Stabilization of Second-Order LTI Switched Systems

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Abstract

In this paper, the problem of asymptotically stabilizing switched systems consisting of second-order LTI subsystems is studied and solved. In particular, necessary and sufficient stabilizability conditions are derived and a design procedure to construct stabilizing switching laws is introduced. Switching is needed for the stabilization of a switched system if none of its subsystems is stable. Switched systems consisting of subsystems with unstable foci are studied first and stabilizing conic switching control laws for such systems are introduced. In particular, necessary and sufficient conditions for asymptotic stabilizability are derived for such systems. This result is then extended to switched systems with unstable nodes and saddle points. If a switched system is asymptotically stabilizable, then using the conic switching approach introduced earlier, asymptotically stabilizing switching control laws can be obtained. Furthermore, the conic switching laws derived in the paper are shown to be robust. Finally, the stabilizing method described here which is based on the conic switching control approach is briefly compared with a matrix inequality based design method. Miscellaneous examples are presented to show the effectiveness of the method throughout the paper.

1 Introduction

A switched system is a particular kind of hybrid system. By the term switched system, we mean a system that consists of several subsystems and a switching law that specifies which subsystem dynamics will be followed at each instant of time. Currently, switched systems are under extensive investigation in the literature because many real-world processes can be modeled as switched systems. Such processes and systems include chemical processes, electrical circuits, computer controlled systems, intelligent control systems, to name a few.

As in the study of ordinary dynamical systems, the issue of stability of switched systems is considered to be of great importance. Note that various switching laws will make the behavior of a switched system quite different from the behaviors of its components. A switched system can be unstable even if all of its subsystems are stable, or a switched system can be stable even if none of its subsystems is stable; examples can be found in [2, 4, 10]. Conventional analysis and design approaches must be extended to be applicable in the study of switched systems.

Currently, there are three stability problems extensively studied by researchers in the field [10], they are:

(i). Stability under arbitrary switching laws.

(ii). Stability under restricted (but useful) classes of switching laws.

(iii). Design of stabilizing switching laws.

There are many results for Problem (i). Basically, in this problem, all the subsystems are assumed to be stable. Various results adopting a common Lyapunov function in state space representations for the stability of such switched systems can be found in literatures [9, 13, 14, 15, 18, 19]. And in Chapter 3 of [5], a frequency domain result is also presented.

Since the class of switched systems that can be stabilized by arbitrary switching laws is, in general, very limited, Problem (ii) received a lot of attention from many researchers.

For switched systems consisting of stable subsystems, the simplest class of switching laws are those with slow switching dwell time. Results on dwell time switching can be found in [12].

In [2], Branicky formally defined Lyapunov-like functions and provided an analysis result for stability of a switched system using multiple Lyapunov functions. This result is a natural extension of conventional Lyapunov stability theory and proves to be a useful tool for the stability analysis of switched systems. In [16], the asymptotic stability of a switched system consisting of m LTI subsystems is studied by using one quadratic Lyapunov-like function for each subsystem. Sufficient conditions for global asymptotic stability are obtained. It is noted that all these methodologies can also be applied to switched systems consisting of unstable subsystems.

In [17], Linear Matrix Inequality (LMI) problems are formulated for the stability analysis of switched systems consisting of linear subsystems. The method searches for multiple Lyapunov functions. If multiple Lyapunov functions can be found, then the switched system is stable. The paper extend the idea in [16]. Note in [17], it is assumed that the switching control law has been given and so the LMI approach is not design oriented in this case. In [7], LMI problems are also formulated as to show the stability of the switched systems with affine subsystems. LMIs provide a good way of finding sufficient conditions for stability in switched systems.

The two problems metioned above put emphasis on the stability analysis of switched systems, while Problem (iii) emphasize the design problem for switched systems. Recently, the design problem has been receiving increasingly more attention.

Following the idea of one common Lyapunov function, Wicks, Peleties and DeCarlo in [21, 23] use single Lyapunov function techniques for the design of stabilizing switching laws. Another systematic way to solve the design problem is to try to find a stabilizing switching law and prove stability using multiple Lyapunov functions. LMI problems may be formulated in order to solve the design problem [22].

While methods using Lyapunov function(s) are quite general and provide sufficient conditions for the stabilizability of switched systems, yet it is not easy to solve these problems. And if we cannot solve the problem using single or multiple Lyapunov functions, we cannot say much about the stabilizability of the switched system. The focus in this paper is on the stabilizability analysis and design of stabilizing switching laws for switched systems consisting of second-order LTI subsystems. In [24], the authors had already provided asymptotic stabilization results for switched systems consisting of second-order LTI subsystems with unstable foci. The current paper provides a more complete version of [24] and also extends the result to more general second-order switched systems with unstable subsystems. The methodology is extended to switched systems consisting of subsystems with unstable nodes and saddle points. Necessary and sufficient conditions for asymptotic stabilizability for such systems can be obtained. If a switched system is asymptotically stabilizable, then a stabilizing switching control law can be derived. It is also shown that the switching control law derived in this paper is robust.

The methodology used in this paper is based on geometric intuition and it has two distinct advantages: (1) Unlike other currently available methodologies, the methodology here provides necessary and sufficient conditions for asymptotic stabilizability of the switched systems we study, rather than just sufficient conditions. (2) The methodology is easy to be implemented and requires little amount of computation.

The structure of the paper is as follows. In Section 2, we provide some definitions and preliminaries that are helpful for our studies later in the paper. In Section 3, we study switched systems consisting of subsystems with unstable foci and derive stabilizing conic switching control laws. In particular, necessary and sufficient conditions for asymptotic stabilizability are derived for such systems. Then the result is extended to switched systems with unstable nodes and saddle points in Section 4 and Section 5, respectively. If a switched system is asymptotically stabilizable, an asymptotically stabilizing switching control law can also be obtained. It is shown in Section 6 that the conic switching law is robust. In Section 7, we compare the method derived here to a matrix inequality based design method. Section 8 concludes the paper. Throughout the paper, miscellaneous examples are presented to show the effectiveness of the method for the class of switched systems we study in the paper.

2 Some Preliminaries

A switched system is a system that consists of several subsystems and a switching law that specifies which subsystem dynamics will be followed by the system trajectory at each instant of time. The center of interest of this paper is on switched systems consisting of second-order LTI subsystems. A *switched system consisting of second-order LTI subsystems* is a system described by system equations:

$$\dot{x} = A_{i(t)}x \tag{2.1}$$

$$i(t) = \phi(x(t), i(t^-), t), \tag{2.2}$$

where $x \in \mathbb{R}^2$, $A_i \in \mathbb{R}^{2\times 2}$ and $i(t) \in I = \{1, 2, \cdots, N\}$. $\phi : \mathbb{R}^2 \times I \times \mathbb{R} \to I$ describes the switching law. It is assumed that the switching law can only generate a finite number of switches in any finite time period.

In the following, mathematical definitions and some related preliminaries will be introduced.

For a second-order LTI system $\dot{x} = Ax$, $A \in \mathbb{R}^{2\times 2}$, we will say that the *direction of vector field* (or the direction of the solution) of the system at $x \neq 0$ is clockwise (resp. counterclockwise) if, starting from x, the system has the potential to drive the trajectory clockwise (resp. counterclockwise) from x within a very small amount of time. Notice that from Figure 1., the vector field $f(x) = Ax$ at x is clockwise (resp. counterclockwise) if and only if the absolute value of the angle between x and $(x_2, -x_1)^T$ is smaller than or equal to $\frac{\pi}{2}$ (resp. greater than or equal to $\frac{\pi}{2}$). Therefore, we formally define the direction of the vector field at x as follows:

Definition 2.1 *The direction of the vector field of a second-order LTI system $\dot{x} = Ax$ at a point $x \neq 0$ is said to be clockwise (counterclockwise) if the dot product of $(x_2, -x_1)^T$ and Ax is greater than or equal to zero (less than or equal to zero), i.e., $(x_2, -x_1)Ax \geq 0$ ($(x_2, -x_1)Ax \leq 0$).*

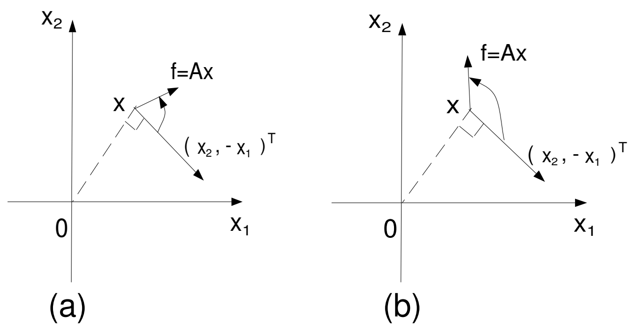

Figure 1: The directions of vector field Ax at x, (a) clockwise, (b) counterclockwise.

2.1 Direction of the Solution of $\dot{x} = Ax$ with Focus at the Origin

The solution of the LTI system

$$\dot{x} = Ax, \ x(0) = x_0 \tag{2.3}$$

is well known to be

$$x(t) = e^{At}x_0 \text{ for } t \geq 0. \tag{2.4}$$

When $x = (x_1, x_2)^T$, $A \in \mathbb{R}^{2\times 2}$, the trajectory of the solution can be shown on the $\mathbb{R}^2$ plane. In this subsection, we are interested in the case when A has two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$. The origin $(0, 0)^T$ is often called a stable($\alpha < 0$) or an unstable($\alpha > 0$) focus of system (2.3). And the solution

(2.4) is a logarithmic spiral on the $X_2 - X_1$ plane. The question we are concerned with here is how to decide, without plotting the trajectory, whether the trajectory will be a spiral around the origin in clockwise or counterclockwise direction. In the following, coordinate transformations are considered so that this question can be answered.

Let us first consider the simplest case where $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. We have the following Lemma for this case.

Lemma 2.2 *For the LTI autonomous system $\dot{x}(t) = Ax(t)$ with focus, where $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, the solution with $x(0) = x_0 \neq 0$ has the following properties:*

- *If $\alpha < 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin clockwise.*
- *If $\alpha < 0$ and $\beta < 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin counterclockwise.*
- *If $\alpha > 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that diverges to ∞ clockwise.*
- *If $\alpha > 0$ and $\beta < 0$, then the solution $x(t) = e^{At}x_0$is a logarithmic spiral that diverges to ∞ counter-clockwise.*

Proof: We just prove the $\alpha < 0$ and $\beta > 0$ case, the others can be shown via similar arguments.

From linear system theory, we know that

$$e^{At} = e^{\alpha t} \begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix}.$$

Notice that the initial state $x_0 = [x_{01},\ x_{02}]^T = [r_0 \cos \theta_0,\ r_0 \sin \theta_0]^T$, where $r_0 = \sqrt{x_{01}^2 + x_{02}^2}$ and $\cos \theta_0 = \frac{x_{01}}{r_0}$, $\sin \theta_0 = \frac{x_{02}}{r_0}$. In other words, the representation of x_0 in the polar coordinate system is (r_0, θ_0).

Now we get

$$\begin{aligned} x(t) &= e^{At}x_0 \\ &= r_0 e^{\alpha t} \begin{bmatrix} \cos \theta_0 \cos \beta t + \sin \theta_0 \sin \beta t \\ -\cos \theta_0 \sin \beta t + \sin \theta_0 \cos \beta t \end{bmatrix} \\ &= r_0 e^{\alpha t} \begin{bmatrix} \cos(\theta_0 - \beta t) \\ \sin(\theta_0 - \beta t) \end{bmatrix}. \end{aligned}$$

Observe that the representation of $x(t)$ in the polar coordinate system is $(r_0 e^{\alpha t}, \theta_0 - \beta t)$, so it is clear that if $\alpha < 0$ and $\beta > 0$, $x(t)$ is a logarithmic spiral that converges to the origin clockwise, i.e., $\|x(t)\|_2 \to 0$ as $t \to \infty$. □

Suppose now that $\mathbf{e}_{X_1}$ and $\mathbf{e}_{X_2}$ are two unit vectors on $\mathbb{R}^2$ for X_1 and X_2 axes, respectively. If a point x has coordinate representation $x = [x_1,\ x_2]^T$, the corresponding vector on $\mathbb{R}^2$ is

$$x = \begin{bmatrix} \mathbf{e}_{X_1}, & \mathbf{e}_{X_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.5}$$

If we choose another coordinate system $Y_2 - Y_1$ with unit vectors $\mathbf{e}_{Y_1}$ and $\mathbf{e}_{Y_2}$ for Y_1 and Y_2 axes, respectively, and

$$\begin{bmatrix} \mathbf{e}_{Y_1}, & \mathbf{e}_{Y_2} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{X_1}, & \mathbf{e}_{X_2} \end{bmatrix} T, \text{ where } T \in \mathbb{R}^{2\times 2} \text{ nonsingular}, \tag{2.6}$$

then $y = T^{-1}x$ will be the representation of x in the $Y_2 - Y_1$ coordinate system.

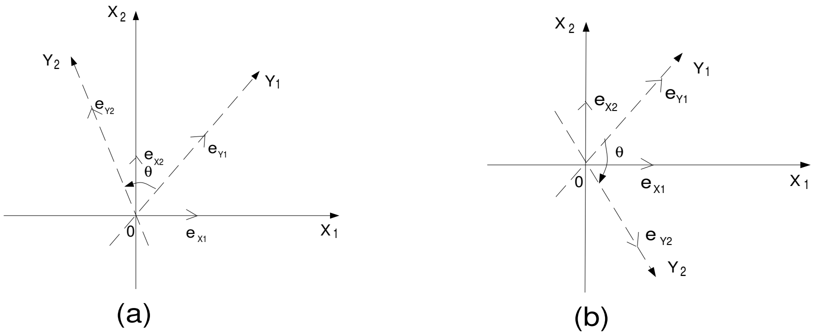

Figure 2: (a) $Y_2 - Y_1$ and $X_2 - X_1$ coordinate system agree in direction. (b) $Y_2 - Y_1$ and $X_2 - X_1$ coordinate system disagree in direction.

Definition 2.3 *The coordinate systems $Y_2 - Y_1$ and $X_2 - X_1$ are said to* **agree (disagree) in direction** *if the directional angle θ confined to $-\pi \le \theta < \pi$ from Y_1 axis to Y_2 axis satisfies $0 < \theta < \pi$ ($-\pi < \theta < 0$) on $X_2 - X_1$ plane, with θ measured counterclockwise.*

Figure 2 shows $Y_2 - Y_1$ and $X_2 - X_1$ that agree and disagree in direction.

A simple criterion to decide the agreement or disagreement issue is to look at the linear transformation matrix T by the following Lemma.

Lemma 2.4 *The coordinate transforamtion in (2.6) results in a coordinate system $Y_2 - Y_1$ that agrees (disagrees) in direction with $X_2 - X_1$ system if $detT > 0$ ($detT < 0$). In the following, we also say that T agrees (disagrees) in direction with $X_2 - X_1$ system.*

Proof: Suppose

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

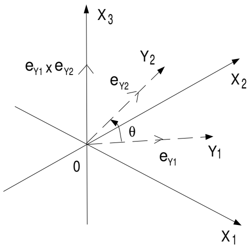

Figure 3: Coordinate system $X_3 - X_2 - X_1$ in $\mathbb{R}^3$.

