On the Reachability of a Class of Second-Order Switched Systems

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Abstract

In this paper, the reachability problem for a class of second-order LTI switched systems is solved. The reachability problem is first explored for switched systems consisting of two subsystems and switching control laws are proposed that can drive the system state from an initial point to a target point via finitely many switches. The method is then extended to the case of several subsystems. The robustness issue is also addressed briefly and the relationship between stabilizability and reachability is discussed.

1 Introduction

A switched system is a system that consists of several subsystems and a switching law that specifies which subsystem dynamics will be followed by the system trajectory at each instant of time. Switchings may be time-driven (i.e., a switching happens at specific time instants) or event-driven (i.e., a switching happens when some internal or external event takes place). Recently, there has been increasing interest in the stability analysis and design of such systems (see, e.g., [1, 2, 3, 4, 9, 10, 13, 14]). [6] provides a survey of recent development of stability and design of switched systems.

In this paper, we are interested in *switched systems consisting of second-order LTI subsystems with foci at the origin.* (A second-order system $\dot{x} = Ax$ is with focus at the origin if A has eigenvalues $\alpha_i \pm \beta_i j$, $\beta_i \neq 0$, see Chapter 1 of [5] for details.)

Definition 1.1 *A switched system consisting of second-order LTI subsystems with foci at the origin is a system described by system equations:*

$$\dot{x} = A_{i(t)} x \tag{1.1}$$

$$i(t) = \phi(x(t), i(t^-), t), \tag{1.2}$$

where $x \in \mathbb{R}^2$, $i(t) \in I = \{1, 2, \cdots, N\}$ and $A_i \in \mathbb{R}^{2\times 2}$ with foci at the origin. $\phi : \mathbb{R}^2 \times I \times \mathbb{R} \to I$ describes the switching law. It is assumed that the switching law can only generate a finite number of switches in any finite time period.

In Definition 1.1, the switching law is to be designed in control synthesis. In this paper, we consider the system (1.1) and design the switching control law for the reachability of the switched system.

There are some results on second-order switched systems in the literature, primarily on stability analysis and design (see, e.g., [7, 11]). The existing results on reachability are primarily analysis results (e.g. [8]). [12] provides controllability results for a class of hybrid systems.

Here based on our earlier results on stabilization of switched systems consisting of second-order LTI subsystems [13, 14], we study the reachability of the class of LTI subsystems defined in Definition 1.1 and we derive appropriate switching control laws.

For the class of switched systems we are interested in, we define a switched system to be reachable as follows.

Definition 1.2 (Reachability) *The switched system (1.1) is said to be* **reachable** *if for any non-equilibrium initial point $x_i \neq 0$ and any non-equilibrium target point $x_t \neq 0$, there exists a switching control law that can generate a finite switching sequence to drive the system state from x_i to x_t in finite amount of time.*

This paper is an expanded version of [15]. The outline of this paper is as follows. In Section 2, we review some results from [13] which we then use in our detailed study of reachability problem of switched systems with two unstable subsystems. Switching control laws, which can drive the system state from x_i to x_t, are also derived. Section 3 discusses the reachability problem for switched systems with two stable subsystems. Section 4 discusses the reachability problem for switched systems with one stable and one unstable subsystems. The switching control laws are extended to switched systems with several subsystems in Section 5. Section 6 briefly addresses the robustness issue of the switching control laws and establishes the relationship between stabilizability and reachability. Section 7 contains concluding remarks.

2 Reachability of Switched Systems with Two Unstable Subsystems

In this section, we consider the reachability problem for the switched system (1.1) consisting of two subsystems with unstable foci and derive switching control laws.

2.1 Unstable Subsystems

We first review some stabilization results from [13]. Note that these results are constructive and that the stabilizability conditions derived are necessary and sufficient. We shall say that a subsystem is of clockwise (counterclockwise) direction if starting from any nonzero initial condition in the phase plane its trajectory is a spiral around the origin in the clockwise (counterclockwise) direction (see [13]).

Some notations and definitions need to be mentioned here. Let $x = (x_1, x_2)^T$ be a nonzero point on $\mathbb{R}^2$ plane, and denote

$$f_1 = A_1 x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \; f_2 = A_2 x = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}.$$

We view x, f_1 and f_2 as vectors in $\mathbb{R}^2$ and define $\theta_i, i = 1, 2$ to be the angle between x and f_i measured counterclockwise with respect to x (θ_i is confined to $-\pi \leq \theta_i < \pi$). So θ_i is positive (negative) if vector f_i is to the counterclockwise (clockwise) side of x. Also as in [13] we define the regions

$$\begin{aligned} E_{is} &= \{x | -\pi \leq \theta_i(f_i(x)) \leq -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq \theta_i(f_i(x)) < \pi\} \\ &= \{x | x^T f_i(x) = x^T A_i x \leq 0\}, \; i = 1, 2, \end{aligned}$$

$$\begin{aligned} E_{iu} &= \{x | -\frac{\pi}{2} \leq \theta_i(f_i(x)) \leq 0 \text{ or } 0 \leq \theta_i(f_i(x)) \leq \frac{\pi}{2}\} \\ &= \{x | x^T f_i(x) = x^T A_i x \geq 0\}, \; i = 1, 2. \end{aligned}$$

To design stabilizing switching control laws, we identify the following two distinct cases.

