frac{1}{2}(x^{\mathbf{u}})_0, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(\mathbf{u}^\theta y)\rangle + \\
-\frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), \mathbf{u}\mathbf{u}^{-\theta}(y^{\mathbf{u}^\theta})_0\rangle + \\
\frac{1}{2}\langle (x^{\mathbf{u}})_- + (x^{\mathbf{u}})_+, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(y^{\mathbf{u}^\theta}))\rangle \\
-\frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), Ad(\mathbf{u}\mathbf{u}^{-\theta})((y^{\mathbf{u}^\theta})_- + ((y^{\mathbf{u}^\theta})_+)\rangle
\end{gathered}$$

$$= \frac{1}{2}\langle x^{\mathbf{u}}, \mathbf{u}\mathbf{u}^{-\theta}\mathcal{H}(y^{\mathbf{u}^\theta}))\rangle - \frac{1}{2}\langle \mathcal{H}(x^{\mathbf{u}}), Ad(\mathbf{u}\mathbf{u}^{-\theta})(y^{\mathbf{u}^\theta})\rangle$$

This implies the Theorem in the case $\mathbf{w}_1 = 1$.

□

We now consider the modifications necessary to deal with a general $\mathbf{w}_1$.

Lemma 4.7. *Suppose that $x \in \mathfrak{p}$. Then*

$$\begin{gathered}
Ad(\mathbf{w}_1 g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(x)\}_{\mathfrak{p}}). \\
= -i(1 - C\sigma)(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-) - ((ix^{\mathbf{u}})_{t_{\mathbf{w}}}) \\
+iA\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)
\end{gathered}$$

We calculate:

$$Ad(\mathbf{w}_1 g_0)(\{Ad(u^{-1}) \circ \mathcal{H} \circ Ad(u)(x)\}_{\mathfrak{p}})$$

$$= \frac{1}{2} Ad(\mathbf{w}_1 g_0)(Ad(u)^{-1} \circ \mathcal{H} \circ Ad(u)(x) - Ad(u^{-\theta}) \circ \mathcal{H} \circ Ad(u^{\theta})(x^{\theta}))$$

(Note: $x^{\theta} = -x$, and use the Lemma)

$$= \frac{1}{2}(Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{\mathbf{w}_1 g_0}) + Ad(\mathbf{w}(\mathbf{la})^{\sigma}) \circ \mathcal{H} \circ Ad((\mathbf{la})^{-\sigma})(x^{\mathbf{w}_1^{\sigma} g_0}))$$

$$= \frac{1}{2}(Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{\mathbf{w}_1 g_0}) + Ad(\mathbf{w}) \circ \sigma \circ Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1}(x^{\mathbf{w}_1 g_0}))$$

(Note: $\sigma(x^{g_0}) = x^{g_0}$, and $\sigma \circ \mathcal{H} = \mathcal{H} \circ \sigma$)

$$= \frac{1}{2}((Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H})(x^{\mathbf{w}_1 g_0}) +$$

$$Ad(\mathbf{w}) \circ \sigma \circ (Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H})(x^{\mathbf{w}_1 g_0})) + \frac{1}{2}(\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(x^{\mathbf{w}_1 g_0}).$$

Lemma 4.8. *Let p_-, p_0, and p_+ denote the projections corresponding to the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Then*

(a)

$$Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H} = i\mathcal{Z}$$

where $\mathcal{Z} : \mathfrak{g} \to \mathfrak{n}^- + \mathfrak{h}$ equals

$$-p_0 \circ Ad(\mathbf{la})^{-1} \circ p_+ + 2p_- \circ Ad(\mathbf{la}) \circ p_+ \circ Ad(\mathbf{la})^{-1} \circ p_+ +$$

$$p_- Ad(\mathbf{la}) \circ p_0 \circ Ad(\mathbf{la})^{-1} \circ p_+ + p_- \circ Ad(\mathbf{la}) \circ p_0.$$

(b)

$$\mathcal{Z}(x^{\mathbf{w}_1 g_0}) = -((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+)_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-.$$

(c) For an element $\chi \in \mathfrak{g}_0^{\mathbf{w}_1}$,

$$\frac{1}{2}((\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(\chi))$$

$$= -i(1 - C\sigma)\chi_- + iA\sigma(\chi_-).$$

Proof. Relative to the triangular decomposition, written in the order $\mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$, $Ad(\mathbf{la})$, $\mathcal{H}$, and $Ad(\mathbf{la})^{-1}$ are represented as 3×3 matrices,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \alpha' & 1 & 0 \\ \gamma & \delta' & \delta \end{pmatrix}, \quad \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} A & 0 & 0 \\ A' & 1 & 0 \\ C & D' & D \end{pmatrix},$$

respectively, where $\alpha = p_+ \circ Ad(\mathbf{la}) \circ p_+$, etc. Note that

$$\alpha' A + A', \quad \gamma A + \delta' A' + \delta C, \quad \text{and} \quad \delta' + \delta D'$$

all vanish. Then

$$Ad(\mathbf{la}) \circ \mathcal{H} \circ Ad(\mathbf{la})^{-1} - \mathcal{H} = \begin{pmatrix} 0 & 0 & 0 \\ i\alpha' A & 0 & 0 \\ i(\gamma A - \delta C) & -i\delta D' & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ i\alpha' A & 0 & 0 \\ i(2\gamma A + \delta' A') & -i\delta D' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -iA' & 0 & 0 \\ i(2\gamma A + \delta' A') & i\delta' & 0 \end{pmatrix}$$

This last expression is equivalent to (a).