By adding another axis X_3 which is perpendicular to the plane $X_2 - X_1$ as shown in Figure 3, we get a coordinate system $X_3 - X_2 - X_1$ in $\mathbb{R}^3$. By analytical geometry, the cross product

$$\begin{aligned} \mathbf{e}_{Y_1} \times \mathbf{e}_{Y_2} &= det \begin{bmatrix} \mathbf{e}_{X_1} & \mathbf{e}_{X_2} & \mathbf{e}_{X_3} \\ t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \end{bmatrix} \\ &= det \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \mathbf{e}_{X_3} \\ &= (detT)\mathbf{e}_{X_3}. \end{aligned} \tag{2.7}$$

Since the cross product, in this case, can also be written as

$$\mathbf{e}_{Y_1} \times \mathbf{e}_{Y_2} = (\|\mathbf{e}_{Y_1}\|_2 \|\mathbf{e}_{Y_2}\|_2 \sin\theta)\mathbf{e}_{X_3}, \tag{2.8}$$

so by (2.7), (2.8), we conclude that $\sin\theta > 0$ iff $detT > 0$. Therefore, if $detT > 0$ ($detT < 0$), we have $0 < \theta < \pi$ ($-\pi < \theta < 0$), that is, $Y_2 - Y_1$ system agrees (disagrees) in direction with $X_2 - X_1$ system. □

Now we come to the general case. Suppose $A \in \mathbb{R}^{2\times 2}$ has two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$. By linear algebra, we know that there exists nonsingular T such that:

$$A = T\Lambda T^{-1}, \text{ where } \Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \tag{2.9}$$

If $detT < 0$, we can let $T_1 = TE_{12}$, with $E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to make $detT_1 > 0$, and

$$A = T_1\Lambda_1 T_1^{-1}, \text{ where } \Lambda_1 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{2.10}$$

So for simplicity, we assume that $detT > 0$ in (2.9), i.e., T agrees with the direction of $X_2 - X_1$ system. By letting $y = T^{-1}x$, (2.3) can be reduced to

$$\dot{y}(t) = \Lambda y(t), \; y(0) = T^{-1}x(0) = T^{-1}x_0. \tag{2.11}$$

Up to this stage, we can use the following Theorem to decide the direction of the solution for (2.3).

Theorem 2.5 *Let $A \in \mathbb{R}^{2\times 2}$ have two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$, and $A = T\Lambda T^{-1}$ as in (2.9), where T agrees with the direction of $X_2 - X_1$ system. Then the direction of the solution (clockwise or counterclockwise) of (2.3) is the same as the direction of the solution of (2.11).*

Proof: Since T agrees with the direction of $X_2 - X_1$ system, so $Y_2 - Y_1$ system agrees with the direction of $X_2 - X_1$ system. Therefore a solution whose direction is clockwise (counterclockwise) in $Y_2 - Y_1$ system will also be clockwise (counterclockwise) in $X_2 - X_1$ system and vice versa. □

By Theorem 2.5, we can find the direction of a solution in $X_2 - X_1$ system by reducing it to a problem in $Y_2 - Y_1$ system and then using Lemma 2.2. In the following, we would say that a subsystem $\dot{x}(t) = A_i x(t)$ is *of clockwise (counterclockwise) direction* if the direction of its solution has clockwise (counterclockwise) direction.

Example 2.6 *Consider $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$, we have $A = T\Lambda T^{-1}$ with $T = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} -2 & 5 \\ -5 & -2 \end{bmatrix}$. By Theorem 2.5 and Lemma 2.2, the solution of (2.3) associated with this A converges to the origin clockwise.*

2.2 Other Preliminaries for $\dot{x} = Ax$ with Focus

Consider two second-order systems $\dot{x}(t) = A_1 x(t)$, $\dot{x}(t) = A_2 x(t)$ with foci. Suppose $x_0 \neq 0$ is an initial point. Let $C^+_{A_1}$, $C^-_{A_1}$ be the set of points on the trajectory of system 1 starting from x_0 and evolving for $t \geq 0$ and $t \leq 0$ respectively. Similarly, we can define $C^+_{A_2}$ and $C^-_{A_2}$. We have the following results.

Lemma 2.7 *If $A_1 = cA_2$, $c > 0$, then $C^+_{A_1} = C^+_{A_2}$.*

Proof: Let's just consider the case $A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, since for general A_1, we can always use linear transformation to transform A_1 into this form.

From linear systems theory, we know that

$$C^+_{A_1} = \{x = r_0 e^{\alpha t} \begin{bmatrix} \cos(\theta_0 - \beta t) \\ \sin(\theta_0 - \beta t) \end{bmatrix} | t \geq 0\},$$

$$C^+_{A_2} = \{x = r_0 e^{c\alpha t} \begin{bmatrix} \cos(\theta_0 - c\beta t) \\ \sin(\theta_0 - c\beta t) \end{bmatrix} | t \geq 0\},$$

where $x_0 = [x_{01},\ x_{02}]^T = [r_0 \cos\theta_0,\ r_0 \sin\theta_0]^T$. By letting $t' = ct$ for the definition of $C^+_{A_2}$, we can get

$$C^+_{A_2} = \{x = r_0 e^{\alpha t'} \begin{bmatrix} \cos(\theta_0 - \beta t') \\ \sin(\theta_0 - c\beta t') \end{bmatrix} | t' \geq 0\}.$$

So we get $C^+_{A_1} = C^+_{A_2}$. □

Note that from the proof of the above lemma, we can also conclude that if $A_1 = -cA_2$, $c > 0$, then $C^+_{A_1} = C^-_{A_2}$ and $C^-_{A_1} = C^+_{A_2}$.

Lemma 2.8 *If there does not exist $c \neq 0$ such that $A_1 = cA_2$, then there exists at most two lines through the origin in $\mathbb{R}^2$ plane such that any point x of these lines satisfies $A_1 x = kA_2 x$, for some $k \in \mathbb{R}$, $k \neq 0$, i.e., the directions of the two vector fields for the two systems at x are either of the same or of opposite directions.*

Proof: Assume $A_1 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (We can use linear transformation to achieve this).

Suppose for some $x = [x_1, x_2]^T \in \mathbb{R}^2$, $\exists k$ such that $A_1 x = kA_2 x$, then

$$\frac{\alpha x_1 + \beta x_2}{-\beta x_1 + \alpha x_2} = \frac{a_{11} x_1 + a_{12} x_2}{a_{21} x_1 + a_{22} x_2}, \tag{2.12}$$

where $-\beta x_1 + \alpha x_2 \neq 0$. The above relationship can be tranformed into

$$(\alpha a_{21} + a_{11}\beta) x_1^2 + (\alpha a_{22} + \beta a_{21} - \alpha a_{11} + \beta a_{12}) x_1 x_2 + (\beta a_{22} - a_{12}\alpha) x_2^2 = 0, \tag{2.13}$$

which even does not necessarily require $-\beta x_1 + \alpha x_2 \neq 0$.

Now notice that if there are any coefficients of x_1^2, $x_1 x_2$, x_2^2 not equal to zero, then there are at most two lines through the origin on which (2.13) is true.

We claim that the case that all the three coefficients are equal to zero is not possible. If otherwise, we have

$$\alpha a_{21} + a_{11}\beta = 0 \tag{2.14}$$

$$\beta a_{22} - a_{12}\alpha = 0 \tag{2.15}$$

$$\alpha a_{22} + \beta a_{21} - \alpha a_{11} + \beta a_{12} = 0. \tag{2.16}$$

Then we get $a_{11} = -\frac{\alpha}{\beta} a_{21}$ from (2.14), $a_{12} = \frac{\beta}{\alpha} a_{22}$ from (2.15) and substitute these two equations into (2.16), we can show $a_{22} = -\frac{\alpha}{\beta} a_{21}$, $a_{11} = a_{22}$, $a_{12} = \frac{\beta}{\alpha} a_{22}$ after some calculation. This means that there exists a $c \neq 0$ such that $A_1 = cA_2$, which is a contradiction to the conditions of the lemma. □

Remark: The conclusion of Lemma 2.8 is also true for general second-order LTI systems with A_1 and A_2, not necessarily with foci.

3 Stabilization of Second-Order LTI Switched Systems with Foci

We first study switched systems consisting of two second-order LTI subsystems with unstable foci and then present the results for several subsystems. Necessary and sufficient conditions for asymptotic stabilizability of such switched systems are derived. If a switched system is asymptotically stabilizable, an asymptotically stabilizing switching control law can also be obtained.

3.1 Two Subsystems with Unstable Foci

For switched systems consisting of two subsystems with unstable foci at the origin, we will study the vector fields of both subsystems and give an intuitive way of obtaining a stabilizing switching control law. In the following, we will first consider switched systems with subsystems of the same direction and then consider switched systems with subsystems of opposite directions.

3.1.1 Two Subsystems of the Same Direction

We will consider the following switched systems,

$$\dot{x}(t) = A_1 x(t),\ \dot{x}(t) = A_2 x(t), \tag{3.1}$$

whose subsystems are both with unstable foci and are both of clockwise directions. For switched systems whose subsystems are both of counterclockwise directions, all the following discussion can be applied in a completely analogous manner.

Let $x = (x_1, x_2)^T$ be a nonzero point on $\mathbb{R}^2$ plane, and let

$$f_1 = A_1 x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},\ f_2 = A_2 x = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}. \tag{3.2}$$

We can view x, f_1 and f_2 as vectors in $\mathbb{R}^2$ and we denote $\theta_i, i = 1, 2$ to be the angle from x to f_i measured counterclockwise with respect to x (θ_i is confined to $-\pi \leq \theta_i < \pi$). So in this case, when the ith subsystem is of clockwise direction, $-\pi \leq \theta_i \leq 0$ (Figure 4(a)). Since if it were otherwise, assume that f_i (here we use f_1 for illustration) is on the other side of x as shown in Figure 4(b), then in sufficiently small elapsed time dt, the trajectory will travel in the counterclockwise direction, which contradicts our assumption.

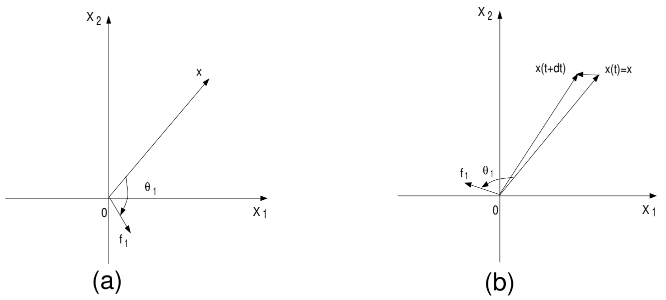

Figure 4: (a) The angle θ_1 from x to f_1. (b) f_1 is on the other side of vector x.

Note: In the following, we will take $a_1x_1 + a_2x_2$ as the dot product of vectors f_1 and x. And we will also take $a_1x_2 - a_2x_1$ as the dot product of vectors f_1 and $y = (x_2, -x_1)^T$. In this way, we can provide geometrical interpretation for the above expressions (refer to Figure 5), i.e., $a_1x_1 + a_2x_2 = x^T f_1 = \|x\|_2 \|f_1\|_2 \cos\theta_1$ and $a_1x_2 - a_2x_1 = y^T f_1 = \|y\|_2 \|f_1\|_2 \cos(\theta_1 + \frac{\pi}{2}) = \|x\|_2 \|f_1\|_2 \cos(\theta_1 + \frac{\pi}{2})$. Similar argument also applies to f_2.

Now we design a switching control law that asymptotically stabilizes the switched system. In other words, by our switching control law, we want to drive the trajectory closer and closer to the origin, i.e., $\|x(t)\|_2 \to 0$

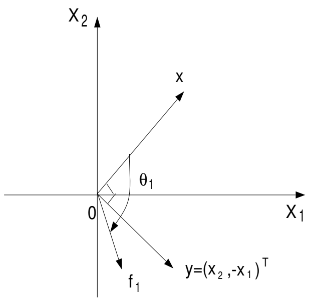

Figure 5: Dot Product.

as $t \to \infty$. In the following, we try to obtain such a switching law by always choosing a subsystem which has the potential to drive the trajectory closer to the origin.

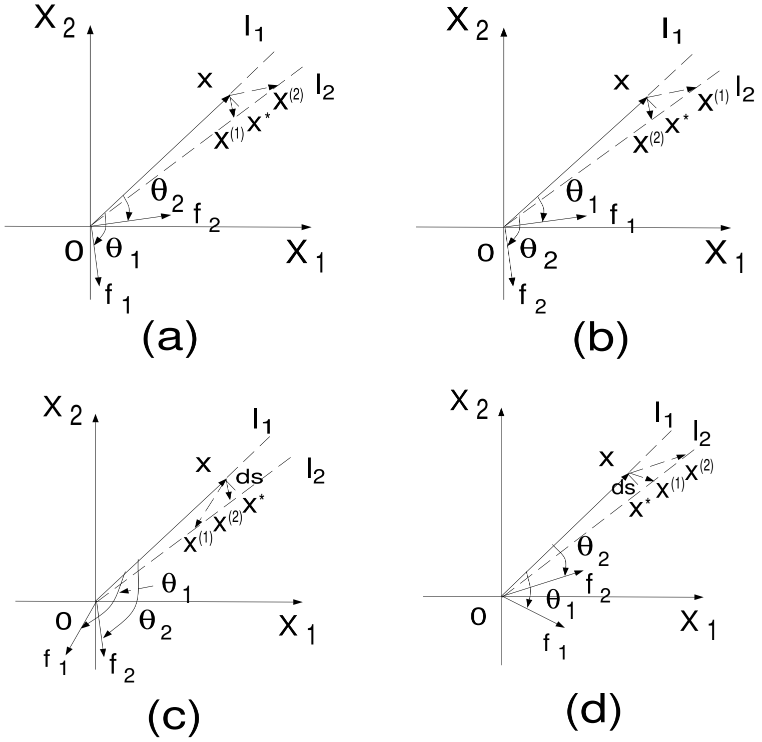

Figure 6: The four different cases.

For a nonzero $x \in \mathbb{R}^2$, let l_1 be a ray determined by the vector x. Let l_2 be a ray sufficiently close to l_1 as shown in Figure 6(a). The trajectory will intersect l_2 at the point $x^{(1)}$ if it follows subsystem 1, or it will intersect l_2 at $x^{(2)}$ if it follows subsystem 2. Let x^* be the point where l_2 is intersected by a line perpendicular to l_1. Let $\|x^{(1)}\|_2$, $\|x^{(2)}\|_2$ be the distances from $x^{(1)}$, $x^{(2)}$ to the origin, respectively. And let

$ds = \|x^* - x\|_2$ (Figure 6). Also we define the following conic regions

$$
\begin{aligned}
E_{is} &= \{x| -\pi \le \theta_i(f_i(x)) \le -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \le \theta_i(f_i(x)) < \pi\} \\
&= \{x| x^T f_i(x) = x^T A_i x \le 0\},\ i = 1, 2 \qquad (3.3) \\
E_{iu} &= \{x| -\frac{\pi}{2} \le \theta_i(f_i(x)) \le \frac{\pi}{2}\} \\
&= \{x| x^T f_i(x) = x^T A_i x \ge 0\},\ i = 1, 2. \qquad (3.4)
\end{aligned}
$$

Now we shall associate a better subsystem to x as follows:

Case 1: Assume that

$$x \in E_{1s} \cap E_{2u}.$$

In this case, as shown in Figure 6(a), $\|x^{(1)}\|_2 \le \|x^*\|_2 \le \|x^{(2)}\|_2$ when ds is sufficiently small. So we can choose subsystem 1 in order to drive the trajectory closer to the origin. We denote the conic region $E_{1s} \cap E_{2u}$ as Ω_1.

Case 2: Assume that

$$x \in E_{1u} \cap E_{2s}.$$

In this case, as shown in Figure 6(b), we can choose subsystem 2 by a similar argument as in Case 1. We denote the conic region $E_{1u} \cap E_{2s}$ as Ω_2.