Case 1. Two Subsystems of the Same Direction

Without loss of generality, assume that both subsystems of (1.1) are of clockwise direction. Note that similar results can be applied to two subsystems both of counterclockwise direction.

Define the following conic regions.

$$\begin{aligned}
\Omega_1 &= E_{1s} \cap E_{2u}, \\
\Omega_2 &= E_{1u} \cap E_{2s}, \\
\Omega_3 &= E_{1s} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \le 0\}, \\
\Omega_4 &= E_{1s} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \ge 0\}, \\
\Omega_5 &= E_{1u} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \le 0\}, \\
\Omega_6 &= E_{1u} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \ge 0\}.
\end{aligned}$$

The **conic switching law** proposed in [13] is as follows: switch the switched system to subsystem 1 whenever the system state enters Ω_1, Ω_3, Ω_5 and switch to subsystem 2 whenever the system state enters Ω_2, Ω_4, Ω_6.

The following theorem is concerned with the stabilizability of the switched system (see [13]). Note that this is a necessary and sufficient condition as opposed to other literature results.

Theorem 2.1 *Let l_1 be a ray that goes through the origin. Let $x_0 \neq 0$ be on l_1. Let x^* be the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law. The switched system (1.1) consisting of two subsystems with unstable foci and of the same direction is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$ by the conic switching law.*

The following example illustrates how the conic switching law works.

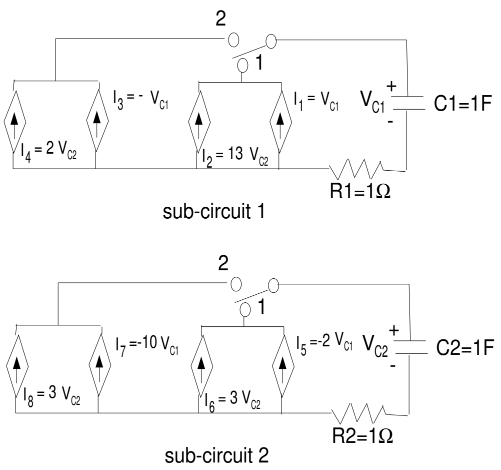

Figure 1: A simple circuit system consisting of two dependent sub-circuits for Example 2.2.

Example 2.2 *Figure 1 shows a simple circuit system consisting of two dependent sub-circuits. For every sub-circuit, there is a switch to connect the sub-circuit to one of the two voltage dependent* *current source circuits. So there are four possible switch combinations which would provide us with four subsystems. Here we assume that the switches for both sub-circuits can only be both at position 1 or both at position 2 simultaneously. This reduces the number of possible switch combinations to two. It is not difficult to derive the following differential equations for the circuit.*

Both switches are at position 1:

$$\begin{bmatrix} \dot{V}_{c_1} \\ \dot{V}_{c_2} \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix}. \tag{2.1}$$

Both switches are at position 2:

$$\begin{bmatrix} \dot{V}_{c_1} \\ \dot{V}_{c_2} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -10 & 3 \end{bmatrix} \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix}. \tag{2.2}$$

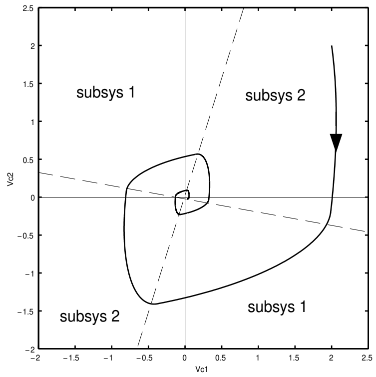

Figure 2: The trajectory of the switched circuit system by using the conic switching law $[(V_{c_1}(0), V_{c_2}(0)]^T = [2, 2]^T)$.

The switched system consists of two unstable subsystems with foci at the origin and is asymptotically stabilizable by the conic switching law proposed above. Figure 2 shows the system trajectory from initial point $[(V_{c_1}(0), V_{c_2}(0)]^T = [2, 2]^T$.

$\square$

Case 2. Two Subsystems of Opposite Directions

Assume that subsystem 1 is of clockwise direction while subsystem 2 is of counterclockwise direction.

We introduce the following conic regions.

$$\begin{aligned}
\Omega_1 &= E_{1s} \cap E_{2s}, \\
\Omega_2 &= E_{1u} \cap E_{2u}, \\
\Omega_3 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\
\Omega_4 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \\
\Omega_5 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\
\Omega_6 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}.
\end{aligned}$$

The following theorem is concerned with the stabilizability of the switched system (see [13]).

Theorem 2.3 *The switched system (1.1) consisting of two subsystems with unstable foci and of opposite directions is asymptotically stabilizable if and only if* $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$.

If the switched system is asymptotically stabilizable, then the conic switching law can also be obtained as in [13] which makes the system asymptotically stable.

The **conic switching law** is as follows: first, by following subsystem 1, force the trajectory into the interior of one of the conic regions Ω_1, Ω_3, Ω_5(there must be one available), and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region.

Remark: The conic switching laws can also be extended to switched systems consisting of second-order LTI subsystems which are not necessarily with foci (see [13]). □

2.2 Reachability Results

Consider the switched system (1.1) consisting of two unstable subsystems with foci. In the following discussion, we assume that the switched system is asymptotically stabilizable.

Assume that $x_i \neq 0$ and $x_t \neq 0$ are given non-equilibrium initial and target points on $\mathbb{R}^2$ plane, respectively. We want to find a switching control law so as to drive the state of the system from x_i to x_t. We consider the following two cases.