For part (b),

$$\begin{gathered}
\mathcal{Z}(x^{\mathbf{w}_1 g_0}) = \\
-p_0 \circ Ad(\mathbf{la})^{-1}((x^{\mathbf{w}_1 g_0})_+) + 2p_- \circ Ad(\mathbf{la}) \circ p_+ \circ Ad(\mathbf{la})^{-1}((x^{\mathbf{w}_1 g_0})_+) + \\
p_- Ad(\mathbf{la}) \circ p_0 \circ Ad(\mathbf{la})^{-1}((x^{\mathbf{w}_1 g_0})_+) + p_- \circ Ad(\mathbf{la})((x^{\mathbf{w}_1 g_0})_0)
\end{gathered}$$

$$\begin{gathered}
= -((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+)_- + \\
(Ad(\mathbf{la})((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0))_- + (Ad(\mathbf{la})((x^{\mathbf{w}_1 g_0})_0)_-
\end{gathered}$$

$$\begin{gathered}
= -((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + \\
(Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- .
\end{gathered}$$

Part (c) follows from the identities (4.22) and (4.23) (with the sign on the RHS changed, because χ in those identities is in $i\mathfrak{g}_0^{\mathbf{w}_1}$). ☐

(Continuation of Proof of Lemma(4.8)). Returning to our calculation

$$\begin{gathered}
= \frac{1}{2}(i\mathcal{Z}(x^{\mathbf{w}_1 g_0}) + Ad(\mathbf{w}) \circ \sigma(i\mathcal{Z}(x^{\mathbf{w}_1 g_0})) + \frac{1}{2}(\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(x^{\mathbf{w}_1 g_0}) \\
= \frac{i}{2}(-((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- \\
-Ad(\mathbf{w}) \circ \sigma(-((x^{\mathbf{u}})_0 - (x^{\mathbf{w}_1 g_0})_0) + 2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0))_-)) \\
+\frac{1}{2}(\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(x^{\mathbf{w}_1 g_0}) \\
= \frac{i}{2}(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-) - i(x^{\mathbf{u}})_{\mathfrak{t}_w} \\
-Ad(\mathbf{w}) \circ \sigma(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-)) \\
+\frac{1}{2}(\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(x^{\mathbf{w}_1 g_0}).
\end{gathered}$$

$$\begin{gathered}
= \frac{i}{2}(1 - C\sigma)(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-) - i(x^{\mathbf{u}})_{\mathfrak{t}_w} \\
-\frac{i}{2}A\sigma(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-)) \\
+\frac{1}{2}(\mathcal{H} + Ad(\mathbf{w}) \circ \sigma \circ \mathcal{H})(x^{\mathbf{w}_1 g_0}).
\end{gathered}$$

(Using (c) of the lemma(4.9))

$$\begin{gathered}
= \frac{i}{2}(1 - C\sigma)(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-) - i(x^{\mathbf{u}})_{\mathfrak{t}_w} \\
-\frac{i}{2}A\sigma(2(Ad(\mathbf{la})((x^{\mathbf{u}})_+))_- + Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-)) + \\
(-i(1 - C\sigma) + iA\sigma)(Ad(\mathbf{la})((x^{\mathbf{u}})_-) + (Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + (Ad(\mathbf{la})((x^{\mathbf{u}})_+))_-)
\end{gathered}$$

$$\begin{gathered}
= -\frac{i}{2}(1 - C\sigma)((Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-) - i(x^{\mathbf{u}})_{\mathfrak{t}_w} \\
+\frac{i}{2}A\sigma(Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-)) \\
-i(1 - C\sigma)(Ad(\mathbf{la})((x^{\mathbf{u}})_-)) + \\
iA\sigma(Ad(\mathbf{la})((x^{\mathbf{u}})_-))
\end{gathered}$$

This proves the Lemma. ☐

Proof of Theorem.

$$\begin{aligned}
&Ad(\mathbf{w}_1 g_0)(-i\{Ad(u^{-1})\circ\mathcal{H}\circ Ad(u)(y)\}_{\mathfrak{p}})\\
&= -(1-C\sigma)(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-) - ((iy^{\mathbf{u}})_{t_{\mathbf{w}}})\\
&+A\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\\
&= \chi(y)
\end{aligned}$$

$$\mathcal{H}_{\mathbf{w}}\chi = i\chi_+ - i\frac{1+C\sigma}{1-C\sigma}\chi_- = i\chi_+ + iL$$

$$\begin{aligned}
&\mathcal{H}_{\mathbf{w}}\circ Ad(\mathbf{w}_1 g_0)(-i\{Ad(u^{-1})\circ\mathcal{H}\circ Ad(u)(x)\}_{\mathfrak{p}})\\
&= i\frac{1+C\sigma}{1-C\sigma}((1-C\sigma)(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-))\\
&+iA\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)
\end{aligned}$$

We calculate:

$$\langle\mathcal{H}_{\mathbf{w}}\circ Ad(\mathbf{w}_1 g_0)(-i\{Ad(u^{-1})\circ\mathcal{H}\circ Ad(u)(x)\}_{\mathfrak{p}}), Ad(\mathbf{w}_1 g_0)(-i\{Ad(u^{-1})\circ\mathcal{H}\circ Ad(u)(y)\}_{\mathfrak{p}})\rangle =$$

$$\begin{aligned}
&= i\langle(1+C\sigma)(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
&A\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
&-i\langle A\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-),\\
&(1-C\sigma)(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle\\
&= I + II
\end{aligned}$$

where

$$\begin{aligned}
I = {} & i\langle(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
&A\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
&-i\langle A\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-),\\
&(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle
\end{aligned}$$

$$\begin{aligned}
II = {} & i\langle C\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
&A\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
&-i\langle A\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-),\\
&-C\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle
\end{aligned}$$

We will show below that II vanishes.