Case 3: Assume that

$$x \in E_{1s} \cap E_{2s}.$$

In this case, as shown in Figure 6(c), consider

$$\frac{\|x\|_2 - \|x^{(1)}\|_2}{\|x\|_2 - \|x^{(2)}\|_2} \text{ as } ds \to 0.$$

Let dt_1 and dt_2 be the time the system would take to intersect l_2 along the vector f_1 and f_2, respectively. When ds is sufficiently small, it is not difficult to see that we approximately have

$$dt_1 = \frac{ds}{\|f_1\|_2 \cos(\theta_1 + \frac{\pi}{2})} = \frac{ds}{\frac{(a_1x_2 - a_2x_1)}{\sqrt{x_2^2 + x_1^2}}}, \qquad (3.5)$$

$$dt_2 = \frac{ds}{\|f_2\|_2 \cos(\theta_2 + \frac{\pi}{2})} = \frac{ds}{\frac{(a_3x_2 - a_4x_1)}{\sqrt{x_2^2 + x_1^2}}}. \qquad (3.6)$$

By (3.5) and (3.6) and using L'Hospital Rule, we obtain

$$
\begin{aligned}
&\lim_{ds \to 0} \frac{\|x\|_2 - \|x^{(1)}\|_2}{\|x\|_2 - \|x^{(2)}\|_2} \\
= &\lim_{ds \to 0} \frac{\sqrt{x_1^2 + x_2^2} - \sqrt{(x_1 + a_1 dt_1)^2 + (x_2 + a_2 dt_1)^2}}{\sqrt{x_1^2 + x_2^2} - \sqrt{(x_1 + a_3 dt_2)^2 + (x_2 + a_4 dt_2)^2}} \\
= &\lim_{ds \to 0} \frac{\frac{(a_1(x_1 + a_1 dt_1) + a_2(x_2 + a_2 dt_1))\sqrt{x_1^2 + x_2^2}}{(a_1x_2 - a_2x_1)\sqrt{(x_1 + a_1 dt_1)^2 + (x_2 + a_2 dt_1)^2}}}{\frac{(a_3(x_1 + a_3 dt_2) + a_4(x_2 + a_4 dt_2))\sqrt{x_1^2 + x_2^2}}{(a_3x_2 - a_4x_1)\sqrt{(x_1 + a_3 dt_2)^2 + (x_2 + a_4 dt_2)^2}}} \\
= &\frac{(a_1x_1 + a_2x_2)(a_3x_2 - a_4x_1)}{(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2)} \qquad (3.7) \\
= &\frac{\cos\theta_1 \cos(\theta_2 + \frac{\pi}{2})}{\cos(\theta_1 + \frac{\pi}{2}) \cos\theta_2} \qquad (3.8)
\end{aligned}
$$

In view of the above, we see that if

$$\lim_{ds\to 0} \frac{\|x\|_2 - \|x^{(1)}\|_2}{\|x\|_2 - \|x^{(2)}\|_2} = \frac{(a_1x_1 + a_2x_2)(a_3x_2 - a_4x_1)}{(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2)} \geq 1, \tag{3.9}$$

or, by simplification (notice $(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2) < 0$ in this case), if

$$a_2a_3 - a_1a_4 \leq 0, \tag{3.10}$$

then we will have $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$ for sufficiently small ds. In this case, we can choose subsystem 1. And we denote the conic region $E_{1s} \cap E_{2s} \cap \{x | a_2a_3 - a_1a_4 \leq 0\}$ as Ω_3.

This condition can also be translated into the following condition (since $\cos(\theta_1 + \frac{\pi}{2})\cos\theta_2 < 0$ in this case)

$$\cos\theta_1 \cos(\theta_2 + \frac{\pi}{2}) - \cos(\theta_1 + \frac{\pi}{2})\cos\theta_2 \leq 0,$$

that is

$$\sin(\theta_1 - \theta_2) \leq 0,$$

that is

$$\theta_1 \leq \theta_2. \tag{3.11}$$

So in this case, if $\theta_1 \leq \theta_2$, we choose subsystem 1. Similarly, if $a_2a_3 - a_1a_4 \geq 0$, i.e., $\theta_1 \geq \theta_2$, we choose subsystem 2 and we denote the conic region $E_{1s} \cap E_{2s} \cap \{x | a_2a_3 - a_1a_4 \geq 0\}$ as Ω_4.

Case 4: Assume that

$$x \in E_{1u} \cap E_{2u}.$$

In this case, as shown in Figure 6(d), consider

$$\frac{\|x^{(1)}\|_2 - \|x\|_2}{\|x^{(2)}\|_2 - \|x\|_2} \text{ as } ds \to 0.$$

In a manner analogous to the derivation of (3.8) above, we obtain

$$\begin{aligned}
&\lim_{ds\to 0} \frac{\|x^{(1)}\|_2 - \|x\|_2}{\|x^{(2)}\|_2 - \|x\|_2} \\
= \; & \frac{(a_1x_1 + a_2x_2)(a_3x_2 - a_4x_1)}{(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2)} & (3.12) \\
= \; & \frac{\cos\theta_1 \cos(\theta_2 + \frac{\pi}{2})}{\cos(\theta_1 + \frac{\pi}{2})\cos\theta_2} & (3.13)
\end{aligned}$$

Then as in Case 3, we see that if

$$\lim_{ds\to 0} \frac{\|x^{(1)}\|_2 - \|x\|_2}{\|x^{(2)}\|_2 - \|x\|_2} = \frac{(a_1x_1 + a_2x_2)(a_3x_2 - a_4x_1)}{(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2)} \leq 1, \tag{3.14}$$

or, by simplification (notice $(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2) > 0$ in this case), if

$$a_2a_3 - a_1a_4 \leq 0, \tag{3.15}$$

then we will have $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$ for sufficiently small ds, so we can choose subsystem 1. And we denote the conic region $E_{1u} \cap E_{2u} \cap \{x | a_2a_3 - a_1a_4 \leq 0\}$ as Ω_5.

Similarly to Case 3, this condition can also be translated into the following condition (since $\cos(\theta_1 + \frac{\pi}{2})\cos\theta_2 > 0$ in this case)

$$\theta_1 \leq \theta_2. \tag{3.16}$$

So in this case, if $\theta_1 \leq \theta_2$, we choose subsystem 1. Similarly, if $a_2a_3 - a_1a_4 \geq 0$, i.e., $\theta_1 \geq \theta_2$, we choose subsystem 2 and we denote the conic region $E_{1u} \cap E_{2u} \cap \{x | a_2a_3 - a_1a_4 \geq 0\}$ as Ω_6. $\square$

Notice that Ω_i, $i = 1, 2, \cdots, 6$, are all conic regions that partition the $\mathbb{R}^2$ plane. We can associate with each region a subsystem according to the above discussion. *In essence, we associate with each point on $\mathbb{R}^2$ a subsystem i whose θ_i is smaller than the angle generated by the other subsystem.* This partition of the $\mathbb{R}^2$ plane is of particular importance here. In the following, we call this switching law (that uses the partition and the associated subsystems) the **conic switching law**. We will show that by using the conic switching law, we can decide whether the system (3.1) is asymptotically stabilizable or not. If (3.1) is asymptotically stabilizable, the partition and associated subsystems also provide a stabilizing switching control law.

Note: If at some time instant t, the trajectory of the switched system is at point x and $\theta_1 = \theta_2$, then the point x is on the boundary of two conic regions. In this case, we associate with x the subsystem whose valid conic region is to be entered. This would make the switching function right continuous.

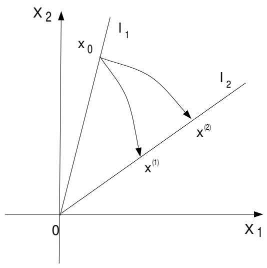

Figure 7: The two rays and related points.

Consider the system (3.1). Let l_1 and l_2 be two different rays that go through the origin and are in the same conic region of one Ω_i. Suppose l_2 is on the clockwise side of l_1 in the conic region and the conic region with angle from l_1 to l_2 is inside Ω_i. Suppose x_0 is on l_1. Let $x^{(1)}$ be the point on l_2 where the trajectory of the system intersects l_2 for the first time if the system evolves according to the subsystem associated with Ω_i corresponding to the conic switching law. Let $x^{(2)}$ be the point on l_2 where the trajectory of the system intersects l_2 for the first time if the system evolves according to the other subsystem. Figure 7 shows the two rays and corresponding points. For the above-mentioned rays and points, we have the following lemma.

Lemma 3.1 *If l_1 and l_2 are in the same conic region Ω_i, $i = 1, 2, \cdots, 6$, then $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.*

Proof: We will prove the cases for Ω_1 and Ω_3, the cases of the other conic regions can be proved by following similar arguments.

The case of Ω_1: Let $r_1(t) = \|x(t)\|_2 - \|x_0\|_2$ and $r_2(t) = \|x(t)\|_2 - \|x_0\|_2$ when the trajectory evolves according to subsystem 1 and subsystem 2, respectively. We have

$$dr_1 = d(\|x(t)\|_2) = \|f_1 \cos\theta_1\|_2 dt = \frac{x^T(t) f_1}{\|x(t)\|_2} dt = \frac{x^T(t) A_1 x(t)}{\|x(t)\|_2} dt,$$

$$dr_2 = d(\|x(t)\|_2) = \|f_2 \cos\theta_2\|_2 dt = \frac{x^T(t) f_2}{\|x(t)\|_2} dt = \frac{x^T(t) A_2 x(t)}{\|x(t)\|_2} dt.$$

Assume t_1 and t_2 are the time the system take to go to $x^{(1)}$ and $x^{(2)}$, respectively. Since in Ω_1, $x^T A_1 x \leq 0$

and $x^T A_2 x \geq 0$, we have

$$r_1(t_1) = \|x^{(1)}\|_2 - \|x_0\|_2 = \int_0^{t_1} \frac{x^T(t)A_1x(t)}{\|x(t)\|_2}dt \leq \int_0^{t_1} 0dt = 0, \tag{3.17}$$

$$r_2(t_2) = \|x^{(2)}\|_2 - \|x_0\|_2 = \int_0^{t_2} \frac{x^T(t)A_2x(t)}{\|x(t)\|_2}dt \geq \int_0^{t_2} 0dt = 0. \tag{3.18}$$

Combining (3.17) and (3.18), we conclude that $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.

The case of Ω_3: If A_1 is a scalar multiple of A_2, then by Lemma 2.7, we conclude that $\|x^{(1)}\|_2 = \|x^{(2)}\|_2$. So in the following, we assume that A_1 is not a scalar multiple of A_2.

Let $\alpha(t)$ be the angle from l_1 to the line on which $x(t)$ lies and let α_1 be the angle from l_1 to l_2. $\alpha(t)$ is measured counterclockwise, so in this case, it is nonpositive as shown in Figure 8. Let $x^{(1)}(t)$ and $x^{(2)}(t)$ be the trajectory of the system if the system follows subsystem 1 and 2, respectively. If we use α as a variable, then we can also denote $x^{(1)}(t)$ and $x^{(2)}(t)$ as $x^{(1)}(\alpha)$ and $x^{(2)}(\alpha)$. In the following, we prove the result by contradiction.

From the discussion of Case 3, we know that for $\alpha \in (-\epsilon, 0)$ with $\epsilon > 0$ very small, $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$.

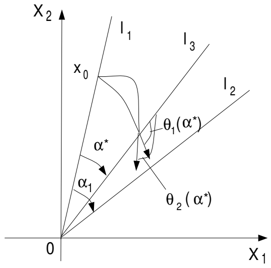

Figure 8: Figure for proof of Lemma 3.1, case Ω_3.

Now suppose at α_1, $\|x^{(1)}(\alpha_1)\|_2 > \|x^{(2)}(\alpha_1)\|_2$, then there exists a

$$\alpha^* = \inf\{\alpha \mid \|x^{(1)}(\alpha_1)\|_2 \leq \|x^{(2)}(\alpha_1)\|_2,\ \alpha \in (\alpha_1, 0)\}. \tag{3.19}$$

By continuity of the trajectories of $x^{(1)}(t)$ and $x^{(2)}(t)$, such a α^* must exist, and at α^*, $\|x^{(1)}(\alpha^*)\|_2 = \|x^{(2)}(\alpha^*)\|_2$. We claim that at α^*,

$$\theta_2(\alpha^*) \leq \theta_1(\alpha^*). \tag{3.20}$$

Since if otherwise, $\theta_2(\alpha^*) > \theta_1(\alpha^*)$, then we will have that for a very small $\epsilon > 0$, $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$ for $\alpha \in (\alpha^* - \epsilon, \alpha^*)$, which is a contradiction to (3.19). So by (3.20) and since $x(\alpha^*)$ is in Ω_3, we must have $\theta_2(\alpha^*) = \theta_1(\alpha^*)$. Yet $x(\alpha^*)$ is in $Int(\Omega_3)$ in this case, and as we have assumed that A_1 is not a scalar multiple of A_2, so it is impossible that for a point in $Int(\Omega_3)$ to have $\theta_2 = \theta_1$ (Since by Lemma 2.8, we can conclude that it is only possible for $\theta_2 = \theta_1$ for points on the boundary of Ω_3), which is a contradiction to our assumption.

So we have proved that in this case, $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$. □

In view of the above lemma, we can prove the following theorem.

Theorem 3.2 *Let l_1 be a ray that goes through the origin. Let x_0 be on l_1. If x^* is the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law, and if x_1 is the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according some arbitrary switching law, then we have $\|x^*\|_2 \leq \|x_1\|_2$.*

Proof: Let α be the angle from l_1 to the line on which $x(t)$ lies, measured counterclockwise. Assume that s is a switching law (of course, by our definition in Section 2, s generates a finite number of switchings in any finite time interval) such that the system switches when α is equal to $0,\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots$. Combining the switching α's above with the switching α's derived by conic switching law and $\alpha-2\pi$. For simplicity of notation, we still denote the combined switching α's as $0,\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots,\alpha-2\pi,\cdots$. Let the trajectory for s be $x_s(\alpha)$ and the trajectory by conic switching law be $x_c(\alpha)$.

Now in the conic region $\alpha_1 \leq \alpha \leq 0$, using Lemma 3.1, it is better to follow the switching law by the conic switching law. So upon arriving α_1, we have $\|x_c(\alpha_1)\|_2 \leq \|x_s(\alpha_1)\|_2$.

In the conic region $\alpha_2 \leq \alpha \leq \alpha_1$, if $\|x_c(\alpha_1)\|_2 = \|x_s(\alpha_1)\|_2$, then by using the above lemmas, it is better to follow the conic switching law. If $\|x_c(\alpha_1)\|_2 < \|x_s(\alpha_1)\|_2$, then it would be even clear that it is better to follow the conic switching law, since every trajectory evolving according to any one subsystem starting from $x_c(\alpha_1)$ is closer to the origin than starting from $x_s(\alpha_1)$. Therefore upon arriving at α_2, we have $\|x_c(\alpha_2)\|_2 \leq \|x_s(\alpha_2)\|_2$.

By induction, we can use similar argument as above to prove that upon arriving at $\alpha-2\pi$, we have $\|x_c(\alpha-2\pi)\|_2 \leq \|x_s(\alpha-2\pi)\|_2$, i.e., $\|x^*\|_2 \leq \|x_1\|_2$. □

Theorem 3.2 implies the following theorem for the necessary and sufficient conditions of the asymptotic stabilizability of switched system (3.1) with subsystems of the same direction.

Theorem 3.3 *The switched system (3.1) with subsystems of the same direction is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$ by the conic switching law, where x^* and x_0 are the same as in Theorem 3.2.*

Example 3.4 *Consider the switched system (3.1) with two subsystems of the same direction with*

$$A_1 = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 13 \\ -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ -10 & 3 \end{bmatrix}.$$

It can be shown that it is asymptotically stabilizable using the conic switching law, though each subsystem has unstable foci. The regions for subsystems are specified and a stabilizing control law is derived. Figure 9 shows the regions for subsystems, the trajectory and the time domain responses of the switched system.

3.1.2 Two Subsystems of Opposite Directions

We will consider the following switched system,

$$\dot{x}(t) = A_1 x(t),\ \dot{x}(t) = A_2 x(t), \tag{3.21}$$

where both subsystems are with unstable foci, and subsystem 1 is of clockwise direction while subsystem 2 is of counterclockwise direction. Figure 10 shows the angle θ_1 ($-\pi \leq \theta_1 \leq 0$) and θ_2 ($0 \leq \theta_2 < \pi$) in this case.