Case 1. Two Subsystems of the Same Direction

Without loss of generality, assume that both subsystems are of the clockwise direction. Since the switched system is asymptotically stabilizable, it is known that the conic switching law will make the system asymptotically stable. Therefore, for this switched system, if we start at x_t, then the state of the system would be closer to the origin clockwise by following the conic switching law for $t \geq 0$.

Now we consider the trajectory starting from x_t with time going backwards, i.e., consider the trajectory C^- for $-t, t \geq 0$ by following the conic switching law. It is clear that this trajectory would be farther from the origin in a counterclockwise fashion.

Let l_i be the ray that goes through the origin and x_i. Let l_t be the ray that goes through the origin and x_t. Let $x^* = x(-t^*)$ be the point on C^- that satisfies the following conditions.

1. The trajectory $E = \{x(-t) \in C^- | 0 < t \leq t^*\}$ intersects l_t at least once.
2. t^* is the minimum possible t such that condition 1 is satisfied and $x(-t^*)$ is on l_i and $\|x(-t^*)\|_2 \geq \|x_i\|_2$.

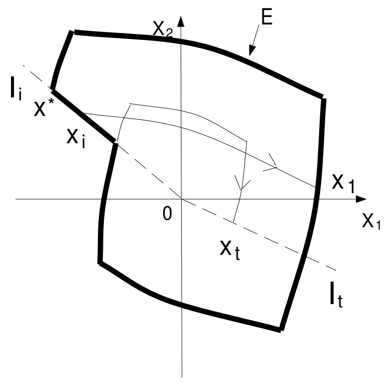

Figure 3: Switching control law for two unstable subsystems of the same direction.

To obtain a switching control sequence, we let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is unstable, it is clear that if the switched system stays at subsystem 1 for sufficiently long time, $x(t)$ will be outside the region formed by E and part of l_i(the region inside the bold curves in Figure 3). By this we conclude that there exists a time instant t_1 such that the trajectory intersects E for the first time, i.e., $x_1 = x(t_1) \in E$ (Since the trajectory is clockwise, it cannot intersect the bold part of l_i except the two end points).

So by the above discussion, we can adopt the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following the trajectory of subsystem 1 until it intersects E for the first time at t_1 on $x_1 = x(t_1)$.*

Step 2. *After it reaches $x_1 = x(t_1)$, let the system evolve following the conic switching law.*

It is quite clear that such a switching control law can drive the system state from x_i to x_t via only a finite number of switches.

Example 2.4 *Consider the circuit system as in Example 2.2. In view of the above discussion, the switched system is reachable since it is asymptotically stabilizable. Figure 4 shows the system trajectory from the initial point $x_i = [-2, -2]^T$ to the target point $x_t = [-0.5, 0.5]^T$ and the corresponding switching sequence obtained by using the switching control law proposed above.*

□

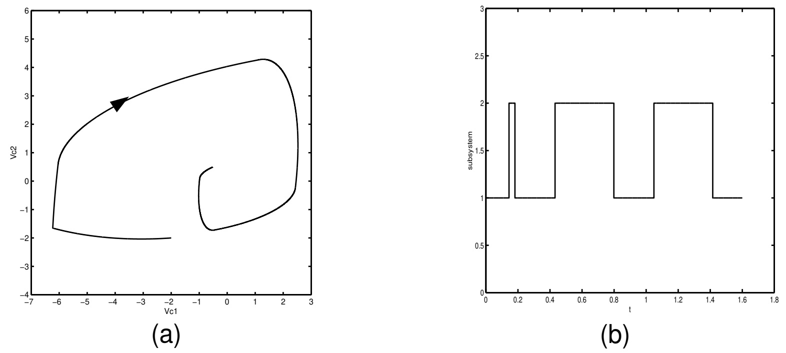

Figure 4: (a). The trajectory of the switched circuit system from $x_i = [-2, -2]^T$ to $x_t = [-0.5, 0.5]^T$. (b). The corresponding switching sequence.

Case 2. Two Subsystems of Opposite Directions

Assume that subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Since the switched system is asymptotically stabilizable, we know that $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$ (see [13]). We assume Ω is one conic region from the possible nonempty sets Ω_1, Ω_3 and Ω_5 which is between the two rays l_1 and l_2(Figure 5).

Consider the following trajectory C^-. Starting from x_t, let the system trajectory go backwards in time by following subsystem 1 until it intersects the ray l_2 for the first time by x^* at $-t^*$, then let the trajectory going backwards in time following the conic switching law in Ω. In this way, we can get the trajectory C^- as depicted in Figure 5.

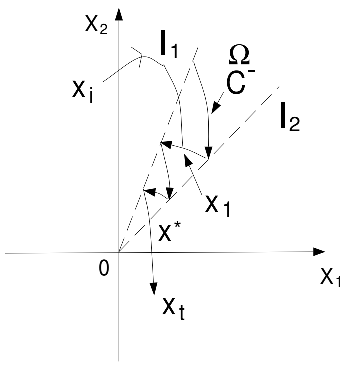

Figure 5: Switching control law for two unstable subsystems of opposite directions.

Now let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is unstable, it is clear that if the switched system stays at subsystem 1 for sufficiently long time, $x(t)$ will intersect C^-. By this we conclude that there exists a time instant t_1 such that the trajectory intersects C^- for the first time at $x_1 = x(t_1)$.