In evaluating I, we initially replace $A\sigma$ by $p_+ \circ Ad(\mathbf{w}) \circ \sigma$. We can omit the p_+, because $\langle \mathfrak{n}^-, \mathfrak{h} + \mathfrak{n}^- \rangle = 0$. We also write

$$(Ad(\mathbf{la})((x^{\mathbf{u}})_0))_- = Ad(\mathbf{la})((x^{\mathbf{u}})_0) - (x^{\mathbf{u}})_0,$$

and similarly for y. We can write

$$\begin{gathered}
I = i\langle(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
-i\langle Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-),\\
(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0) + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle
\end{gathered}$$

because

$$\begin{gathered}
-i(\langle(\frac{1}{2}(x^{\mathbf{u}})_0), Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
-i\langle Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_- + Ad(\mathbf{la})((x^{\mathbf{u}})_-), (\frac{1}{2}(y^{\mathbf{u}})_0)\rangle) = 0.
\end{gathered}$$

We can further simplify

$$\begin{gathered}
I = i\langle(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0) + Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle\\
-i\langle Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-),\\
(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0) + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle
\end{gathered}$$

because

$$\begin{gathered}
i\langle(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0) + Ad(\mathbf{la})((x^{\mathbf{u}})_-)),\\
Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}(y^{\mathbf{u}})_0)\rangle\\
-i\langle Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)),\\
(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0) + Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle
\end{gathered}$$

$$\begin{gathered}
= i\langle(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)),\\
Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}(y^{\mathbf{u}})_0)\rangle\\
-i\langle Ad(\mathbf{w}) \circ \sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)),\\
(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0))\rangle = 0
\end{gathered}$$

We now see that

$$I = i\langle(\frac{1}{2}(x^{\mathbf{u}})_0 + (x^{\mathbf{u}})_-),$$

$$Ad(\mathbf{uu}^{-\theta})(\frac{1}{2}((y^{\mathbf{u}^{\theta}})_0)+((y^{\mathbf{u}^{\theta}})_+)\rangle$$

$$-i\langle Ad(\mathbf{uu}^{-\theta})(\frac{1}{2}((x^{\mathbf{u}^{\theta}})_0)+((x^{\mathbf{u}^{\theta}})_+),$$

$$(\frac{1}{2}((y^{\mathbf{u}})_0)+((y^{\mathbf{u}})_-))\rangle$$

(apply θ to the second Killing form pairing, and use invariance)

$$I=i\langle(\frac{1}{2}(x^{\mathbf{u}})_0+(x^{\mathbf{u}})_-),$$

$$Ad(\mathbf{uu}^{-\theta})(\frac{1}{2}((y^{\mathbf{u}^{\theta}})_0)+((y^{\mathbf{u}^{\theta}})_+)\rangle$$

$$-i\langle(\frac{1}{2}((x^{\mathbf{u}})_0)+((x^{\mathbf{u}})_+),$$

$$Ad(\mathbf{uu}^{-\theta})(\frac{1}{2}((y^{\mathbf{u}^{\theta}})_0)+((y^{\mathbf{u}^{\theta}})_-))\rangle$$

This now is the same as $(*)$ in the $\mathbf{w}_1=1$ case. So the same argument proves the Theorem, assuming that we can show that II vanishes.

$$(-i)II=\langle C\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((x^{\mathbf{u}})_-)),$$

$$A\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle$$

$$+\langle A\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((x^{\mathbf{u}})_-),$$

$$C\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle$$

$$=\langle C\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((x^{\mathbf{u}})_-)),$$

$$Ad(\mathbf{w})\circ\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle$$

$$+\langle A\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((x^{\mathbf{u}})_-),$$

$$Ad(\mathbf{w})\circ\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((y^{\mathbf{u}})_-))\rangle$$

$$=\langle Ad(\mathbf{w})\circ\sigma(\frac{1}{2}Ad(\mathbf{la})((x^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((x^{\mathbf{u}})_-)),$$

$$Ad(\mathbf{w})\circ\sigma(\frac{1}{2}Ad(\mathbf{la})((y^{\mathbf{u}})_0)_-+Ad(\mathbf{la})((y^{\mathbf{u}})_-)\rangle=0.$$

□

5. The Group Case

Let K be a simply connected compact Lie group. With respect to the invariant metric induced by the Killing form, K may be viewed as a compact symmetric space X. In this case, the diagram in (0.1) specializes to

$$\begin{tikzcd}
& G = K^{\mathbb{C}} \times K^{\mathbb{C}} & \\
K^{\mathbb{C}} \simeq G_0 \arrow[ur] & & U = K \times K \arrow[ul] \\
& \Delta(K) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where $\Delta(K) = \{(k,k) \colon k \in K\}$ is the diagonal embedding of K in U, and $G_0 = \{g_0 = (g, g^{-*}) \colon g \in K^{\mathbb{C}}\}$. The involution θ in this case is the outer automorphism $\theta((g_1, g_2)) = (g_2, g_1)$. Here

$$X_0 = G_0/\Delta(K) \simeq K^{\mathbb{C}}/K, \text{ and } X = U/\Delta(K) \simeq K,$$

where the latter isometry is $(k_1, k_2)\Delta(K) \mapsto k = k_1 k_2^{-1}$.

To distinguish between $\mathfrak{g}$ and $\mathfrak{k}^{\mathbb{C}}$, we will adopt the (admittedly cumbersome) convention of denoting structures associated with $\mathfrak{k}^{\mathbb{C}}$ using superchecks.

We fix a triangular decomposition

$$\check{\mathfrak{g}} = \mathfrak{k}^{\mathbb{C}} = \check{\mathfrak{n}}_- + \check{\mathfrak{h}} + \check{\mathfrak{n}}_+. \tag{5.1}$$

This induces a θ-stable triangular decomposition for $\mathfrak{g}$

$$\mathfrak{g} = \underbrace{(\check{\mathfrak{n}}^- \times \check{n}^-)}_{\mathfrak{n}^-} + \underbrace{(\check{\mathfrak{h}} \times \check{\mathfrak{h}})}_{\mathfrak{h}} + \underbrace{(\check{\mathfrak{n}}^+ \times \check{\mathfrak{n}}^+)}_{\mathfrak{n}^+} \tag{5.2}$$

Let $\check{\mathfrak{a}} = \check{\mathfrak{h}}_{\mathbb{R}}$ and $\check{\mathfrak{t}} = i\check{\mathfrak{a}}$. Then

$$\mathfrak{t}_0 = \{(x,x) : x \in \check{\mathfrak{t}}\}, \quad \text{and} \quad \mathfrak{a}_0 = \{(y,-y) : y \in \check{\mathfrak{a}}\}.$$

The standard Poisson Lie groups structure on $U = K \times K$ induced by the decomposition in (5.2) is then the product Poisson Lie group structure for the standard Poisson Lie group structure on K induced by the decomposition (5.1).