We would like to design a switching law that asymptotically stabilizes the switched system. By our switching law, we want to lead the trajectory closer and closer to the origin, i.e., $\|x(t)\|_2 \to 0$ as $t \to \infty$. In the following, we will design a switching law by choosing a subsystem which has the potential to drive the trajectory closer to the origin. We first define the following conic regions

$$\Omega_1 = E_{1s} \cap E_{2s}, \tag{3.22}$$
$$\Omega_2 = E_{1u} \cap E_{2u}, \tag{3.23}$$
$$\Omega_3 = E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \tag{3.24}$$
$$\Omega_4 = E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \tag{3.25}$$
$$\Omega_5 = E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \tag{3.26}$$
$$\Omega_6 = E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \tag{3.27}$$

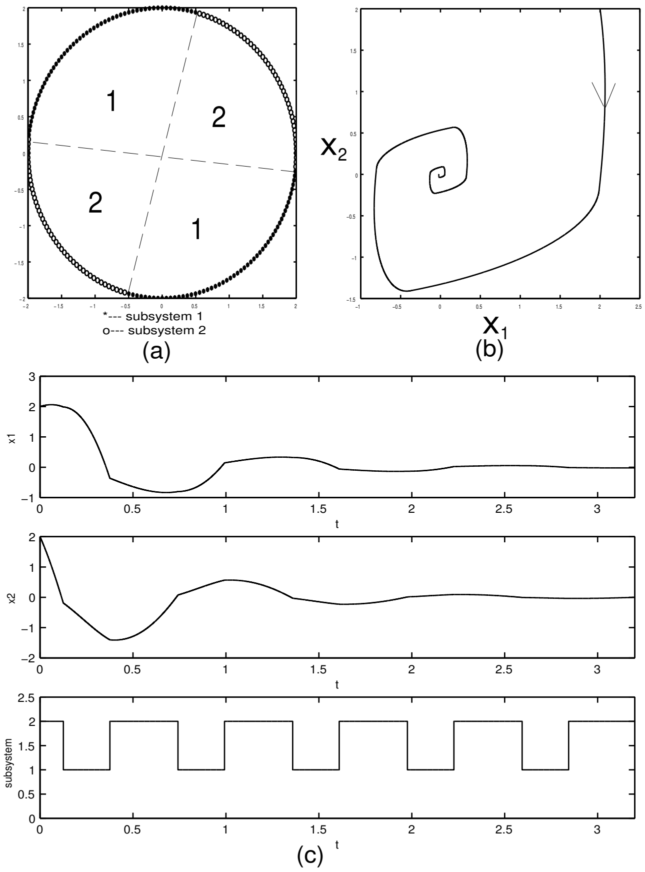

Figure 9: Example 3.4: (a) The regions for subsystems. (b) The trajectory of the system. (c) Time domain responses. ($x_0 = (2,2)^T$)

where E_{is}, E_{iu} are as defined in (3.3) and (3.4). Notice, the Ω_i's in this case are different from Ω_i's in the previous subsection.

Case 1: If $Int(\Omega_1) \neq \emptyset$, it is clear that any trajectory of the switched system that is totally inside Ω_1 is stable (Figure 11(a)). Now if the initial point of the switched system is outside Ω_1, we can follow the trajectory of any one subsystem to go into Ω_1. Then we can use the **conic switching law**, i.e., *switch to another subsystem upon intersecting the boundary of* Ω_1, as this will give us a trajectory that is totally inside Ω_1 and it is bounded. If there exists $x \in \Omega_1$, such that $|\theta_i(f_i(x))| \neq \frac{\pi}{2}$ for some $i = 1$, or 2, then the switched system is asymptotically stablizable by the conic switching law.

Case 2: If $Int(\Omega_3) \neq \emptyset$. In this case, assume l_1 and l_2 be two rays that go through the origin and are in the same conic region Ω_3, and that the conic region with angle from l_1 to l_2 is inside Ω_3 (Figure 11(b)). Suppose l_2 is on the clockwise side of l_1 in the conic region and suppose x_0 is on l_2. Then we have the following Lemma.

Lemma 3.5 *Let the switched system follow subsystem 2 until the trajectory intersects l_1 for the first time at x_1 and then let the system switch to subsystem 1 and evolve according to subsystem 1. Suppose x^* is a point on l_2 where the trajectory intersects l_2 for the first time after the switching. Then we have $\|x^*\|_2 \leq \|x_0\|_2$.*

Proof: Figure 11(b) shows the case. It is quite clear that if we consider $\dot{x}(t) = A_3x(t)$ with $x(0) = x_1$, where $A_3 = -A_2$, then the trajectory is exactly the same as the trajectory from x_0 to x_1 but evolving in opposite direction. Also notice that $f_3 = (-a_3, -a_4)^T$.

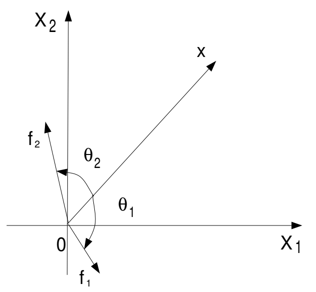

Figure 10: The angle θ_i from x to f_i, $i = 1, 2$.

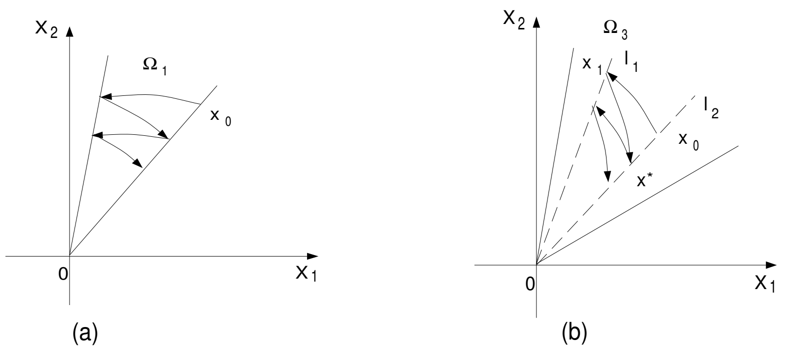

Figure 11: (a) Case 1:$Int(\Omega_1) \neq \emptyset$. (b) Case 2:$Int(\Omega_3) \neq \emptyset$.

Now consider $\dot{x}(t) = A_1 x(t)$ and $\dot{x}(t) = A_3 x(t)$, they are in the same direction. And since the conic region of angle from l_1 to l_2 are inside Ω_3 defined in (3.10), it is also inside $E_{1s} \cap E_{3s} \cap \{x | a_2(-a_3) - a_1(-a_4) \leq 0\}$. Therefore, by lemma 3.1, we obtain $\|x^*\|_2 \leq \|x_0\|_2$. □

According to Lemma 3.5, the **conic switching law** (i.e., *switch to another subsystem upon intersecting the boundary of* Ω_3 so as to keep the trajectory inside Ω_3 (Figure 11(b))) will give us a trajectory that is totally inside Ω_3 so it is bounded. If there exists $x \in \Omega_3$, such that $|\theta_1(f_1)| + |\theta_2(f_2)| \neq \pi$ at x, then the switched system is asymptotically stabilizable by the conic switching law.

Remark: By Lemma 2.8, if $Int(\Omega_3) \neq \emptyset$, then $|\theta_1(f_1)| + |\theta_2(f_2)| = \pi$ can be true everywhere in $Int(\Omega_3)$ if and only if $A_1 = kA_2$ for some $k < 0$. Yet if $A_1 = kA_2$ for some $k < 0$, subsystem 1 must be stable since subsystem 2 is unstable, which contradicts our assumption of the instability of both subsystems. Therefore, the condition $|\theta_1(f_1)| + |\theta_2(f_2)| \neq \pi$ can be easily checked to be satisfied.

Case 3: Let $Int(\Omega_5) \neq \emptyset$. Then similar argument as in Case 2 can be applied and we find that the system can be stabilized by **conic switching law** similar to the one in Case 2 to keep the trajectory inside Ω_5. □

The following theorem shows us that the system can only be stabilized for the above-mentioned three cases.

Theorem 3.6 *The switched system (3.21) with two subsystems of opposite directions is asymptotically stabilizable if and only if* $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$.

Proof: First we claim that A_1 must not be a scalar multiple of A_2 in this case. Since if otherwise, subsystem 1 must be stable since subsystem 2 is unstable, which contradicts the assumption of instability of both subsystems. So if $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$, there must exist $x \in Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5)$ such that $|\theta_1(f_1)| + |\theta_2(f_2)| \neq \pi$.

Sufficiency: By using the conic switching laws associated with Ω_i, $i = 1, 3, 5$, if $Int(\Omega_i) \neq \emptyset$, the sufficiency is quite clear from the above discussion.

Necessity. We prove by contradiction. If the switched system is asymptotically stabilizable but $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) = \emptyset$. Assume x_0 lies on ray l_1. Let α be the angle from l_1 to the line on which $x(t)$ lies, measured counterclockwise. Assume s is an arbitrary switching law such that the system switches when α is equal to $0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ and starting from subsystem 2. And let the correponding points at switching moment be $x_0, x_1, x_2, \cdots, x_n, \cdots$. (The similar argument can be applied to an arbitrary switching law starting from subsystem 1.)

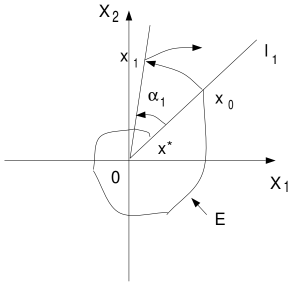

Figure 12: Figure for the proof of Theorem 3.6.

Now we consider the trajectory of subsystem 2 and go backward in time, i.e., $x(-t)$. Assume x^* is the point where the trajectory intersects l_1 for the first time (at time $-t^*$) (Figure 12). Let

$$E = \{x(t)| - t^* \leq t \leq 0\}.$$

If we consider the trajectory from x_0 to x_1, it is clear that for any points x on the trajectory between x_0 to x_1, $\|x\|_2$ would be greater than or equal to the norm of the corresponding points on E. (x' on E corresponds to x_1 when they are on the same ray through the origin).

Since $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) = \emptyset$, we can show by induction that any x on the trajectory between x_k and x_{k+1} would have a norm greater than or equal to the norm of the corresponding points on E. Therefore, any x on the trajectory of the switched system would have a norm than the smallest norm of the points on E. Therefore, the switched system is not asymptotically stablizable by any s, which is a contradiction to our assumption. $\square$

Remark: Notice that in Ω_1, Ω_3 and Ω_5, the angle $\theta_{f_1 f_2}$ from f_1 to f_2(i.e., $\theta_{f_1 f_2} = \theta_2 - \theta_1$) is greater than or equal to π.

Example 3.7 *Consider the switched system (3.21) with two subsystems of different directions with*

$$A_1 = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 20 \\ -20 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 52 \\ -8 & 6 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 20 \\ -20 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 11 & -10 \\ 50 & -9 \end{bmatrix}.$$

It is asymptotically stabilizable since the region $Int(\Omega_1)$ is nonempty. Figure 13 shows the conic regions, the trajectory and the time domain responses of the switched system.

3.1.3 Several Subsystems

Now we study stabilizing switching laws for switched systems with several second-order LTI subsystems with unstable foci

$$\dot{x}(t) = A_i x(t), \ i = 1, 2, \cdots, N. \tag{3.28}$$

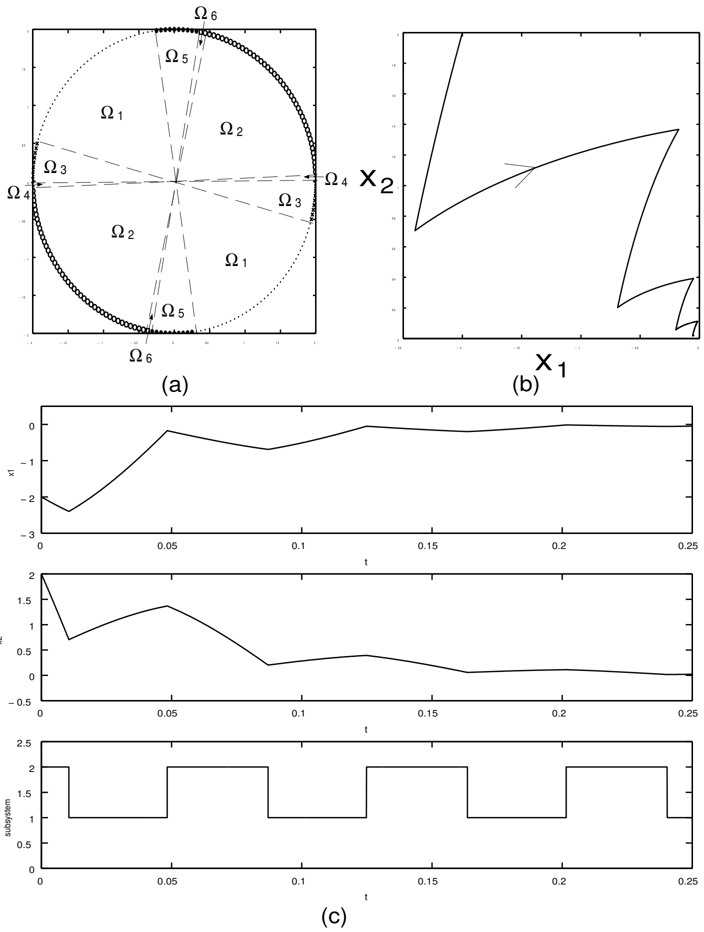

Figure 13: Example 3.7: (a) The conic regions. (b) The trajectory of the system. (c) Time domain responses. $(x_0 = (-2, 2)^T)$

We would consider the following cases.

All Subsystems of the Same Direction

Assume that all the N subsystems are of the clockwise direction (similar argument can also be applied to several subsystems of the counterclockwise direction). Notice that from the discussion in Section 3.1.1 (of two subsystems of the same direction), we find the conic switching law by associating with each point on $\mathbb{R}^2$ a subsystem i with smaller θ_i. As an extension of the conic switching law there, we obtain the **conic switching law** by associating with each point on $\mathbb{R}^2$ a subsystem i where θ_i is the smallest (if some $\theta_j = \theta_i$, we may choose subsystem similarly to the conic switching law in Section 3.1.1).

Similar argument as in Section 3.1.1 can be adopted for several subsystems to show the following theorem.

Theorem 3.8 *Let l_1 be a ray that goes through the origin. Let x_0 be on l_1 and consider the conic switching law. Let x^* on l_1 be the point where the trajectory intersects l_1 for the first time after leaving x_0. The switched system (3.28) is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$.*

Not All Subsystems of the Same Direction

Assume that $K(K > 0)$ subsystems are of clockwise direction and $M(M > 0)$ subsystems are of counterclockwise direction $(K + M = N)$. Combining the previous results, the following theorem can be obtained.

Theorem 3.9 *The switched system (3.28) with $K(K > 0)$ subsystems $S_1^-, \cdots, S_K^-$ of clockwise direction and $M(M > 0)$ subsystems $S_1^+, \cdots, S_M^+$ of counterclockwise direction is asymptotically stabilizable if and only if at least one of the following three conditions holds:*

1. *The switched system consisting of $S_1^-, \cdots, S_K^-$ is asymptotically stabilizable.*
2. *The switched system consisting of $S_1^+, \cdots, S_M^+$ is asymptotically stabilizable.*
3. *There exist i and j with $1 \le i \le K$ and $1 \le j \le M$ such that the switched system consisting of two subsystems S_i^- and S_j^+ is asymptotically stabilizable.*

Proof: "If": If any one of the statement 1, 2, 3 holds, then we can use the conic switching control laws developed previously to asymptotically stabilize the switched system.

"Only if": If the switched system is asymptotically stabilizable, and if statement 1 or 2 holds, then the only if part is also true.