By the above discussion, we can adopt the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following the trajectory of subsystem 1 until it intersects C^- for the first time at t_1 on $x_1 = x(t_1)$.*

Step 2. *After it reaches $x_1 = x(t_1)$, let the system evolve following the corresponding subsystems associated with the points on the trajectory C^-.*

It is clear that such a switching control law can drive the system state from x_i to x_t via only a finite number of switches. □

3 Reachability of Switched Systems with Two Stable Subsystems

In the present section, we consider reachability problem for the switched systems (1.1) consisting of two subsystems with stable foci and derive switching control laws.

3.1 Stable Subsystems

If both subsystems are stable, then the stabilizability of the switched system can be easily established if we simply let the system stay at one subsystem and do not apply any switches. Yet we may ask a converse question: can we find a control law such that the switched system can be "destabilized", in other words, the trajectory of the switched system can be made unbounded. It is not difficult to see that such a control law can be found if the switched system with two unstable subsystems

$$\dot{x} = -A_1 x, \quad \dot{x} = -A_2 x, \tag{3.1}$$

is asymptotically stabilizable. Formally, we defined d-stabilizability as follows.

Definition 3.1 *An switched system (1.1) with two stable subsystems is said to be* **d-stabilizable** *if and only if the corresponding switched system (3.1) with two unstable subsystems is asymptotically stabilizable.*

A close look at the result for two unstable subsystems shows that the d-stabilizability problem is readily solved by some small modification of the aforementioned results. Similarly to Section 2.1, we will discuss this according to the following two cases.

Case 1. Two Subsystems of the Same Direction

Assume that both subsystems are of clockwise direction. Using the same notation as in Case 1 in the Section 2.1, we propose the following **conic switching law**: switch the switched system to subsystem 2 whenever the system state enters Ω_1, Ω_3, Ω_5 and switch to subsystem 1 whenever the system state enters Ω_2, Ω_4, Ω_6.

The following theorem concerns the d-stabilizability of the switched system.

Theorem 3.2 *Let l_1 be a ray that goes through the origin. Let $x_0 \neq 0$ be on l_1. Let x^* be the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law. The switched system (1.1) consisting of two subsystems with stable foci and of the same direction is d-stabilizable if and only if $\|x^*\|_2 > \|x_0\|_2$ by the conic switching law.*

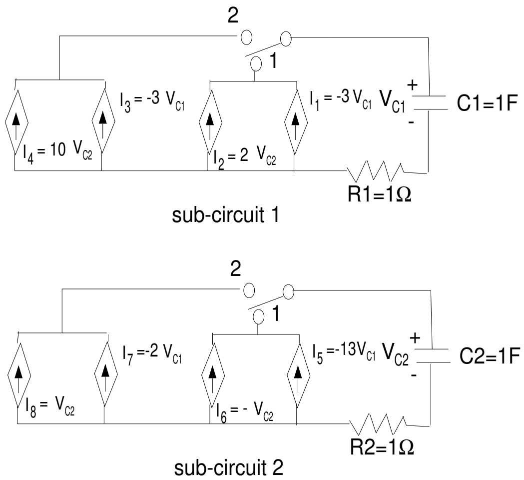

Figure 6: A simple circuit system consisting of two dependent sub-circuits for Example 3.3.

The following example shows how the conic switching law works.

Example 3.3 *Figure 6 shows a simple circuit system consisting of two dependent sub-circuits with coefficients different from Example 2.2. It can be readily obtained that:*

Both switches are at position 1:

$$\begin{bmatrix} \dot{V}_{c_1} \\ \dot{V}_{c_2} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -13 & -1 \end{bmatrix} \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix}. \tag{3.2}$$

Both switches are at position 2:

$$\begin{bmatrix} \dot{V}_{c_1} \\ \dot{V}_{c_2} \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix}. \tag{3.3}$$

The switched system consists of two stable subsystems with foci at the origin and is d-stabilizable by the conic switching law proposed in Section 3.1. Figure 7 shows the system trajectory from initial point $[(V_{c_1}(0), V_{c_2}(0)]^T = [0.2, 0.2]^T$.

□

Case 2. Two Subsystems of Opposite Directions

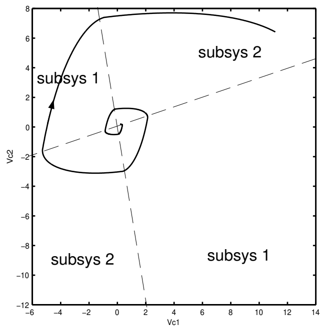

Figure 7: The trajectory of the switched circuit system by using the conic switching law $[(V_{c_1}(0), V_{c_2}(0)]^T = [0.2, 0.2]^T)$.

Assume that subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. With the same notation as in Case 2 of Section 2.1, the following theorem is concerned with the d-stabilizability of the switched system.

Theorem 3.4 *The switched system (1.1) consisting of two subsystems with stable foci and of opposite directions is d-stabilizable if and only if $Int(\Omega_2) \cup Int(\Omega_4) \cup Int(\Omega_6) \neq \emptyset$.*

If a switched system is d-stabilizable, then the conic switching law can also be obtained similarly to the one in Case 2 of Section 2.1.