Let us denote the Poisson Lie group structure on K by π_K and the Evens-Lu homogeneous Poisson structure on $X = K$ by Π_X. The identification of $\mathfrak{k}$ with its dual via the Killing form allows us to view the Hilbert transform $\check{\mathcal{H}}$ associated to (5.1) as an element of $\mathfrak{k} \wedge \mathfrak{k}$. As a bivector field

$$\pi_K = \check{\mathcal{H}}^r - \check{\mathcal{H}}^l$$

where $\check{\mathcal{H}}^r$ (resp. $\check{\mathcal{H}}^l$) denotes the right (resp. left) invariant bivector field on K generated by $\check{H}$. Whereas $\Pi_K = \check{\mathcal{H}}^r + \check{\mathcal{H}}^l$ as shown in [C]. The following proposition illustrates the relationship between these two structures.

Proposition 5.1. *Let $w_0 \in N_K(T)$ be a representative for the longest element of the Weyl group. The map $L_{w_0} \colon K \to K$*

$$K \ni k \mapsto w_0 k \in K \tag{5.3}$$

is a Poisson diffeomorphism carrying the Poisson Lie group structure π_K *onto (the negative of) the Evens-Lu* $(K \times K, \pi_K \oplus \pi_K)$*-homogeneous Poisson structure* Π_X *on* $X = K$.

Proof. Identify the dual of $\mathfrak{k}$ with $\mathfrak{k}$ using the Killing form, and use right translation to trivialize T^*K as $K \times \mathfrak{k}$. From [C], we have that

$$\pi_K((k,\phi),(k,\psi)) = \langle(\check{\mathcal{H}} - \mathrm{Ad}(k) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(k)^{-1})(\phi), \psi\rangle \tag{5.4}$$

and

$$\Pi_K((k,\phi),(k,\psi)) = \langle(\check{\mathcal{H}} + \mathrm{Ad}(k) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(k)^{-1})(\phi), \psi\rangle \tag{5.5}$$

for each (k,ϕ) and (k,ψ) representing contangent vectors at $k \in K$. With the tangent bundle to K trivialized as $K \times \mathfrak{k}$ using right translation, the derivative of (5.3) is $(k, X) \mapsto (w_0 k, \mathrm{Ad}(\mathbf{w}_0)(X))$ and the transpose map is

$$(w_0 k, \phi) \mapsto (k, \mathrm{Ad}(w_0)^{-1}(\phi)).$$

Then $L_{w_0 *}\pi_K((w_0 k,\phi),(w_0 k,\psi))$ is

$$\begin{aligned} &= \langle \check{\mathcal{H}} \circ \mathrm{Ad}(w_0)^{-1}(\phi), \mathrm{Ad}(w_0)^{-1}(\psi)\rangle \\ &\quad -\langle \mathrm{Ad}(k) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(k)^{-1} \circ \mathrm{Ad}(w_0)^{-1}(\phi), \mathrm{Ad}(w_0)^{-1}(\psi)\rangle \\ &= \langle \mathrm{Ad}(w_0) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(w_0)^{-1}(\phi), \psi\rangle \\ &\quad -\langle \mathrm{Ad}(w_0 k) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(\mathbf{w}_0 k)^{-1}(\phi), \psi\rangle. \end{aligned} \tag{5.6}$$

The operator $\check{\mathcal{H}}$ is conjugated to $-\check{\mathcal{H}}$ by $\mathrm{Ad}(w_0)$ as conjugation by $\mathbf{w}_0$ interchanges $\check{\mathfrak{n}}^-$ and $\check{\mathfrak{n}}^+$. Thus (5.6) becomes

$$\begin{aligned} &-\langle(\check{\mathcal{H}} + \mathrm{Ad}(w_0 k) \circ \check{\mathcal{H}} \circ \mathrm{Ad}(w_0 k)^{-1})(\phi), \psi\rangle \\ = \; &-\Pi_K((w_0 k,\phi),(w_0 k,\psi)). \end{aligned}$$

The proof is complete. □

The symplectic leaves of Π_K foliate the strata of the Birkoff decomposition of K induced by (5.1). The top stratum, Σ_1^K consists of those elements admitting a unique factorization as $k = lmau$ where $l \in \check{N}^-$, $m \in \exp(\check{t}) = \check{T}$, $a \in \exp(\check{a}) = \check{A}$ and $u \in \check{N}^+$. This stratum is an open dense subset of K. The symplectic leaf through the identity, $S(1)$, consists of those elements whose factorization has $m = 1$. We will write $\{\Sigma_1^K : m = 1\}$ for this leaf in the following discussion of the Hamiltonian system in this generic component.

The isomorphism in the theorem of section 4 is given by

$$\check{A}\backslash K^{\mathbb{C}}/K \to \{\Sigma_1^K : m = 1\} : \check{A}gK \to k = \check{\mathbf{a}}^{-1}\check{\mathbf{l}}^{-1}(\check{\mathbf{a}}^{-1}\check{\mathbf{l}}^{-1})^{*\theta}$$

where $g = \check{\mathbf{l}}\check{\mathbf{a}}\check{\mathbf{k}}$. In this group case there is an obvious global coordinate

$$\check{A}\backslash K^{\mathbb{C}}/K \leftrightarrow \check{N}^- \leftrightarrow \{\Sigma_1^K : m = 1\} : \check{A}\check{\mathbf{l}}K \leftrightarrow l \leftrightarrow k$$

where $k \in K$ has triangular decomposition $k = la(k)u$, $\check{\mathbf{l}} = \check{\mathbf{a}}l^{-1}\check{\mathbf{a}}^{-1}$, and $a(k) = \check{\mathbf{a}}^{-2}$. From the noncompact perspective, l is essentially a horocycle coordinate, and from the compact perspective, l is a standard affine coordinate for the flag space K/T.