Now we assume that statement 1 and 2 are not true, we want to prove that statement 3 is true. We prove this by contradiction.

Assume that in this case, switched system consisting of any two subsystems S_i^- and S_j^+ is not asymptotically stabilizable. For simplicity of proof, we also assume that there is no $A_i = cA_j, \forall i, j$. Then it is implied that $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) = \emptyset$ for any S_i^- and S_j^+.

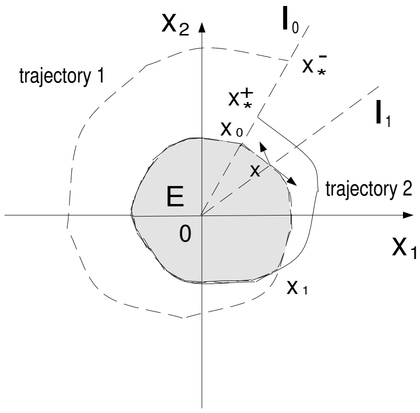

Figure 14: Figure for the proof of Theorem 3.9.

Now consider Figure 14 and assume that x_0 is a nonzero initial point on ray l_0 in $\mathbb{R}^2$. Let trajectory 1 (dotted curve) be the trajectory starting from x_0 and switching among $S_1^-, \cdots, S_K^-$ using the conic switching law for subsystems of clockwise direction. Trajectory 1 intersects l_0 at x_*^- for the first time. Since statement 1 does not hold in this case, we have $\|x_*^-\|_2 \ge \|x_0\|_2$. Similarly, trajectory 2 is the trajectory under conic switching law for $S_1^+, \cdots, S_M^+$, and it intersects l_0 at x_*^+ with $\|x_*^+\|_2 \ge \|x_0\|_2$.

It is clear that trajectory 1 intersects trajectory 2 at some point x_1. Now we prove that the trajectory starting at x_0 cannot enter the interior of the region E bounded by trajectory 1 and trajectory 2 (the shaded region in Figure 14).

Assume that x is on the boundary of E, e.g., x is on trajectory 1. Notice that vector field of any subsystem $S_1^-, \cdots, S_K^-$ cannot lead the trajectory into E since trajectory 1 is already the "best" trajectory using the conic switching laws. While for vector fields of $S_1^+, \cdots, S_M^+$, we have $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) = \emptyset$ for any S_i^- and S_j^+ by our assumption, therefore these vector fields also cannot lead the trajectory into E. The similar arguments can be applied to points on the boundary of E which are on trajectory 2.

Therefore, the trajectory starting at x_0 cannot enter $Int(E)$, so the switched system is not asymptotically stabilizable. This is a contradiction to our assumption of the asymptotic stabilizability of the switched system. So statement 3 must hold. $\square$

4 Stabilization of Second-Order LTI Switched Systems with Nodes

In this section, we will study and design stabilizing switching laws for switched systems consisting of second-order LTI subsystems with unstable nodes. We mainly study switched systems consisting of two subsystems. The results can similarly be applied to several subsystems. The method is an extension of the method used in the previous section.

Let us begin our discussion by looking into the simplest such system where $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$, $c_1 > 0$, $c_2 > 0$. Figure 15 shows the trajectory of the system when $c_2 > c_1$ (note in this section, we generally do not consider the special case $c_1 = c_2 > 0$, yet the case of $c_1 = c_2$ can also be analyzed using similar techniques).

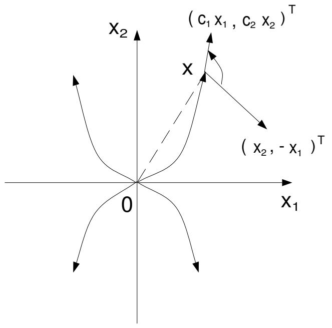

Figure 15: Trajectory of the simplest second-order system with unstable node.

For the trajectory in quadrant I, we have $(x_2, -x_1) \cdot (c_1 x_1, c_2 x_2)^T = c_1 x_1 x_2 - c_2 x_1 x_2 = (c_1 - c_2) x_1 x_2 < 0$. So we conclude that the trajectory travels in the counterclockwise direction in quadrant I. Similarly, we conclude that the trajectories in quadrant II, III and IV travel in the clockwise, counterclockwise and clockwise direction, respectively.

For a general second-order system with unstable nodes, we have

$$A = T \Lambda T^{-1},$$

where $\Lambda = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$, $c_2 > 0$, $c_1 > 0$, $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ and $det(T) > 0$.

If we let unit vectors of the $Y_2 - Y_1$ coordinate system to be

$$[\mathbf{e}_{Y_1}, \mathbf{e}_{Y_2}] = [\mathbf{e}_{X_1}, \mathbf{e}_{X_2}] T,$$

then in the conic regions on the $X_2 - X_1$ plane that correspond to quadrants I, II, III and IV in $Y_2 - Y_1$ plane, the trajectory travels in the same direction as (the solution travels) in the $Y_2 - Y_1$ plane.

Now we consider the switched system

$$\dot{x}(t) = A_1 x(t), \ \dot{x}(t) = A_2 x(t), \tag{4.1}$$

whose both subsystems have unstable nodes at the origin. In the following discussion, we exclude the case $A_1 = cA_2$ for some fixed c. So from Section 3.1.2, whenever $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$, asymptotic stabilizability can be assured. Let

$$E_{ic} = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | (x_2, -x_1) A_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0\}, \ i = 1, 2, \tag{4.2}$$

$$E_{icc} = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | (x_2, -x_1) A_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 0\}, \ i = 1, 2. \tag{4.3}$$

In other words, E_{ic} (resp. E_{icc}) denote the conic regions where the ith subsystem trajectory travels clockwise (resp. counterclokwise). It is not difficult to see that the boundary of the conic regions are given by the two lines

$$\begin{aligned} x_2 t_{11} - x_1 t_{21} &= 0, \\ x_2 t_{12} - x_1 t_{22} &= 0. \end{aligned}$$

With these notation in mind, we let $E_{c,c} = E_{1c} \cap E_{2c}$, which denotes the conic region in which the trajectory of subsystem 1 travels clockwise and the trajectory of subsystem 2 also travels clockwise. In the same manner, we can also define the conic regions $E_{c,cc} = E_{1c} \cap E_{2cc}$, $E_{cc,c} = E_{1cc} \cap E_{2c}$ and $E_{cc,cc} = E_{1cc} \cap E_{2cc}$.

Figure 16, 18, 21 show exhaustively the pictures of six possible arrangements of $E_{c,c}$, $E_{c,cc}$, $E_{cc,c}$ and $E_{cc,cc}$. Without loss of generality, we illustrate these cases by fixing E_{1cc} to be in the I, III quadrants and E_{1c} to be in the II, IV quadrants, respectively. (We can always do this by some linear transformation to both subsystems; the linear transformation would not affect the applicability of what we will discuss in the following).

Case 1. We only discuss Case 1(a) (see Figure 16), since similar argument can be applied to Case 1(b).

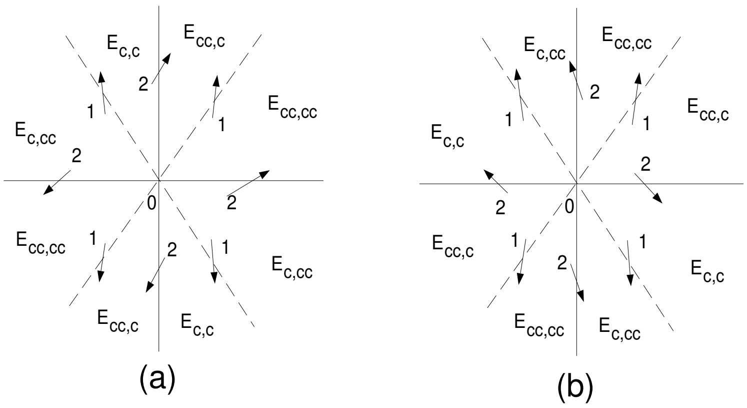

Figure 16: Case 1.

Notice that in this case the two subsystems are of the same direction in $E_{c,c}$, $E_{cc,cc}$, which means that any trajectory starting in the conic region will eventually leave the conic region if the switched system stay at both subsystem 1 and subsystem 2 long enough. We call this kind of conic regions **Type A regions**. Of course, if the switched system stick to subsystem 1 in $E_{c,c}$, it will not leave $E_{c,c}$, yet the system will not be stable, so this kind of strategy of sticking to one subsystem is not of interest to us.

The conic region $E_{cc,c}$ has the property that whenever a trajectory goes into it, the trajectory can never leave it again, so it is called a **Type B region**.

For conic region $E_{c,cc}$, the trajectory of the systems can always leave the conic region from both boundaries under appropriate switchings. The region is called a **Type C region**.

Next we introduce a theorem.

Theorem 4.1 *In Case 1(a), for the Type B region $E_{cc,c}$, we have $Int(E_{cc,c} \cap \Omega_i) = \emptyset$, $i = 1, 3, 5$, where Ω_i are the conic regions defined as in our discussion of two subsystems with unstable foci of opposite directions.*

Proof: Consider the $E_{cc,c}$ part in the I quadrant and note that similar arguments can be applied to $E_{cc,c}$ part in the III quadrant. Assume that the boundary of $E_{cc,c}$ is formed by two rays l_1 and l_2. Also notice that subsystem 1 (resp. 2) is in counterclockwise (resp. clockwise) direction in $E_{cc,c}$ (Figure 17).

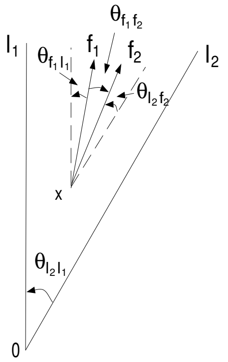

Figure 17: Proof of Theorem 4.1.

We claim that the angle from l_2 to f_2 is greater than zero, i.e., $\theta_{l_2 f_2} > 0$, and the angle from f_1 to l_1 is greater than zero, i.e., $\theta_{f_1 l_1} > 0$ ($\theta_{l_2 f_2}$, $\theta_{f_1 l_1}$ are measured counterclockwise). Therefore

$$|\theta_{f_1 f_2}| = \theta_{l_2 l_1} - \theta_{l_2 f_2} - \theta_{f_1 l_1} < \theta_{l_2 l_1} < \pi.$$

While from the remark after Theorem 3.6 in Section 3.1.2, we know that in $Int(\Omega_i)$, $i = 1, 3, 5$, we have $|\theta_{f_1 f_2}| \geq \pi$, so we conclude that

$$Int(E \cap \Omega_i) = \emptyset.$$

□

In view of Theorem 4.1, once the trajectory goes into the Type B region $E_{cc,c}$, the trajectory cannot be stabilized for any switching sequences. So in general, the switched system is not stabilizable because stabilizability in general requires stabilizability from any initial point.

Also notice that in the Type C region $E_{c,cc}$, similar arguments as in the proof of Theoerm 4.1 can be applied to show that $Int(E_{c,cc} \cap \Omega_i) = \emptyset$, $i = 1, 3, 5$.

So in this case, the switched system cannot be stabilized regardless of the initial point.

Case 2. We only discuss Case 2(a) (see Figure 18) as Case 2(b) is analogous. $E_{c,c}$ and $E_{cc,cc}$ are Type A regions. $E^1_{c,cc}$ is a Type B region and $E^2_{c,cc}$ is a Type C region. In $E_{c,c}$ and $E_{cc,cc}$, the two subsystems are of the same direction, yet $E_{c,c}$ and $E_{cc,cc}$ are not Type A regions. For example, in $E_{c,c}$, if we stick to subsystem 2, we may never pass the boundary of $E_{c,c}$.

By applying similar arguments as in Theorem 4.1, it can be shown that $Int(E^1_{c,cc} \cap \Omega_i) = \emptyset$, $i = 1, 3, 5$. So we conclude that the system is *not asymptotically stabilizable.* Yet, it is still possible for the trajectory starting in the Type C regions to be stabilized by using some switching laws as long as the outgoing region satisfies the condition of the following theorem.

Theorem 4.2 *In Case 2(a), the trajectory starting inside the Type C region $E^2_{c,cc}$ can be asymptotically stabilized if and only if $Int(E^2_{c,cc} \cap \Omega_1) \cup Int(E^2_{c,cc} \cap \Omega_3) \cup Int(E^2_{c,cc} \cap \Omega_5) \neq \emptyset$.*

Proof: "If": If $Int(E^2_{c,cc} \cap \Omega_1) \cup Int(E^2_{c,cc} \cap \Omega_3) \cup Int(E^2_{c,cc} \cap \Omega_5) \neq \emptyset$, then we can adopt the conic switching law as in Section 3.1.2 to asymptotically stabilize the trajectory starting inside $E^2_{c,cc}$.

"Only if": We prove by contradiction. Assume that the trajectory starting inside $E^2_{c,cc}$ can be asymptotically stabilized but $Int(E^2_{c,cc} \cap \Omega_1) \cup Int(E^2_{c,cc} \cap \Omega_3) \cup Int(E^2_{c,cc} \cap \Omega_5) = \emptyset$. Assume the boundary of $E^2_{c,cc}$ is formed by two rays l_1 and l_2, and subsystem 1 (resp. 2) is in clockwise (resp. counterclockwise) direction in $E^2_{c,cc}$ as in Figure 19.

Let F be a region formed by l_1, l_2, the trajectory if the system starts at x_0 and follows subsystem 1, and the trajectory if the system starts at x_0 and follows subsystem 2. We can show that for any switching laws,

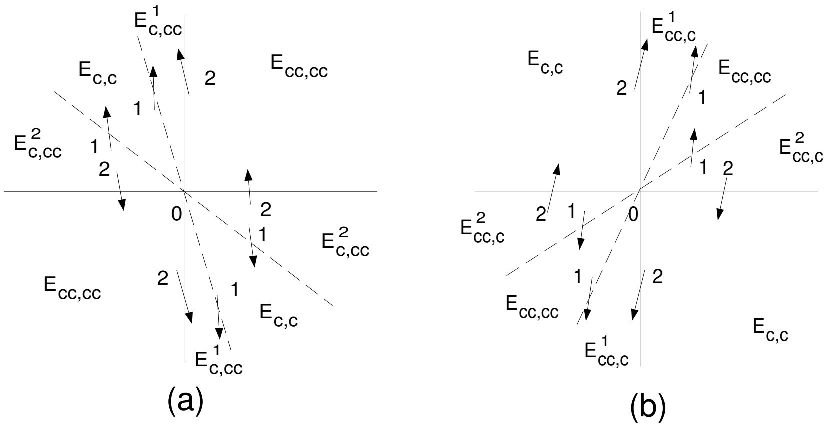

Figure 18: Case 2.

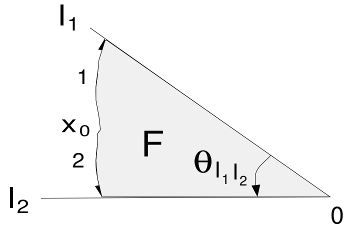

Figure 19: The region F.

the trajectory of the switched system will never enter $Int(F)$ (this can be proved by using similar techniques as in the proof of Theorem 3.6), which is a contradiction to the asymptotic stabilizability of the trajectory. So $Int(E^2_{c,cc} \cap \Omega_1) \cup Int(E^2_{c,cc} \cap \Omega_3) \cup Int(E^2_{c,cc} \cap \Omega_5) \neq \emptyset$. $\square$

In this case, it is possible that the conditions in Theorem 4.2 be satisfied, hence the trajectory starting from inside the Type C regions might be stabilized using the conic switching laws.