The **conic switching law** is as follows: first, by following subsystem 1, force the trajectory into the interior of one of the conic regions Ω_2, Ω_4, Ω_6(there must be one available), and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region. □

3.2 Reachability Results

Consider the switched system (1.1) consisting of two stable subsystems. In the following discussion, we assume that the switched system is d-stabilizable. The switching control law can be obtained by some modifications of the discussion in Section 2.2.

Case 1. Two Subsystems of the Same Direction

Assume that both subsystems are of the clockwise direction. Since the switched system is d-stabilizable, we know that the conic switching law will "destabilize" the system asymptotically. Therefore, for this switched system, if we consider the trajectory starting from x_i with time going forward, i.e., consider the trajectory C^+ for $t, t \geq 0$ by following the conic switching law. It is clear that this trajectory would be farther from the origin in a clockwise fashion.

Let l_i be the ray that goes through the origin and x_i. Let l_t be the ray that goes through the origin and x_t. Let $x^* = x(t^*)$ be the point on C^+ that satisfies the following conditions.

1. The trajectory $E = \{x(t) \in C^+ | 0 < t \leq t^*\}$ intersects l_i at least once.

2. t^* is the minimum possible t such that condition 1 is satisfied and $x(t^*)$ is on l_t and $\|x(t^*)\|_2 \geq \|x_t\|_2$.

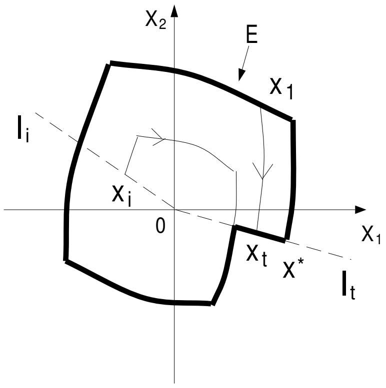

Figure 8: Switching control law for two stable subsystems of the same direction.

To obtain a switching control sequence, we let the system start from x_t and going backwards in time following subsystem 1, since subsystem 1 is stable, the backward trajectory will be unstable. So as the discussion in Case 1 in Section 2.2, the backward trajectory will intersect E for the first time at $x_1 = x(t_1)$.

Then the switching control law is readily available as follows.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following conic switching law until it reaches x_1.*

Step 2. *After it reaches x_1, let the system switch to subsystem 1 and evolve following subsystem 1.*

Such a switching control law can drive the system state from x_i to x_t via only a finite number of switches (Figure 8).

Example 3.5 *Consider the circuit system in Example 3.3. In view of the above discussion, the switched system is reachable since it is d-stabilizable. Figure 9 shows the system trajectory from the initial point $x_i = [0.2, 0.2]^T$ to the target point $x_t = [4, -4]^T$ and the corresponding switching sequence obtained by using the switching control law proposed above.*

□

Case 2. Two Subsystems of Opposite Directions

Assume that subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Since the switched system is d-stabilizable, we know that $Int(\Omega_2) \cup Int(\Omega_4) \cup Int(\Omega_6) \neq \emptyset$.

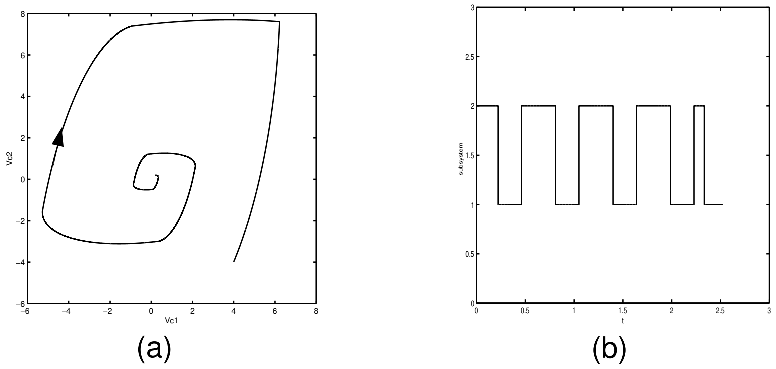

Figure 9: (a). The trajectory of the switched circuit system from $x_i = [0.2, 0.2]^T$ to $x_t = [4, -4]^T$. (b). The corresponding switching sequence.

We assume Ω is one conic region from the possible nonempty sets Ω_2, Ω_4 and Ω_6 which is between the two rays l_1 and l_2(Figure 10).

Consider the following trajectory C^-. Starting from x_t, let the system trajectory go backwards in time by following subsystem 1 until it intersects the ray l_2 for the first time by x^* at $-t^*$, then let the trajectory going backwards in time following the conic switching law in Ω. In this way, we can get the trajectory C^- as depicted in Figure 10.

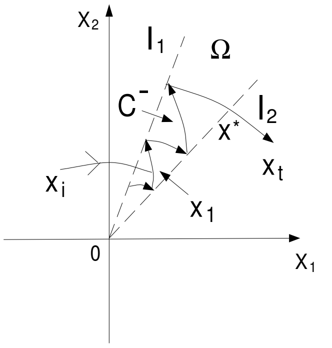

Figure 10: Switching control law for two stable subsystems of opposite directions.

Now let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is stable, it is clear that if the switched system stay at subsystem 1 for sufficiently long time, $x(t)$ will intersect C^-. By this we conclude that there exists a time instant t_1 such that the trajectory intersects C^- for the first time at $x_1 = x(t_1)$.