In the case $K = SU(2)$, this coordinate is given explicitly by

$$l = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \leftrightarrow k(\zeta) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{\zeta} \\ 0 & 1 \end{pmatrix},$$

where $a = (1 + |\zeta|^2)^{-1/2}$. The symplectic form is

$$\frac{i}{2}(1 + |\zeta|^2)^{-1} d\zeta \wedge d\bar{\zeta}$$

The following is a reformulation of results in [Lu] on the standard Poisson structure. We denote the symplectic form on $\check{N}^-$ simply by ω (ω_1, from the noncompact point of view, Π_1^{-1}, from the compact point of view).

In the following statement, $\check{\delta} = \sum \Lambda_j$, the sum of the dominant integral functionals. This equals half the sum of the positive roots, when $\check{\mathfrak{g}}$ is finite dimensional.

Corollary 5.2. *Fix $w \in W$.*

a) The submanifold $\check{N}^- \cap w^{-1}\check{N}^+ w \subset \check{N}^-$ is $\check{T}$-invariant and symplectic.

Fix a minimal factorization $w = r_n..r_1$, in terms of simple reflections r_j corresponding to simple positive roots γ_j. Let $w_j = r_j..r_1$.

b) The map

$$\mathbb{C}^n \to \{\Sigma_w^K : m = 1\} \leftrightarrow N^- \cap w^{-1}N^+ w$$

$$\zeta = (\zeta_n, .., \zeta_1) \to k(\zeta) = w_{n-1}^{-1} i_{\gamma_n}(k_n(\zeta_n) w_{n-1} .. i_{\gamma_1}(k_1(\zeta_1) \leftrightarrow l(\zeta),$$

where $k = l(\zeta)au$, is a diffeomorphism.

c) In these coordinates

$$\omega|_{N^- \cap w^{-1}N^+ w} = \sum \frac{i}{\langle \gamma_j, \gamma_j \rangle} \frac{1}{(1 + |\zeta_j|^2)} d\zeta_j \wedge d\bar{\zeta}_j$$

$$\omega^n / n! = \prod_1^n \frac{\pi}{\langle \check{\delta}, w_{j-1}^{-1} \cdot \gamma_j \rangle} a(k)^{\check{\delta}} d\lambda_{N^{-1} \cap w^{-1} N^+ w}(l),$$

and the momentum map is the restriction of $-\langle \frac{i}{2} log(a), \cdot \rangle$ to this submanifold, where

$$a(k(\zeta)) = \prod (1 + |\zeta_j|^2)^{-\frac{1}{2} w_{j-1}^{-1} h_{\gamma_j} w_{j-1}}.$$

Proof. (First Proof) There exists a minimal factorization of $\mathbf{w}_0$ of the form $\mathbf{w}_0 = r_N..r_{n+1} r_n..r_1$. Translation by $\mathbf{w}_0$, as in the Proposition, maps the top Bruhat leaf, with the standard symplectic structure, to $\{\Sigma_1^K : m = 1\}$, with the Evens-Lu symplectic structure. We can now directly translate Lu's results into our context. The point is that the product structure in [Lu] is compatible with the claimed product structure in the Corollary. □

This proof has the deficiency that it depends on the existence of a longest Weyl group element, a special feature of finite type Kac-Moody groups. However one can modify the arguments of Lu to apply directly to our setting, without reference to a longest Weyl group element. The point is that this second proof is valid also for affine Kac-Moody groups (some of the consequences of this generalization will be pursued in [Pi3]). We will now briefly discuss this second proof, referring to the Appendix for some of our notation (for flag manifolds) and details (concerning the extension to symmetrizable Kac-Moody groups).

Via the projection,

$$K \to K/\check{T} = \check{G}/\check{B}_+,$$

$\{\Sigma_1^K : m = 1\}$ is identified with Σ_1, the unique open Birkhoff stratum in the flag manifold. There is a surjective map

$$SL(2, \mathbb{C}) \times .. \times SL(2, \mathbb{C}) \to w^{-1}\bar{C}_w :$$

$$(g_n,..,g_1) \to w_{n-1}^{-1} i_{\gamma_n}(g_n) w_{n-1} .. w_1^{-1} i_{\gamma_2}(g_2) w_1 i_{\gamma_1}(g_1) \check{B}_+ \tag{5.7}$$

This map has the remarkable property that the notion of generic is compatible with factorization: the preimage of $w^{-1}\bar{C}_w$ is precisely

$$SL(2,\mathbb{C})' \times .. \times SL(2,\mathbb{C})',$$

where

$$SL(2,\mathbb{C})' = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \neq 0 \}.$$

This follows from corollary 6.4.

In terms of the affine coordinate $\check{N}^-$ for $\check{G}/\check{B}^+$,

$$\check{N}^- \cap w^{-1}\check{N}^+ w \leftrightarrow w^{-1}C_w. \tag{5.8}$$

Lu observed that (5.7) and (5.8) induce a diffeomorphism

$$\mathbb{C}^n \leftrightarrow \{\Sigma_1^{SU(2)} : m = 1\} \times .. \times \{\Sigma_1^{SU(2)} : m = 1\} \to N^- \cap w^{-1} N^+ w$$

$$(\zeta_n, .., \zeta_1) \leftrightarrow (k_n(\zeta_n), .., k_1(\zeta_1)) \to l(\zeta)$$

where

$$w_{n-1}^{-1} i_{\gamma_n}(k_n) w_{n-1} .. i_{\gamma_1}(k_1) = l(\zeta) \prod_1^n a_j^{Ad(w_{n-1})(h_{\gamma_j})} u,$$

$u \in \check{N}^+$. This is a unitary cross-section for the map (5.7), versus the more conventional complex analytic cross-section in proposition 6.1.