Example 4.3 *Consider the switched system (4.1) consisting of two subsystems with unstable nodes with*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{8}{\sqrt{68}} & -\frac{8}{\sqrt{73}} \\ -\frac{2}{\sqrt{68}} & -\frac{3}{\sqrt{73}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \frac{8}{\sqrt{68}} & -\frac{8}{\sqrt{73}} \\ -\frac{2}{\sqrt{68}} & -\frac{3}{\sqrt{73}} \end{bmatrix}^{-1} = \begin{bmatrix} -37 & -152 \\ 14.25 & 58 \end{bmatrix}.$$

It can be shown that the trajectory starting in $E^2_{c,cc}$ is asymptotically stabilizable using the conic switching law. The conic regions are specified in Figure 20(a). Figure 23(b) and (c) show the trajectory in Ω_5 and the time domain responses of the switched system.

Case 3. We only discuss Case 3(a) (see Figure 21) as Case 3(b) is analogous. In this case, $E^1_{c,c}$ and $E^2_{c,c}$ are Type A regions. In this case we first claim that $Int(E_{c,cc} \cap \Omega_i) = \emptyset$ and $Int(E_{cc,c} \cap \Omega_i) = \emptyset$, $i = 1, 3, 5$. This can be proved using the similar arguments as in the proof of theorem 4.1. Hence, the trajectory must not always be inside $E_{c,cc}$ ($E_{cc,c}$) in order to possibly make the trajectory stable.

In fact, note that we can adopt similar (one round) test as in the case of two subsystems with unstable foci of the same direction. In particular, the **conic switching law** can be chosen as follows. The corresponding

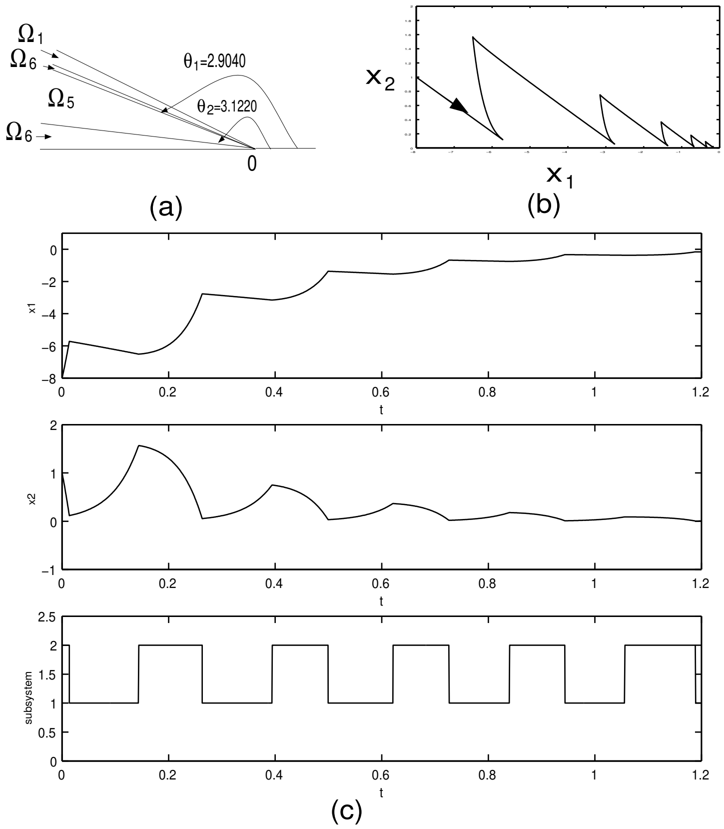

Figure 20: Example 4.3: (a). Conic regions in $E^2_{c,cc}$. (b) The trajectory of the system. (c) Time domain responses. ($x_0 = (-8, 1)^T$)

subsystem can be chosen as subsystem 1 in $E_{c,cc}$, subsystem 2 in $E_{cc,c}$. While in $E_{c,c}$ and $E_{cc,cc}$, the subsystem is chosen similar to the case of unstable foci, the subsystem which has the potential to drive the system trajectory closer to the origin is chose at each point.

The following theorem provides a necessary and sufficient condition for the asymptotical stabilizability of the switched system.

Theorem 4.4 *Let l_1 be a ray that goes through the origin. Let x_0 be on l_1. If x^* is the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law stated above. Then the switched system as in Case 3(a) is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$.*

The conditions in the above theorem is not easy to check. The following corollary provides a simplified sufficient condition to check whether a switched system is not stabilizable.

Corollary 4.5 (The Parallelogram Sufficient Condition) *If in Case 3(a), the boundaries of the regions are denoted as l_{11}, l_{12}, l_{21}, l_{22} as shown in Figure 22. Let x_0 be a nonzero point on l_{21}. If $x_0x_1x_2x_3$ is a parallelogram on $\mathbb{R}^2$, where $x_1 \in l_{22}$, $x_2 \in l_{11}$, $x_3 \in l_{21}$ and $x_0x_1 \parallel x_2x_3 \parallel l_{12}$, $x_1x_2 \parallel x_0x_3$. Then the switched system is not asymptotically stabilizable if $\|x_3\|_2 \geq \|x_0\|_2$*

Proof: Consider the trajectory starting at x_0 and evolve according to the conic switching law. Notice that in $E_{c,cc}$, the point x on the trajectory is always farther from the origin than the corresponding point on x_0x_1

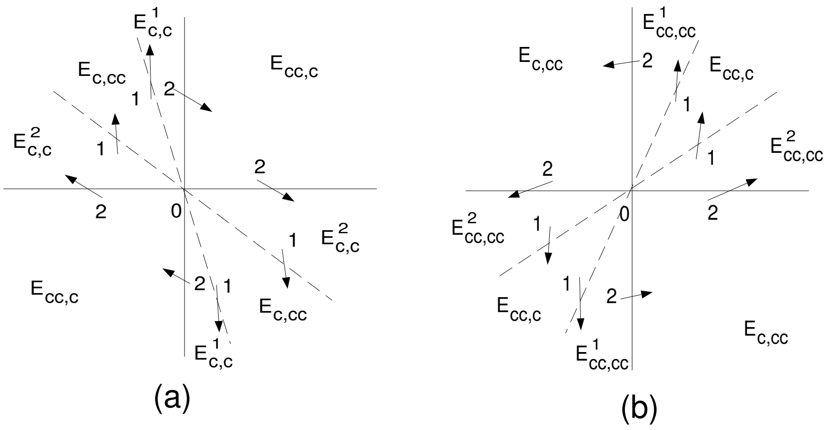

Figure 21: Case 3.

with the same angle, so the trajectory is farther away from the origin than x_0x_1. Similarly, in $E^1_{c,c}$, we can also show that the points on the trajectory are farther from the origin than the corresponding points on x_1x_2 regardless of the switching law employed. And we can have the similar results for $E_{cc,c}$, $E^2_{c,c}$. Therefore, the half trajectory is outside the parallelogram, so if $\|x_3\|_2 \geq \|x_0\|_2$, this would imply $\|x'\|_2 \geq \|x_0\|_2$, where x' is the intersecting point on l_{21} when the system trajectory evolves according to the conic switching law for half round. This means that the switched system is not asymptotic stabilizable by Theorem 4.4. □

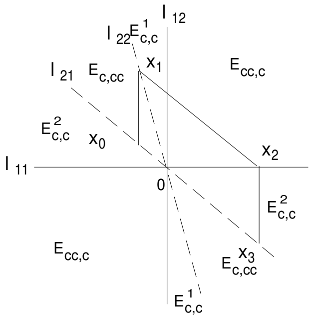

Figure 22: Proof for Corollary 4.5.

Example 4.6 *Consider the switched system (4.1) consisting of two subsystems with unstable nodes with*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{8}{\sqrt{73}} & \frac{8}{\sqrt{68}} \\ -\frac{3}{\sqrt{73}} & -\frac{2}{\sqrt{68}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} \frac{8}{\sqrt{73}} & \frac{8}{\sqrt{68}} \\ -\frac{3}{\sqrt{73}} & -\frac{2}{\sqrt{68}} \end{bmatrix}^{-1} = \begin{bmatrix} 58 & 152 \\ -14.25 & -37 \end{bmatrix}.$$

It can be shown that it is asymptotically stabilizable using the conic switching law, though each subsystem has an unstable node. The regions for the subsystems are specified and a stabilizing control law is derived. Figure 23 shows the regions for subsystems, the trajectory and the time domain responses of the switched system.

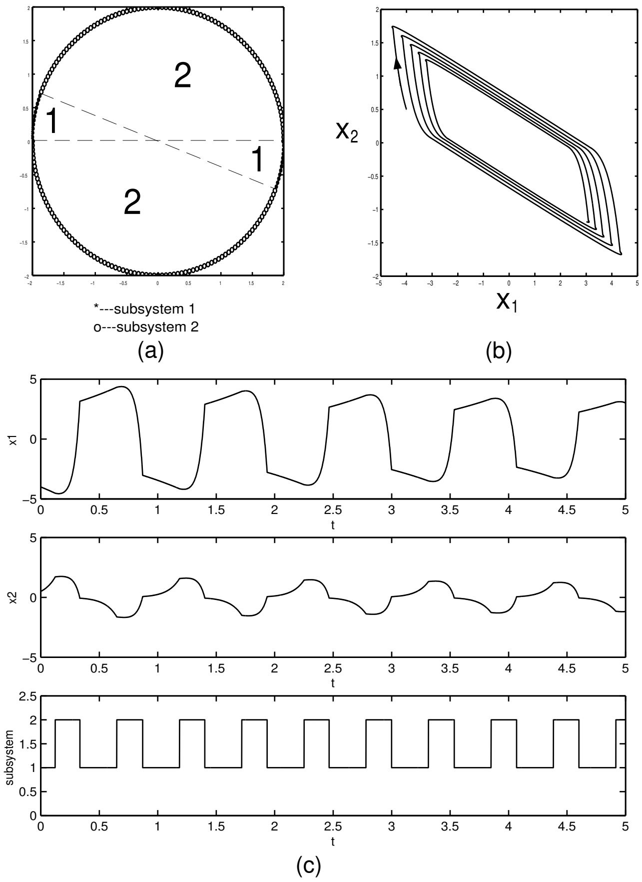

Figure 23: Example 4.6: (a) The regions for subsystems. (b) The trajectory of the system. (c) Time domain responses. $(x_0 = (-4, 0.5)^T)$

5 Stabilization of Second-Order LTI Switched Systems with Saddle Points

In this section, we will study and design stabilizing switching laws for switched systems consisting of second-order LTI subsystems with saddle points. We mainly study switched systems consisting of two subsystems. The results can be similarly applied to several subsystems.

Let us begin our discussion by looking into the simplest such system where $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$. Here we assume $c_1 > 0 > c_2$. Figure 24 shows the trajectory of the system.

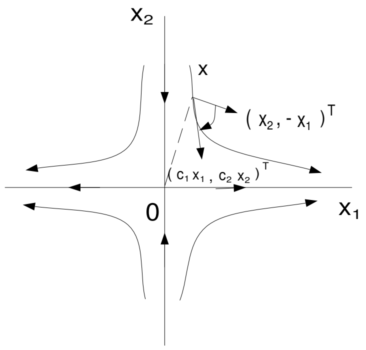

Figure 24: Trajectory of the simplest second-order system with saddle points.

For the trajectory in quadrant I, we have $(x_2, -x_1) \cdot (c_1x_1, c_2x_2)^T = c_1x_1x_2 - c_2x_1x_2 = (c_1 - c_2)x_1x_2 > 0$. So we conclude that at point x the trajectory travels in the clockwise direction in quadrant I. Similarly, we conclude that the trajectories in quadrant II, III and IV travel in the counterclockwise, clockwise and counterclockwise direction, respectively. While on X_2 axis, the trajectory tends toward the origin, we call X_2 axis the *converging line*. On X_1 axis, the trajectory tends toward ∞, we call X_1 axis the *diverging line*.

Now consider the switched system

$$\dot{x}(t) = A_1x(t), \ \dot{x}(t) = A_2x(t), \tag{5.1}$$

whose subsystems are both with saddle points at the origin. Similarly to the previous section, we can define E_{ic}, E_{icc}, $i = 1, 2$ to denote the conic regions where the ith subsystem trajectory travels clockwise (counterclokwise).

Figure 25, 28, 32 show exhaustively the picture of six possible arrangements of $E_{c,c}$, $E_{c,cc}$, $E_{cc,c}$ and $E_{cc,cc}$. As in section 4, without loss of generality, we illustrate these cases by fixing E_{1c} to be in the I, III quadrants and E_{1cc} to be in the II, IV quadrants, respectively. (We can always do this by some linear transformation to both subsystems, the linear transformation would not affect the applicability of what we will discuss in the following, but only affect the position of the converging line and the diverging line).

Case 1. We only discuss Case 1(a) (see Figure 25), since similar arguments can be applied to Case 1(b). In this case, $E_{c,c}$, $E_{cc,cc}$ are Type A regions. $E_{cc,c}$ is a Type B region and $E_{c,cc}$ is a Type C region.

In this case, we have the following theorem.

Theorem 5.1 *In Case 1(a), for the Type C region $E_{c,cc}$, we have $Int(E_{c,cc} \cap \Omega_1) \cup Int(E_{c,cc} \cap \Omega_3) \cup Int(E_{c,cc} \cap \Omega_5) \neq \emptyset$, where Ω_i, $i = 1, 3, 5$, are the conic regions defined as in our discussion of two subsystems with unstable foci of opposite directions.*

Proof: Assume the boundary of $E_{c,cc}$ is formed by two rays l_1 and l_2, and subsystem 1 (resp. 2) is in clockwise (resp. counterclockwise) direction in $E_{c,cc}$ (Figure 26).

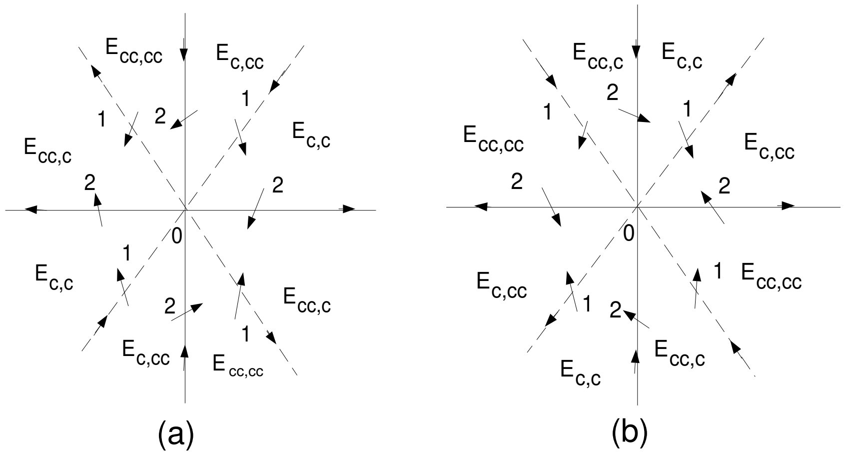

Figure 25: Case 1.

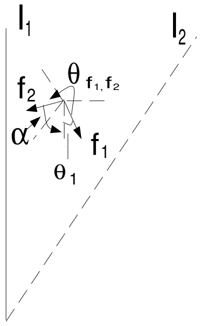

Figure 26: Figure for proof of theorem 5.1.

We claim that at a point x very close to l_1, the angle $\theta_{f_1 f_2}$ from f_1 to f_2 satisfies $|\theta_{f_1 f_2}| > \pi$. This is because when x is very close to l_1, the angle θ_1 between f_1 and l_1 is very small. And therefore, $2\pi - \theta_{f_1 f_2} \simeq \alpha$. Since $\alpha < \pi$ from Figure 26, we have $|\theta_{f_1 f_2}| > \pi$. So we conclude that $Int(E_{c,cc} \cap \Omega_1) \cup Int(E_{c,cc} \cap \Omega_3) \cup Int(E_{c,cc} \cap \Omega_5) \neq \emptyset$. $\square$

For the Type B region $E_{cc,c}$, we have the following theorem.