By the above discussion, we can adopt the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following the trajectory of subsystem 1 until it intersects C^- for the first time at t_1 on $x_1 = x(t_1)$.*

Step 2. *After it reaches $x_1 = x(t_1)$, let the system evolve following the corresponding subsystems associated with the points on the trajectory C^-.*

The switching control law can drive the system state from x_i to x_t via only a finite number of switches. □

4 Reachability of Switched Systems with One Stable and One Unstable Subsystems

If the switched system (1.1) consists of stable subsystem 1 and unstable subsystem 2 (both with foci), we note that the switched system must be both asymptotically stabilizable (by just following subsystem 1) and d-stabilizable (by just following subsystem 2).

For reachability, there are two cases to be discussed, where the first case is similarly to Case 1 in Section 2.2 and the second case is slightly different from Case 2 in Section 2.2.

Case 1. Two Subsystems of the Same Direction

Assume that both subsystems are of clockwise direction. Let the trajectory C^- be the trajectory starting from x_t with time going backwards by following subsystem 1. Let $x^* = x(-t^*)$ be the point on C^- that satisfies:

1. The set $E = \{x(-t) \in C^- | 0 < t \leq t^*\}$ intersects l_t at least once.
2. t^* is the minimum possible t such that condition 1 is satisfied and $x(-t^*)$ is on l_i and $\|x(-t^*)\|_2 \geq \|x_i\|_2$.

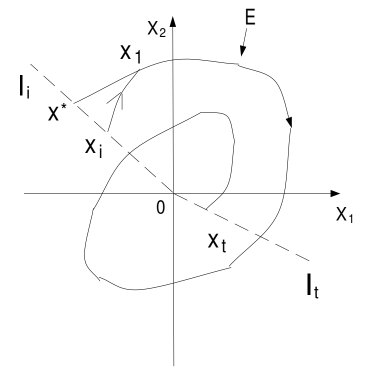

Figure 11: Switching control law for one stable subsystem and one unstable subsystem of the same direction.

Let the system start from x_i at $t = 0$ and follow subsystem 2. Since subsystem 2 is unstable, by the similar reason as in Case 1 in Section 2.2, the trajectory will intersect E at $x_1 = x(t_1)$ for the first time.

We propose the following switching control law.

Switching control law:

Step 1. *Let the system start from x_i at $t = 0$ and follow subsystem 2 until it reaches x_1.*

Step 2. *After it reaches x_1, let the system switch to subsystem 1 and evolve following subsystem 1.*

The switching control law requires only one switch to solve the reachability problem. □

Case 2. Two Subsystems of Opposite Directions

Assume that in (1.1), subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Then we identify the following two sub-cases.

a. If $\exists c > 0$ such that $A_1 = -cA_2$, then from Lemma 1 in Appendix A (see also Section 2 of [13]), we know that $C^+_{A_1} = C^-_{A_2}$, $C^-_{A_1} = C^+_{A_2}$ for any initial point $x_0 \neq 0$. Therefore x_t is not reachable from x_i if x_t is neither on $C^+_{A_1}$ nor on $C^-_{A_1}$.

b. If there does not exist $c > 0$ such that $A_1 = -cA_2$, then from Lemma 2 in Appendix A (see also Section 2 of [13]), there exists at most two lines on which $A_1x = kA_2x$ for some $k \neq 0$. In view of the θ_i introduced in Section 2.1, we can find a conic region in which either $|\theta_1| + |\theta_2| < \pi$ or $|\theta_1| + |\theta_2| > \pi$ holds for every point.

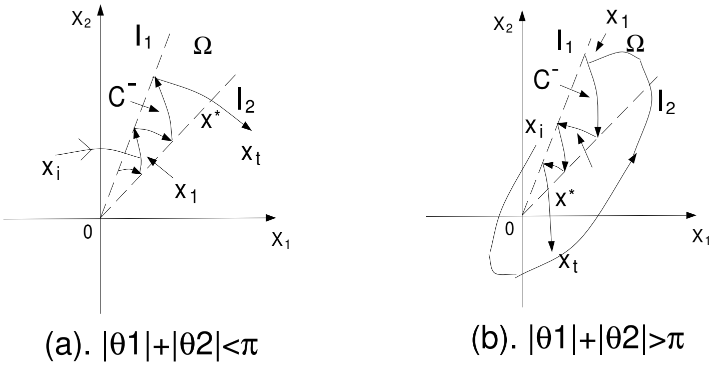

Figure 12: Switching control law for one stable subsystem and one unstable subsystem of opposite directions.

For $|\theta_1| + |\theta_2| < \pi$ case, we can obtain a switching control law similar to Case 2 in Section 3.2 (Figure 12(a)).

For $|\theta_1| + |\theta_2| > \pi$ case, we can obtain a switching control law similar to Case 2 in Section 2.2 except that we choose subsystem 2 between x_i and x_1 (Figure 12(b)). □

5 Several Subsystems

Consider the switched system (1.1) consisting of several second-order LTI subsystems with foci. The reachability results in Sections 2, 3 and 4 are readily extended to several subsystems as shown in the following.

Case 1. All Subsystems with Unstable Foci

We assume that the switched system is asymptotically stabilizable. If all subsystems are of the same direction, then we adopt the similar method as in Case 1 of Section 2.2 to drive x_i to x_t.

If $K(K>0)$ subsystems $S_1^-,\cdots,S_K^-$ are of clockwise direction and $M(M>0)$ subsystems $S_1^+,\cdots,S_M^+$ are of counterclockwise direction $(K+M=N)$, then we can use one of the following methods .