In these coordinates

$$\sum \frac{i}{\langle \gamma_j, \gamma_j \rangle} \frac{1}{(1+|\zeta_j|^2)} d\zeta_j \wedge d\bar{\zeta}_j = \omega|_{N^- \cap w^{-1} N^+ w}$$

which follows from a relatively straightforward modification of the arguments in [Lu].

The last part of (c) follows from proposition 6.5.

6. Appendix

In this appendix we will review some of the ideas in [Lu], relevant to this paper, from a slightly different perspective. The main rationale for including this appendix is that the basic arguments are valid in the more general Kac-Moody category. Throughout this appendix, we will use the notation and basic results in [Kac].

We start with the following data: A is an irreducible symmetrizable generalized Cartan matrix; $\mathfrak{g} = \mathfrak{g}(A)$ is the corresponding Kac-Moody Lie algebra, realized via its standard (Chevalley-Serre) presentation; $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the triangular decomposition; $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ the upper Borel subalgebra; $G = G(A)$ is the algebraic group associated to A by Kac-Peterson; $H, N^{\pm}$ and B are the subgroups of G corresponding to $\mathfrak{h}$, $\mathfrak{n}^{\pm}$, and $\mathfrak{b}$, respectively; K is the "unitary form" of G; $T = K \cap H$ the maximal torus; and $W = N_K(T)/T \simeq N_G(H)/H$ is the Weyl group.

A basic fact is that $(G, B, N_G(H))$ with Weyl group W is an abstract Tits system. This yields a complete determination of all the (parabolic) subgroups between B and G. They are described as follows.

Let Φ be a fixed subset of the simple roots. The subgroup of W generated by the simple reflections corresponding to roots in Φ will be denoted by $W(\Phi)$. The parabolic subgroup corresponding to Φ, $P = P(\Phi)$, is given by $P = BW(\Phi)B$.

Given $\mathbf{w} \in N_K(T)$, we will denote its image in $W/W(\Phi)$ by w.

The basic structural features of G/P which we will need are the Birkhoff and Bruhat decompositions

$$G/P = \bigsqcup \Sigma_w, \qquad \Sigma_w = N^- \mathbf{w} P \tag{6.1}$$

$$G/P = \bigsqcup C_w, \qquad C_w = B\mathbf{w}P, \tag{6.2}$$

respectively, where the indexing set is $W/W(\Phi)$ in both cases. The strata Σ_w are infinite dimensional if $\mathfrak{g}$ is infinite dimensional, while the cells C_w are always finite dimensional. Our initial interest is in the Schubert variety $\bar{C}_w$, the closure of the cell.

Fix $w \in W/W(\Phi)$. We choose a representative $\mathbf{w} \in N(T)$ of minimal length n; for definiteness we will always take $\mathbf{w}$ of the form

$$\mathbf{w} = r_n \cdots r_1 \tag{6.3}$$

where $r_j = i_{\gamma_i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $i_{\gamma_j} : SL_2 \to G$ is the canonical homomorphism of SL_2 onto the root subgroup corresponding to the simple root γ_j.

Proposition 6.1. *For* $\mathbf{w}$ *as in 6.3, the map*

$$r_n exp(\mathfrak{g}_{-\gamma_n}) \times .. \times r_1 exp(\mathfrak{g}_{-\gamma_1}) \to G/P : (p_j) \to p_n..p_1 P$$

is a complex analytic isomorphism onto C_w.

This result is essentially (5) of [Kac] together with Tits's theory. We will sketch a proof for completeness.

Proof of 6.1. Let Δ^+ denote the positive roots, $\Delta^+(\Phi)$ the positive roots which are combinations of elements from Φ. The "Lie algebra of P" is $\mathfrak{p} = \Sigma \mathfrak{g}_{-\beta} \oplus \mathfrak{b}$ where the sum is over $\beta \in \Delta^+(\Phi)$; this is the Lie algebra of P in the sense that it is the subalgebra generated by the root spaces $\mathfrak{g}_\gamma$ for which $exp : \mathfrak{g}_\gamma \to G$ is defined and have image contained in P. The subgroups $exp(\mathfrak{g}_\gamma)$ generate P. We also let $\mathfrak{p}^-$ denote the subalgebra opposite $\mathfrak{p}$: $\mathfrak{p} = \sum \mathfrak{g}_{-\gamma}$, where the sum is over $\gamma \in \Delta^+ \setminus \Delta^+(\Phi)$. The corresponding group will be denoted by P^-.

The cell C_w is the image of the map $N^+ \to G/P : u \to u\mathbf{w}P$. The stability subgroup at $\mathbf{w}P$ is $N^+ \cap \mathbf{w}P\mathbf{w}^{-1}$.

At the Lie algebra level we have the splitting

$$\mathfrak{n}^+ = \mathfrak{n}^+ \cap Ad(\mathbf{w})(\mathfrak{p}) \oplus \mathfrak{n}^+ \cap Ad(\mathbf{w})(\mathfrak{p}^-). \tag{6.4}$$

The second summand equals

$$\mathfrak{n}_w^+ = \oplus \, \mathfrak{g}_\beta \tag{6.5}$$

where the sum is over roots $\beta > 0$ with $w^{-1}\beta \in -(\Delta^+ \setminus \Delta^+(\Phi))$. These roots β are necessarily real, so that $exp : \mathfrak{n}_w^+ \to N_w^+ \subseteq N^+$ is well-defined.