Theorem 5.2 *If $Int(E_{cc,c} \cap \Omega_1) \cup Int(E_{cc,c} \cap \Omega_3) \cup Int(E_{cc,c} \cap \Omega_5) \neq \emptyset$, then any trajectory starting in $E_{cc,cc}$, $E_{c,c}$ and $E_{cc,c}$ can be asymptotically stabilized.*

By theorem 5.1 and 5.2, if the condition of theorem 5.2 holds, then the system can be asymptotically stabilized from any initial point on $\mathbb{R}^2$ and **conic switching law** can also be obtained similar to Section 3.1.2.

Example 5.3 *Consider the switched system (5.1) consisting of two subsystems with saddle points with*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} = \begin{bmatrix} -4.5 & -5.5 \\ -5.5 & -4.5 \end{bmatrix}.$$

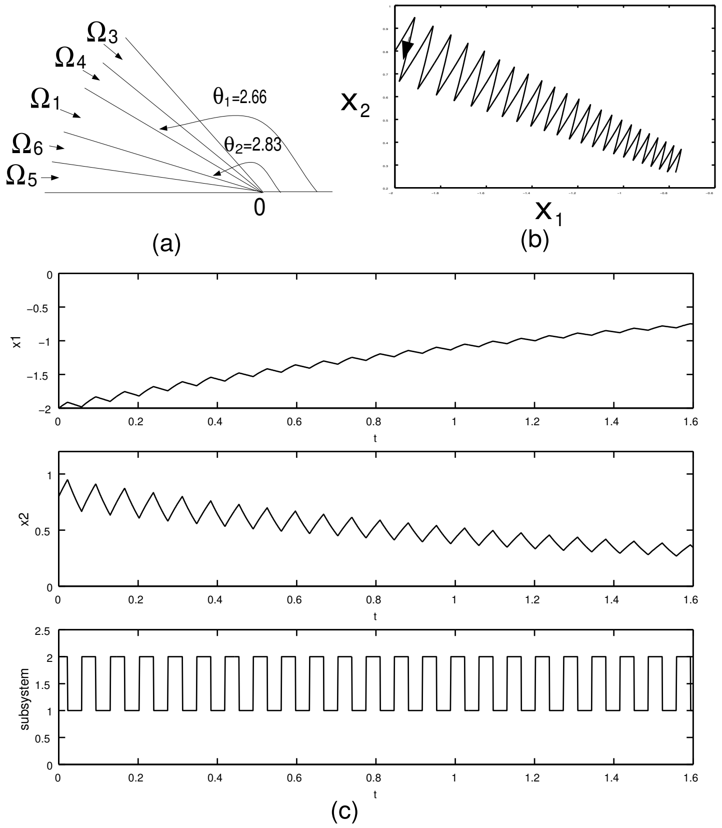

Figure 27: Example 5.3: (a). Conic regions in $E_{cc,c}$. (b) The trajectory of the system. (c) Time domain responses. ($x_0 = (-2, 0.8)^T$)

It can be shown that the trajectory starting in $E_{cc,c}$ is asymptotically stabilizable using the conic switching law. The conic regions are specified in Figure 27(a). Figure 27(b) and (c) show the trajectory in a conic region in Ω_1 and the time domain responses of the switched system.

Case 2. We only discuss Case 2(a) (see Figure 28) as Case 2(b) is analogous. In this case, $E_{c,c}$ and $E_{cc,cc}$ are Type A regions, $E^2_{cc,c}$ is a Type B region and $E^1_{cc,c}$ is a Type C region.

By adopting similar arguments as in Theorem 5.1, we can show that $Int(E^1_{cc,c} \cap \Omega_1) \cup Int(E^1_{cc,c} \cap \Omega_3) \cup Int(E^1_{cc,c} \cap \Omega_5) \neq \emptyset$ (See Figure 29; we can prove that $|\theta_{f_1 f_2}| > \pi$ holds in $E^1_{cc,c}$).

Furthermore, we can also prove that $Int(E^2_{cc,c} \cap \Omega_i) = \emptyset$, $i = 1, 3, 5$ (See Figure 30; we can prove that $|\theta_{f_1 f_2}| < \pi$ holds in $E^2_{cc,c}$).

Therefore, in this case, the system is *not asymptotically stabilizable.* Yet for initial points in $E^1_{cc,c}$, we can use the conic switching law as in Section 3.1.2 to keep the trajectory in $E^1_{cc,c}$ so that the trajectory can be stabilized.

Example 5.4 *Consider the switched system (5.1) consisting of two subsystems with saddle points with*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{4}{5} & -\frac{1}{\sqrt{10}} \\ -\frac{3}{5} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{1}{\sqrt{10}} \\ -\frac{3}{5} & \frac{3}{\sqrt{10}} \end{bmatrix}^{-1} = \begin{bmatrix} 1.6667 & 0.8889 \\ -2.0000 & -1.6667 \end{bmatrix}.$$

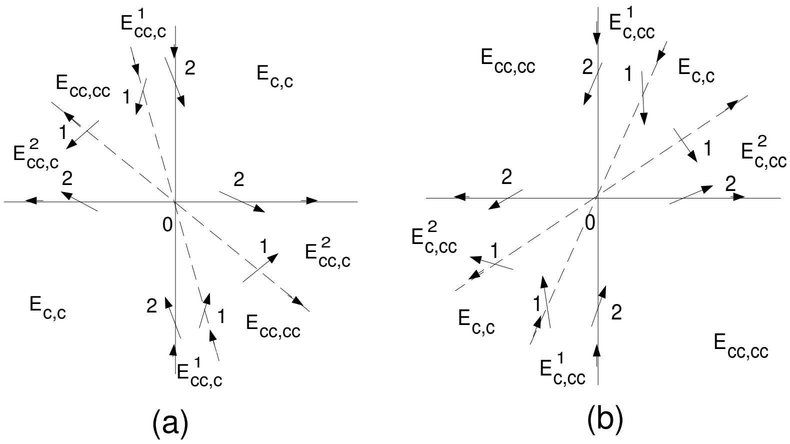

Figure 28: Case 2.

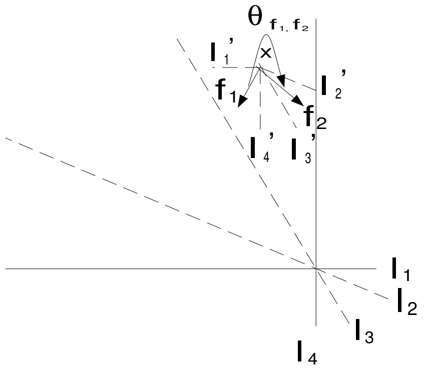

Figure 29: $|\theta_{f_1 f_2}| > \pi$ holds in $E^1_{cc,c}$ in Case 2(a).

It can be shown that the trajectory starting in $E^1_{cc,c}$ is asymptotically stabilizable using the conic switching law since all the points in $E^1_{cc,c}$ belongs to Ω_1. We keep the trajectory between the rays l_1 with angle 1.6 and l_2 with angle 1.8 (Figure 31(a)). Figure 31(b) and (c) show the trajectory in a conic region between l_1 and l_2 and the time domain responses of the switched system.

Case 3. We only discuss Case 3(a) (see Figure 32) as Case 3(b) is analogous. In this case, $E^1_{cc,cc}$ and $E^2_{cc,cc}$ are Type A regions.

In this case, we claim that $Int(E_{c,cc} \cap \Omega_1) \cup Int(E_{c,cc} \cap \Omega_3) \cup Int(E_{c,cc} \cap \Omega_5) \neq \emptyset$. This can be proved using similar arguments as in the proof of Theorem 5.1. Since for point $x \in E_{c,cc}$ which is very close to the X_2 axis, we will have $|\theta_{f_1 f_2}| > \pi$ (See Figure 33).

Furthermore, we claim that the system is *stabilizable* in this case. This is because for any initial point $x \in \mathbb{R}^2$, we can always first choose appropriate switchings such that the system trajectory is driven into $E_{c,cc}$ and then adopt the *conic switching law* as in Section 3.1.2 so as to keep the system trajectory in $Int(E_{c,cc} \cap \Omega_1)$ or $Int(E_{c,cc} \cap \Omega_3)$ or $Int(E_{c,cc} \cap \Omega_5)$. In this way, the system can be asymptotically stabilized.

Example 5.5 *Consider the switched system (5.1) consisting of two subsystems with saddle points with*

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

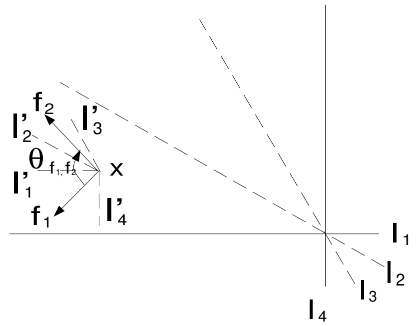

Figure 30: $|\theta_{f_1 f_2}| < \pi$ holds in $E^2_{cc,c}$ in Case 2(a).

$$A_2 = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{10}} & -\frac{2}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{10}} & -\frac{2}{\sqrt{13}} \end{bmatrix}^{-1} = \begin{bmatrix} -1.5714 & -0.8571 \\ 1.7143 & 1.5714 \end{bmatrix}.$$

It can be shown that the trajectory is asymptotically stabilizable starting from any initial point since $Int(E_{c,cc} \cap \Omega_3) \neq \emptyset$ *and trajectories from any initial point can be driven into* $E_{c,cc} \cap \Omega_3$. *In this example, we keep the trajectory between the rays* l_1 *with angle* 1 *and* l_2 *with angle* 1.5. *Figure 34(a) shows the conic regions in* $E_{c,cc}$. *Figure 34(b) and (c) show the trajectory in a conic region between* l_1 *and* l_2 *and the time domain responses of the switched system.*

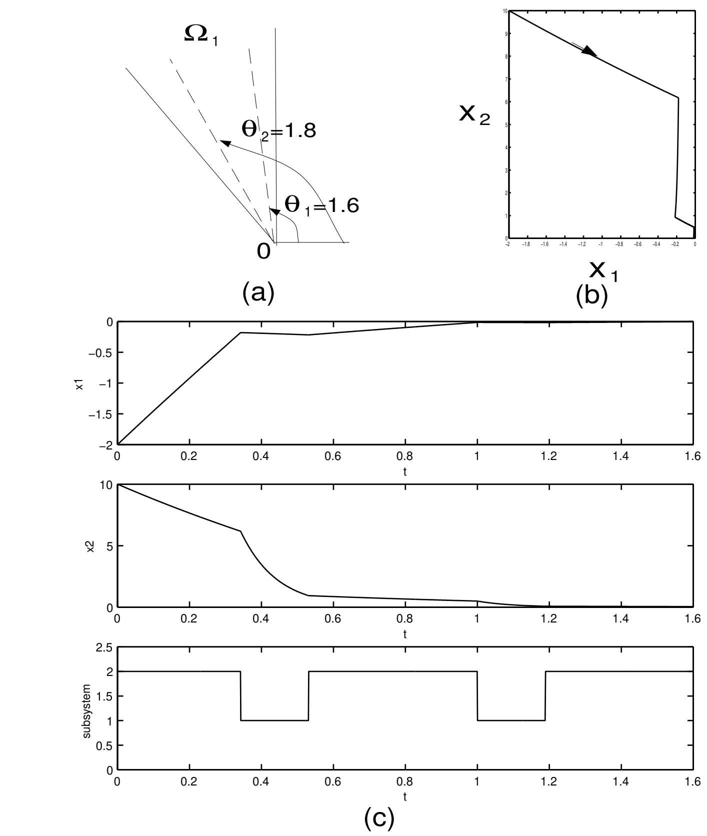

Figure 31: Example 5.4: (a). l_1 and l_2 in $E^1_{cc,c}$. (b) The trajectory of the system. (c) Time domain responses. $(x_0 = (-2, 10)^T)$

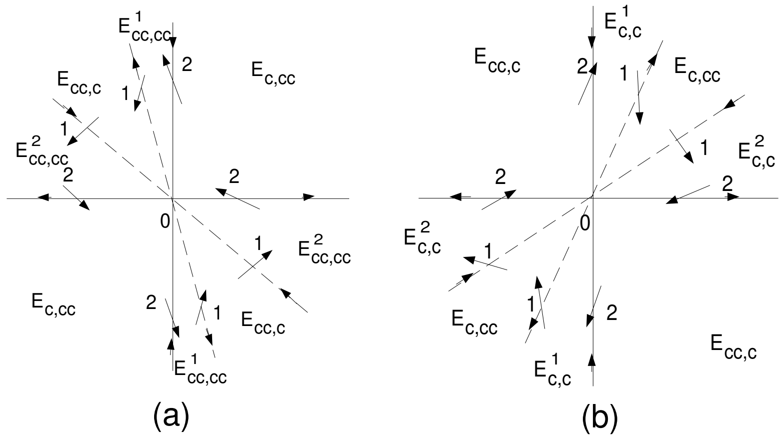

Figure 32: Case 3.

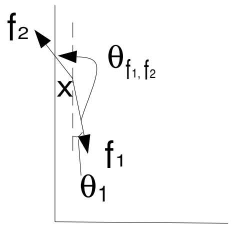

Figure 33: $|\theta_{f_1 f_2}| > \pi$ holds in $E_{c,cc}$ in Case 3(a).

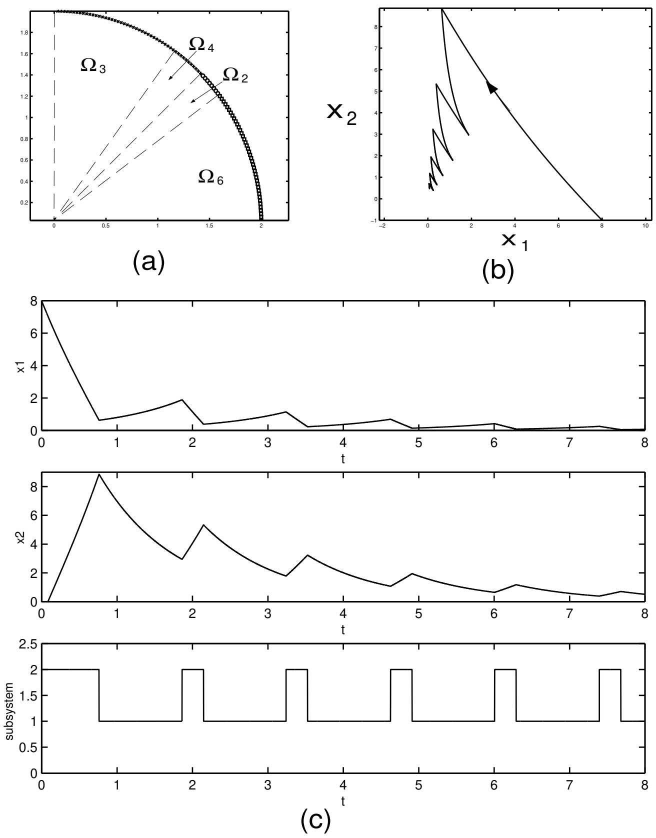

Figure 34: Example 5.5: (a). Conic regions in $E_{c,cc}$. (b) The trajectory of the system. (c) Time domain responses. ($x_0 = (8, -1)^T$)

6 Robustness Analysis of Conic Switching Laws

In the present section, we investigate the robustness problem of the previously proposed switching control law for LTI switched systems (note that a version of these results has appeared in [6]). We focus our attention on *switched systems consisting of two subsystems with foci of opposite directions.* Similar arguments can be applied to switched systems consisting of combination of other subsystems as discussed in the previous sections.

We call the conic regions in Section 3.1.2, Ω_1, Ω_3, Ω_5, *safe regions*, since the existence of a nonempty interior of such regions guarantees the existence of a stabilizing switching control law.

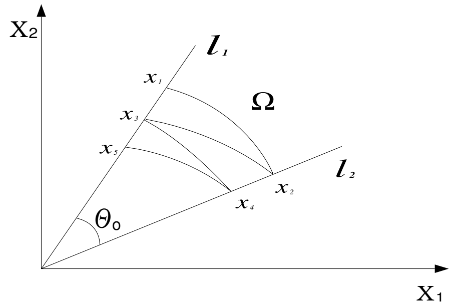

Figure 35: Switching happens on the boundary of the safe region Ω: l_1 and l_2.