1. If $S_1^-,\cdots,S_K^-$ are asymptotically stabilizable, then adopt the similar method as in Case 1 of Section 2.2.
2. If $S_1^+,\cdots,S_M^+$ are asymptotically stabilizable, then adopt the similar method as in Case 1 of Section 2.2.
3. If there exists S_i^- and S_j^+ such that the switched system consisting of S_i^- and S_j^+ is asymptotically stabilizable, then adopt the similar method as in Case 2 of Section 2.2.

Case 2. All Subsystems with Stable Foci

The discussion can be carried out similar to the above case.

Case 3. K Subsystems with Stable Foci and M Subsystems with Unstable Foci

In this case, as long as there is one stable subsystem and one unstable subsystem satisfying the condition of Case 1 or the condition (b) in Case 2 of Section 4, we can always adopt the method therein.

6 Robustness, Stabilizability and Reachability

6.1 Robustness of the Switching Control Law

In [3], the authors proved the robustness of the stabilizing switching laws proposed in [13, 14] for switched systems consisting of second-order LTI subsystems. The paper indicated that if the switched system is asymptotically stabilizable, then there exists marginal conic regions that include the switching boundaries of conic switching laws such that the open-loop switched system is asymptotically stable if the switchings happen not exactly on the switching boundaries but inside those conic regions.

For the switching control laws derived in the previous several sections. It is not difficult to show that for any $\epsilon>0$, there exists $\delta>0$ and marginal conic regions including the switching boundaries of the conic switching laws, such that a neighborhood $B(x_t,\epsilon)$ of the target state can be reached even when we switch in a neighborhood $B(x_1,\delta)$ of x_1 and in the marginal conic regions. This establishes the robustness of the switching control laws.

6.2 Stabilizability and Reachability

Now we state the relationship between stabilizability and reachability. Without loss of generality, we just consider switched systems (1.1) with two subsystems.

If a switched system is reachable, then it must be asymptotically stabilizable and d-stabilizable. This is not difficult to show. For asymptotic stabilizability, we can start from x_i and drive the state to x_t which are on l_i and $\|x_t\|_2 \leq q\|x_i\|_2$, $q < 1$. Continuing this way, we can asymptotically stabilize the system. d-stabilizability can be similarly shown.

Combined with the results in Sections 2, 3 and 4, we can readily prove the following theorem.

Theorem 6.1 *Consider the second-order switched systems (1.1) consisting of two LTI subsystems with foci. If there does not exist $c \neq 0$ such that $A_1 = cA_2$, then the switched system is asymptotically stabilizable and d-stabilizable if and only if it is reachable.*

Theorem 6.1 provides us with a necessary and sufficient condition for the reachability of the switched systems.

7 Conclusions

This paper considers the reachability problem for switched systems consisting of second-order LTI subsystems with foci. The problem is concerned with whether we can find a switching control law to drive the state from $x_i \neq 0$ to $x_t \neq 0$. Necessary and sufficient conditions for reachability is also obtained.

The method to obtain a switching control law is intuitive and constructive. It is based on the conic switching laws proposed in [13, 14]. We discussed various cases according to the stability and directions of subsystems and proposed switching control law for each case.

The method in this paper is also applicable to second-order switched systems consisting of subsystems not necessarily with foci (e.g., with nodes, or saddle points). As long as the switched system is asymptotically stabilizable and d-stabilizable, we can always design switching control laws for reachability similarly to the methods described in this paper.

Appendix

Consider two second-order systems $\dot{x}(t) = A_1x(t)$, $\dot{x}(t) = A_2x(t)$ with foci. Suppose $x_0 \neq 0$ is an initial point. Let $C^+_{A_1}$, $C^-_{A_1}$ be the set of points on the trajectory of system 1 starting from x_0 and evolving for $t \geq 0$ and $t \leq 0$ respectively. Similarly, we can define $C^+_{A_2}$ and $C^-_{A_2}$. We have the following results.

Lemma 1 *If $A_1 = cA_2$, $c > 0$, then $C^+_{A_1} = C^+_{A_2}$.*

Proof: Let's just consider the case $A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, since for general A_1, we can always use linear transformation to transform A_1 into this form.

From linear systems theory, we know that

$$C^+_{A_1} = \{x = r_0 e^{\alpha t} \begin{bmatrix} \cos(\theta_0 - \beta t) \\ \sin(\theta_0 - \beta t) \end{bmatrix} | t \geq 0\},$$

$$C^+_{A_2} = \{x = r_0 e^{c\alpha t} \begin{bmatrix} \cos(\theta_0 - c\beta t) \\ \sin(\theta_0 - c\beta t) \end{bmatrix} | t \geq 0\},$$

where $x_0 = [x_{01},\ x_{02}]^T = [r_0 \cos\theta_0,\ r_0 \sin\theta_0]^T$. By letting $t' = ct$ for the definition of $C^+_{A_2}$, we can get

$$C^+_{A_2} = \{x = r_0 e^{\alpha t'} \begin{bmatrix} \cos(\theta_0 - \beta t') \\ \sin(\theta_0 - c\beta t') \end{bmatrix} | t' \geq 0\}.$$

So we get $C^+_{A_1} = C^+_{A_2}$. □

Note that from the above lemma, we can also conclude that if $A_1 = -cA_2$, $c > 0$, then $C^+_{A_1} = C^-_{A_2}$ and $C^-_{A_1} = C^+_{A_2}$.