For $q \in \mathbb{Z}^+$ let N_q^+ denote the subgroup corresponding to $\mathfrak{n}_q^+ = span\{\mathfrak{g}_\beta : height(\beta) \geq q\}$. Then N^+/N_q^+ is a finite dimensional nilpotent Lie group, and it is also simply connected. By taking q sufficiently large and considering the splitting (6.4) modulo $\mathfrak{n}_q^+$, we conclude by finite dimensional considerations that each element in N^+ has a unique factorization $n = n_1 n_2$, where $n_1 \in N_w^+$ and $n_2 \in N^+ \cap \mathbf{w}P\mathbf{w}^{-1}$:

$$N^+ \simeq N_w^+ \times (N^+ \cap \mathbf{w}P\mathbf{w}^{-1}). \tag{6.6}$$

The important point here is that modulo N_q^+ we can control $N^+ \cap \mathbf{w}P\mathbf{w}^{-1}$ by the exponential map.

The following lemma is standard.

Lemma 6.2. *In terms of the minimal factorization* $\mathbf{w} = r_n \cdots r_1$*, the roots* $\beta > 0$ *with* $\mathbf{w}^{-1}\beta < 0$ *are given by*

$$\beta_j = r_n \cdots r_{j+1}(\gamma_j) = r_n \cdots r_j(-\gamma_j), \qquad 1 \le j \le n.$$

Because $\mathbf{w}$ is a representative of $w \in W/W(\Phi)$ of minimal length, all of these β_j satisfy $\mathbf{w}^{-1}\beta_j \in -(\Delta^+ \setminus \Delta^+(\Phi))$. Otherwise, if say $\mathbf{w}^{-1}\beta_j \in -\Delta^+(\Phi)$, then

$$\mathbf{w}^{-1} r_{\beta_j} \mathbf{w} = r_1 \cdots r_{j-1} r_j r_{j-1} \cdots r_1 \in N(T) \cap P \tag{6.7}$$

and $\mathbf{w}' = \mathbf{w}(\mathbf{w}^{-1} r_{\beta_j} \mathbf{w}) = r_n \cdots \hat{r}_j \cdots r_1$ would be a representative of w of length $< n$ (here we have used the fact that $W(\Phi) = N(T) \cap P/T$, which follows from the Bruhat decomposition). For future reference we note this proves that

$$N_w^+ = N^+ \cap (N^-)^w = N^+ \cap (P^-)^w \tag{6.8}$$

and (2.4) shows that

$$N_w^+ \times w \cong C_w. \tag{6.9}$$

Now for any $1 \le p \le q \le n$, $\bigoplus_{p \le j \le p} \mathfrak{g}_{\beta_j}$ is a subalgebra of $\mathfrak{n}_w^+$. Thus by (6.6)

$$exp(\mathfrak{g}_{\beta_n}) \times \cdots \times exp(\mathfrak{g}_{\beta_1}) \times w \cong C_w. \tag{6.10}$$

This yields Proposition 6.1 when we write

$$exp(\mathfrak{g}_{\beta_j}) = r_n \cdots r_j exp(\mathfrak{g}_{\gamma_j}) r_j \cdots r_n. \tag{6.11}$$

□

We now note several important corollaries of Proposition 6.1.

For each j, let P_j denote the parabolic subgroup $i_{\gamma_j}(SL_2)B$. Let

$$\Gamma_{\mathbf{w}} = P_n \times_B \cdots \times_B P_1/B \tag{6.12}$$

where

$$P_n \times \cdots \times P_1 \times B \times \cdots \times B \to P_n \times \cdots \times P_1 \tag{6.13}$$

is given by

$$(p_j) \times (b_j) \to (p_n b_n, b_n^{-1} p_{n-1} b_{n-1}, \cdots, b_2^{-1} p_1 b_1). \tag{6.14}$$

We have written "$\Gamma_{\mathbf{w}}$" instead of "Γ_w" to indicate that this compact complex manifold depends upon the factorization.

Corollary 6.3. *The map*

$$\Gamma_{\mathbf{w}} \to \bar{C}_w : (p_j) \to p_n \cdots p_1 P$$

is a (Bott-Samelson) desingularization of $\bar{C}_w$*.*

Let

$$SL_2' = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) : a \ne 0\}. \tag{6.15}$$

Corollary 6.4. *Consider the surjective map*

$$SL_2 \times \cdots \times SL_2 \to \bar{C}_w : (g_j) \to r_n i_{\gamma_n}(g_n) \cdots r_1 i_{\gamma_1}(g_1) P.$$

The inverse image of C_w *is* $SL_2' \times \cdots \times SL_2'$*.*

Proof of Corollary 6.3. Let $\sigma = r_{n-1} \cdots r_1$. It suffices to show that for the natural actions

$$r_n i_{\gamma_n}(SL_2') \times C_{\bar\sigma} \to C_w, \tag{6.16}$$

$$r_n i_{\gamma_n}(SL_2 \setminus SL_2') \times C_{\bar\sigma} \to \bar C_w, \tag{6.17}$$

and

$$r_n i_{\gamma_n}(SL_2) \times (\bar C_{\bar\sigma} \setminus C_{\bar\sigma}) \to \bar C_w \setminus C_w. \tag{6.18}$$

The first line, (6.16), follows from Proposition 6.1 since $i_{\gamma_n}(SL_2') \subseteq exp(-\mathfrak{g}_{-\gamma_n})B$ and $B \times C_{\bar\sigma} \subseteq C_{\bar\sigma}$. The second line follows from

$$r_n i_{\gamma_n}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \cdot C_{\bar\sigma} = i_{\gamma_n}\begin{pmatrix} c & b \\ 0 & d \end{pmatrix} \cdot C_{\bar\sigma} \subseteq C_{\bar\sigma}. \tag{6.19}$$

For the third line it's clear that the image of the left hand side is a union of cells, since we can replace $r_n i_{\gamma_n}(SL_2)$ by P_n. This image is at most $n-1$ dimensional. Therefore it must have null intersection with C_w. □