Lemma 2 *If there does not exist $c \neq 0$ such that $A_1 = cA_2$, then there exists at most two lines through the origin in $\mathbb{R}^2$ plane such that any point x on these lines satisfies $A_1 x = kA_2 x$, for some $k \in \mathbb{R}$, $k \neq 0$, i.e., the directions of the two vector fields for the two systems at x are either of exactly the same or of opposite directions.*

Proof: Assume $A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (We can use linear transformation to achieve this).

Suppose for some $x = [x_1, x_2]^T \in \mathbb{R}^2$, $\exists k$ such that $A_1 x = kA_2 x$, then

$$\frac{\alpha x_1 + \beta x_2}{-\beta x_1 + \alpha x_2} = \frac{a_{11} x_1 + a_{12} x_2}{a_{21} x_1 + a_{22} x_2}, \tag{1}$$

where $-\beta x_1 + \alpha x_2 \neq 0$. The above relationship can be transformed into

$$(\alpha a_{21} + a_{11}\beta) x_1^2 + (\alpha a_{22} + \beta a_{21} - \alpha a_{11} + \beta a_{12}) x_1 x_2 + (\beta a_{22} - a_{12}\alpha) x_2^2 = 0, \tag{2}$$

which even does not require $-\beta x_1 + \alpha x_2 \neq 0$.

Now notice that if there are any coefficients of x_1^2, $x_1 x_2$, x_2^2 not equal to zero, then there are at most two lines through the origin on which (2) is true.

We claim that the case that all the three coefficients are equal to zero is not possible. If otherwise, we have

$$\begin{aligned} \alpha a_{21} + a_{11}\beta &= 0 && (3) \\ \beta a_{22} - a_{12}\alpha &= 0 && (4) \\ \alpha a_{22} + \beta a_{21} - \alpha a_{11} + \beta a_{12} &= 0. && (5) \end{aligned}$$

Then we get $a_{11} = -\frac{\alpha}{\beta}a_{21}$ from (3), $a_{12} = \frac{\beta}{\alpha}a_{22}$ from (4) and substitute these two equations into (5), we can show $a_{22} = -\frac{\alpha}{\beta}a_{21}$, $a_{11} = a_{22}$, $a_{12} = \frac{\beta}{\alpha}a_{22}$ after some calculation. This means that there exists a $c \neq 0$ such that $A_1 = cA_2$, which is a contradiction to the conditions of the lemma.□

Remark: The conclusion of Lemma 2 is also true for general second-order LTI systems with A_1 and A_2 not necessarily with foci.

References

[1] M. S. Branicky, *Stability of switched and hybrid systems*, in Proc. 33rd Conf. Decision and Control, 1994, pp.3498-3503.

[2] M. S. Branicky, *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems*, IEEE Trans. on Automatic Control, vol. 43, 1998, pp.475-482.

[3] B. Hu, X. Xu, A. N. Michel and P. J. Antsaklis, *Robustness of stabilizing control laws for a class of second-order switched systems*, to appear, in Proc. 1999 American Control Conf., 1999.

[4] M. Johansson and A. Rantzer, *Computation of piecewise quadratic Lyapunov functions for hybrid systems*, IEEE Trans. on Automatic Control, vol. 43, 1998, pp.555-559.

[5] H. K. Khalil, *Nonlinear systems*, second edition, Prentice-Hall, 1996.

[6] D. Liberzon, A. S. Morse, *Benchmark problems in stability and design of switched systems*, submitted to IEEE Control Systems Magazine.

[7] K. A. Loparo, J. T. Aslanis, O. IIajek, *Analysis of switching linear systems in the plane, part 1: Local behavior of trajectories and local cycle geometry*, J. Optim. Theory Appl., vol. 52, 1987, pp. 365-394.

[8] K. A. Loparo, J. T. Aslanis, O. IIajek, *Analysis of switching linear systems in the plane, part 1: Global behavior of trajectories, controllability and attainability*, J. Optim. Theory Appl., vol. 52, 1987, pp. 395-427.

[9] P. Peleties and R. DeCarlo, *Asymptotic stability of m-switched systems using Lyapunov-like functions*, in Proc. of American Control Conf., pp.1679-1684, Boston, June 1991.

[10] S. Pettersson, B. Lennartson, *Stability and robustness for hybrid systems*, in Proc. 35th Conf. Decision and Control, 1996, pp. 1202-1207.

[11] R. N. Shorten, K. S. Narendra, *A sufficient condition for the existence of a common Lyapunov function for two second-order linear systems*, in Proc. 36th Conf. Decision and Control, 1997, pp. 3521-3522.

[12] J. H. van Schuppen, *A sufficient condition for controllability of a class of hybrid systems*, in Hybrid Systems: Computation and Control, LNCS 1386, pp. 374-383, Springer 1998.

[13] X. Xu, P. J. Antsaklis, *Stabilization of second-Order LTI switched systems*, ISIS Technical Report isis-99-001, Department of Electrical Engineering, University of Notre Dame, January 1999. (`http://www.nd.edu/~isis/tech.html`)

[14] X. Xu, P. J. Antsaklis, *Design of stabilizing control laws for second-order switched systems*, to appear, in Proc. of 14th IFAC world congress, 1999.

[15] X. Xu, P. J. Antsaklis, *On the reachability of a class of second-order switched systems*, to appear, in Proc. of American Control Conf., 1999.