Fix an integral functional $\lambda \in \mathfrak{h}^*$ which is antidominant. Denote the (algebraic) lowest weight module corresponding to λ by $L(\lambda)$, and a lowest weight vector by σ_λ. Let Φ denote the simple roots γ for which $\lambda(h_\gamma) = 0$, where h_γ is the coroot, $P = P(\Phi)$ the corresponding parabolic subgroup. The Borel-Weil theorem in this context realizes $L(\lambda)$ as the space of strongly regular functions on G satisfying

$$f(gp) = f(g)\lambda(p)^{-1} \tag{6.20}$$

for all $g \in G$ and $p \in P$, where we have implicitly identified λ with the character of P given by

$$\lambda(u_1 w \exp(x) u_2) = \exp \lambda(x) \tag{6.21}$$

for $x \in \mathfrak{h},\ u_1, u_2 \in N^+,\ w \in W(\Phi)$. Thus we can view $L(\lambda)$ as a space of sections of the line bundle

$$\mathcal{L}_\lambda = G \times_\lambda \mathbb{C} \to G/P. \tag{6.22}$$

If $\mathfrak{g}$ is of finite type, then $L(\lambda) = H^0(\mathcal{L}_\lambda)$; if $\mathfrak{g}$ is affine (and untwisted), then $L(\lambda)$ consists of the holomorphic sections of finite energy, as in [PS].

Normalize σ_λ by $\sigma_\lambda(1) = 1$.

Proposition 6.5. *Let $w \in W/W(\Phi)$, and let $w = r_n \cdots r_1$ be a representative of minimal length n. Let $w_j = r_j..r_1$. The positive roots mapped to negative roots by w are given by*

$$\tau_j = w_{j-1}^{-1}(\gamma_j), \qquad 1 \le j \le n;$$

let $\lambda_j = -\lambda(h_{\tau_j})$, where h_τ is the coroot corresponding to τ. Then

$$\sigma_\lambda^w(r_n i_{\gamma_n}(g_n) \cdots r_1 i_{\gamma_1}(g_1)) =$$

$$\sigma_\lambda(w_{n-1}^{-1} i_{\gamma_n}(g_n) w_{n-1}..w_1^{-1} i_{\gamma_2}(g_2) w_1 i_{\gamma_1}(g_1)) = \prod_1^n a_j^{\lambda_j}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$.

Proof. Proof of Proposition 6.5 The claim about the τ_j follows from lemma 6.2. None of these roots lie in $\Delta^+(\Phi)$, by the same argument as follows (A.8). Thus each $\lambda_j > 0$. It follows that $\Pi a_j^{\lambda_j}$ is nonzero precisely on the set $SL_2' \times \cdots \times SL_2'$.

Now σ_λ^w, viewed as a section of $\mathcal{L}_\lambda \to G/P$, is nonzero precisely on the w-translate of the largest strata,

$$w\Sigma_0 = wP^-P = (P^-)^w wP. \tag{6.23}$$

We claim the intersection of this with $\bar{C}_w$ is C_w. In one direction

$$C_w = \left(N^+ \cap (P^-)^w\right) wP \subseteq (P^-)^w wP \tag{6.24}$$

by (6.8). On the other hand $(N^+ \cap (P^-)^w)$ is a closed finite dimensional subgroup of $(P^-)^w$. Since $(P^-)^w$ is topologically equivalent to $w\Sigma_0$, the limit points of C_w must be in the complement of $w\Sigma_0$. This establishes the other direction.

It now follows from Proposition 6.1 that σ_λ^w is also nonzero precisely on $SL_2' \times \cdots \times SL_2'$, viewed as a function of $(g_n, \cdots, g_1)$.

Write

$$\sigma_\lambda\left(w_{n-1}^{-1} i_{\gamma_n}(g_n) w_{n-1} w_{n-2}^{-1} i_{\gamma_{n-1}}(g_{n-1}) w_{n-2} \cdots i_{\gamma_1}(g_1)\right)$$

$$= \sigma_\lambda\left(i_{\tau_n}(g_n) i_{\tau_{n-1}}(g_{n-1}) \cdots i_{\tau_1}(g_1)\right) \tag{6.25}$$

where $i_{\tau_i}(\cdot) = w_{i-1}^{-1} i_{\gamma_i}(\cdot) w_{i-1}$. Because

$$w_{i-1}^{-1}(\gamma_i) > 0, \tag{6.26}$$

$i_{\tau_j} : SL_2 \to G$ is a homomorphism onto the root subgroup corresponding to τ_j which is compatible with the canonical triangular decompositions.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2'$, write $g = LDU$, where

$$L = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix}, \quad D = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}. \tag{6.27}$$

Then for $(g_j) \in SL_2' \times \cdots \times SL_2'$, (6.25) equals

$$\sigma_\lambda(i_{\tau_n}(L_n D_n U_n) \cdots i_{\tau_1}(L_1 D_1 U_1))$$

$$= \sigma_\lambda(i_{\tau_n}(L_n U_n') i_{\tau_{n-1}}(L_{n-1}' U_{n-1}') \cdots i_{\tau_1}(L_1' U_1') i_{\tau_n}(D_n) \cdots i_{\tau_1}(D_1))$$

$$= \sigma_\lambda(i_{\tau_n}(L_n U_n') \cdots i_{\tau_1}(L_1' U_1')) \Pi a_j^{\lambda_j} \tag{6.28}$$

where each L_j' (U_j') has the same form as L_j (U_j, respectively). This follows from the fact that H normalizes each $exp(\mathfrak{g}_{\pm r})$.

Now each $L_j' U_j' \in SL_2'$, so that $i_{\tau_n}(L_n U_n') \cdots i_{\tau_1}(L_1' U_1')$ is in Σ_0. We now conclude that

$$\sigma_\lambda(i_{\tau_n}(L_n U_n') \cdots i_{\tau_1}(L_1' U_1')) = 1, \tag{6.29}$$

by the fundamental theorem of algebra, since this is polynomial and never vanishes. □

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