The reason that the conic switching law applies lies in the fact that there exists a safe region Ω (see Figure 35) such that for every point $x_1 \in l_1 \subset \partial\Omega$, by following an appropriate subsystem (for example, we assume subsystem A_1 in the subsequent discussion), the trajectory will intersect another boundary at $x_2 \in l_2 \subset \partial\Omega$; then switch to another subsystem A_2 until it intersects l_1 again at $x_3 \in l_1$. It is satisfied that $x_3 = qx_1$ for some constant $0 < q < 1$. From this fact, we know that the above switching control law exponentially stabilizes the entire switched system.

In this section, we study the robustness of conic switching laws for switched systems (3.21). We assume the following conic switching law: for any $x_0 \in \mathbb{R}^2$, subsystem A_2 is first activated until the trajectory intersects l_1, and then proceeds following the procedure described above.

The first robustness question we will study is stated as follows.

Robustness Question 1: From the previous discussions in Section 3.1.2, it is required that the switchings happen exactly at the time when a trajectory intersects l_1 or l_2. Can this requirement be more flexible? This gives rise to the following question: Are there any conic regions R_1 and R_2 that include l_1 and l_2, respectively (see Figure 36), so that any switching that happens inside these two regions will lead to asymptotic stability?

It is clear from Figure 36 that such marginal regions are characterized by angles $\theta_{ij} > 0$, $i, j = 1, 2$, and in fact

$$R_i \quad = \quad \{x \in \mathbb{R}^2 | x = [r\cos\theta, r\sin\theta]^T, -\theta_{i1} < \theta < \theta_{i2}, 0 < r < \infty\}, \quad i = 1, 2.$$

We need to show the existence of θ_{ij} that guarantees the robustness of the switching control law.

To answer the above questions, for solution beginning from any initial condition (t_0, x_0), we assume that the trajectory follows subsystem A_2 for $t_1 - t_0$ time until it switches at $x_1' = e^{A_2(t_1-t_0)}x_0 \in R_1$. Then it follows subsystem A_1 for $t_2 - t_1$ time until it switches at $x_2' = e^{A_1(t_2-t_1)}x_1' \in R_2$. Later on it switches back to subsystem A_2 for $t_3 - t_2$ time until it arrives at $x_3' = e^{A_2(t_3-t_2)}x_2' \in R_1$, and so forth.

Assume that from any point $x_1 \in l_1$, it takes T_1 time to get $x_2 = e^{A_1T_1}x_1 \in l_2$ while following subsystem A_1 and it takes T_2 time to get back to l_1 at $x_3 = e^{A_2T_2}x_2 = e^{A_2T_2}e^{A_1T_1}x_1 \in l_1$. Clearly, T_1 and T_2 are independent of the choice of x_1. As before, we assume that $x_3 = qx_1$ for some constant $0 < q < 1$. That is, $e^{A_2T_2}e^{A_1T_1}x_1 = qx_1$, which implies that q is an eigenvalue of the matrix $e^{A_2T_2}e^{A_1T_1}$.

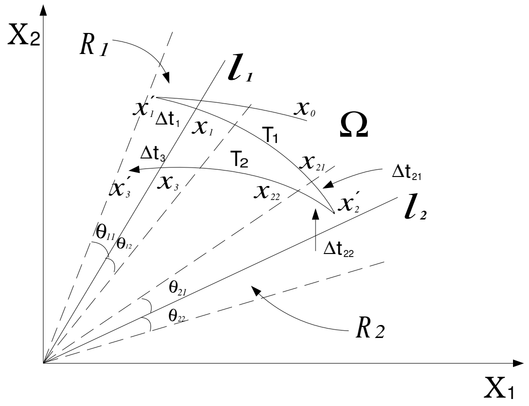

Figure 36: Switching happens inside the conic regions R_1 and R_2.

It is also clear that there exist quantities Δt_1, Δt_{21}, Δt_{22}, Δt_3, which might be negative, and points $x_{21}, x_{22} \in l_2$ and $x_1, x_3 \in l_1$ such that $x_1 = e^{A_1 \Delta t_1} x_1' \in l_1$, $x_2' = e^{A_1 \Delta t_{21}} x_{21}$, $x_{22} = e^{A_2 \Delta t_{22}} x_2' \in l_2$, $x_3 = e^{A_2 (T_2 + \Delta t_3)} x_{22} \in l_1$, $x_3' = e^{A_2 \Delta t_3} x_3$, where $t_2 - t_1 = \Delta t_1 + T_1 + \Delta t_{21}$, $t_3 - t_2 = \Delta t_{22} + T_2 + \Delta t_3$.

Denoting $\theta \overset{\triangle}{=} (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ (see Fig. 2), we have the following observation.

Observation: There exists a nonnegative continuous function $c(\theta) \geq 0$ satisfying $\lim_{\theta \to 0} c(\theta) = c(0) = 0$ such that

$$\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \quad \leq \quad c(\theta). \tag{6.1}$$

This observation can be easily proved using the polar coordinate representation of the subsystems.

Now due to quasi-periodicity of the switching law (i.e., it switches back and forth for almost the same periods of time T_1 and T_2, respectively), it suffices to show that there exist switching regions R_1, R_2 (i.e., θ_{11}, θ_{12}, θ_{21}, $\theta_{21} \geq 0$) such that no matter when the switchings happen inside regions R_1 and R_2, it is true that

$$\|x_3'\| \quad \leq \quad q_1 \|x_1'\| \text{ with a constant } 0 < q_1 < 1. \tag{6.2}$$

To see this, we compute

$$\begin{aligned}
x_3' &= e^{A_2(t_3 - t_2)} e^{A_1(t_2 - t_1)} x_1' \\
&= e^{A_2(\Delta t_{22} + T_2 + \Delta t_3)} e^{A_1(\Delta t_1 + T_1 + \Delta t_{21})} x_1' \\
&= e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} e^{A_1(\Delta t_1 + \Delta t_{21})} x_1' \\
&= e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} x_1' + e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (e^{A_1(\Delta t_1 + \Delta t_{21})} - I) x_1' \\
&= e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} x_1 + e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (I - e^{A_1 \Delta t_1}) x_1' \\
&\quad + e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (e^{A_1(\Delta t_1 + \Delta t_{21})} - I) x_1' \\
&= q e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_1 \Delta t_1} x_1' + e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (I - e^{A_1 \Delta t_1}) x_1' \\
&\quad + e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (e^{A_1(\Delta t_1 + \Delta t_{21})} - I) x_1',
\end{aligned} \tag{6.3}$$

where I denotes the identity matrix. It is now not difficult to see that $\exists \epsilon > 0$ such that when $\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \leq \epsilon$, we have

$$\|e^{A_2(\Delta t_{22} + \Delta t_3) e^{A_1 \Delta t_1}}\| \leq 1 + \frac{1-q}{8q}, \quad \|e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (I - e^{A_1 \Delta t_1})\| \leq \frac{1-q}{8},$$
$$\|e^{A_2(\Delta t_{22} + \Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (e^{A_1(\Delta t_1 + \Delta t_{21})} - I)\| \leq \frac{1-q}{4}.$$

Therefore, $\|x_3'\| \le (q + \frac{1-q}{8} + \frac{1-q}{8} + \frac{1-q}{4})\|x_1'\| = \frac{1+q}{2}\|x_1'\|$. If we let $q_1 = \frac{1+q}{2}$, since $0 < q_1 < 1$, (6.2) holds.

Now by (6.1), we know that there exist constants $\theta_{11}^0, \theta_{12}^0, \theta_{21}^0, \theta_{22}^0 > 0$ such that whenever $0 \le \theta_{ij} \le \theta_{ij}^0$, $i, j = 1, 2$, $c(\theta) \le \epsilon$. Therefore, (6.2) holds.

By induction, whenever switchings happen inside these conic regions, we always have $\|x_{2k+1}'\| \le \frac{1+q}{2}\|x_{2k-1}'\|$, for $k \in N \triangleq \{1, 2, \cdots\}$. Therefore, $x_{2k+1}' \to 0$ as $k \to \infty$. Since the trajectory between x_{2k-1}' and x_{2k+1}' for $k \ge 0$ is uniformly bounded by $\|x_{2k-1}'\|$ (denote $x_{-1}' = x_0$) and since the traveling time periods are uniformly bounded as well (for example, we can pick a bound like $\max\{T_0, 2(T_1 + T_2)\}$ for small θ_{ij}, $i, j = 1, 2$), we conclude that $\|x(t)\| \le c_0(\frac{1+q}{2})^{t-t_0}\|x_0\|$ for some constant $c_0 > 0$ (which depends on θ_{ij}, $i, j = 1, 2$). Therefore the entire switched system is exponentially stable. This proves that our conic switching law is robust in the sense introduced in Question 1. □

Another robustness question of interest is as follows.

Robustness Question 2: If switchings happen inside the strip regions R_3 and R_4 (see Figure 37), the best we can hope for is that the switching control law would drive the trajectory to the vicinity of the origin exponentially, but not to the origin. Since once the trajectory enters into the dark shaded region (see Figure 37), either subsystems can be chosen, and clearly the arbitrary choice of switching may force the trajectory to go outwards.

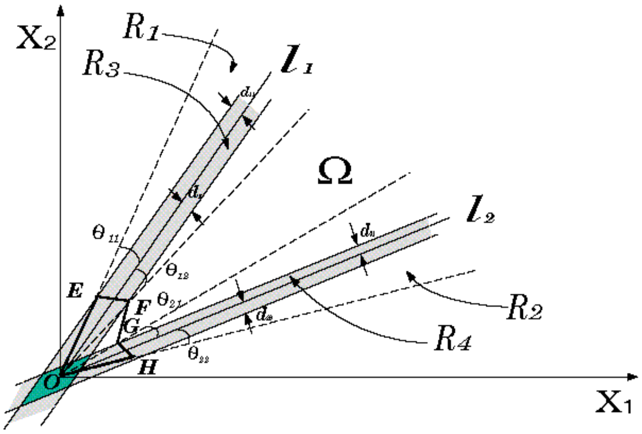

Figure 37: Switching happens inside the strip regions R_3 and R_4.

The reason that we can make the trajectory move to the vicinity of the origin is simple. From the answer to Question 1, we know that there exist $\theta_{ij} > 0$, $i, j = 1, 2$, such that when a switching happens inside the conic regions R_1 and R_2, the trajectory converges to the origin exponentially. Now pick $d_{ij} > 0$, $i, j = 1, 2$, sufficiently small (the choice depends on how close to the origin we require to move). From Figure 37, we know that there exist intersecting points E, F, G, H. Clearly, the trajectory will finally move into the polygon region $\overline{OEFGH}$ since the strips beyond this region are all inside the conic regions R_1 and R_2 respectively. □

However, once the trajectory goes into the polygon $\overline{OEFGH}$, it may leave this region if we still employ the strip switching control law. For this reason, the robustness property discussed in Question 1 is usually of more interest to us.

Example 6.1 *Consider the switched system consisting of two unstable subsystems with foci and of opposite directions, given by*

$$\dot{x}(t) = A_1 x(t), \ \dot{x}(t) = A_2 x(t), \tag{6.4}$$

with

$$A_1 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -8 \\ 4 & -3 \end{bmatrix}.$$

After some calculations, we can determine that the interior of Ω_5 is nonempty and Ω_5 is bounded by the two lines $l_1: \; x_2 = 0.9087x_1$ with angle 42.261^o and $l_2: \; x_2 = -20.9087x_1$ with angle 92.738^o. The trajectory of the system under the conic switching law is shown in Fig. 4(a).

We determine that $\|x_3\| = q\|x_1\|$, $q = 0.786$, $T_1 = 0.293$, and $T_2 = 0.2269$. Now if we let

$$\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \;\; \leq \;\; 0.0022,$$

we find that

$$\begin{aligned} \|e^{A_2(\Delta t_{22}+\Delta t_3)}\| &\leq 1.0244 \leq 1.0340 = 1 + \frac{1-q}{8q}, \\ \|e^{A_2(\Delta t_{22}+\Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (I - e^{A_1 \Delta t_1})\| &\leq 0.0263 \leq 0.0267 \leq \frac{1-q}{8}, \\ \|e^{A_2(\Delta t_{22}+\Delta t_3)} e^{A_2 T_2} e^{A_1 T_1} (e^{A_1(\Delta t_1 + \Delta t_{21})} - I)\| &\leq 0.0526 \leq 0.0535 = \frac{1-q}{4}. \end{aligned}$$

Corresponding to the above result, we find that if we choose conic regions R_1 and R_2 with $\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22} = 0.2269^o$, then according to the discussion in Robustness Question 1, the switched system is robust with respect to variation on switchings. The trajectory of the system in this case is shown in Figure 38(b).

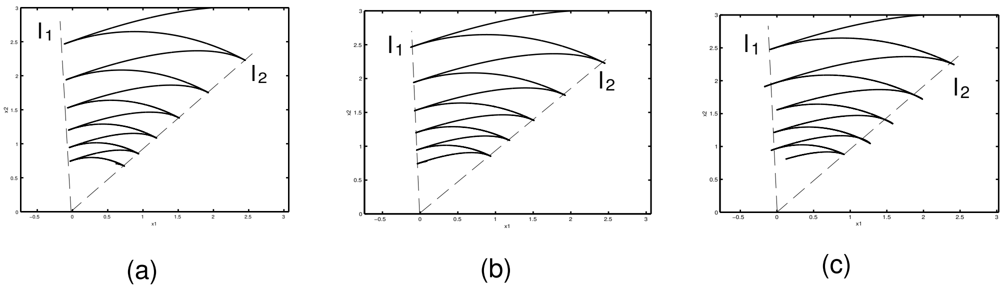

Figure 38: Trajectories for the switching system in Example 6.1.

Since the estimates in the discussion of Robustness Question 1 typically are quite conservative, we may try larger disturbances than the one given above. If we choose conic regions R_1 and R_2 with $\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22} = 3.0^o$, then we will find that the system is still exponentially stable under the conic switching law. The trajectory of the system is shown in Figure 38(c).

7 Comparison with Matrix Inequality Approaches

In [17], Pettersson introduced a LMI based stability analysis method. In Section 4.2 of [17], the $\mathcal{S}$-procedure[1] is used to formulate the LMI problem. Here we briefly describe the LMI problem with one candidate Lyapunov function for each subsystem.

LMI Analysis Problem *Find* $P_i > 0, \alpha_i \geq 0$ *and* $\alpha_{ij} \geq 0,\ i,j = 1, \cdots, N$, *such that*

1. $A_i^T P_i + P_i A_i - \alpha_i Q_i \leq 0,\ i = 1, \cdots, N.$ (7.1)
2. $(P_j - P_i) - \alpha_{ij} Q_{ij} \leq 0,\ i,j = 1, \cdots, N,\ i \neq j.$ (7.2)

In (7.1)-(7.2), Q_i which determines the conic region

$$R_i = \{x \in \mathbb{R}^n | i\text{-th subsystem is valid}\} = \{x \in \mathbb{R}^n | x^T Q_i x \leq 0\}$$

and Q_{ij} which determines the conic region

$$S_{ij} = \{x \in \mathbb{R}^n | \text{switching from } i\text{-th to } j\text{-th subsystem is valid at } x\} = \{x \in \mathbb{R}^n | x^T Q_{ij} x \leq 0\}$$

are assumed to be given.

For the design problem, we do not know R_i and S_{ij} beforehand and we want to design those regions. We can transform this problem to the following matrix inequality design problem.

Matrix Inequality Design Problem *Find* $P_i > 0, Q_i = Q_i^T, Q_{ij} = Q_{ij}^T, \alpha_i \geq 0, \alpha_{ij} \geq 0, \beta_{ij} \geq 0, \gamma_{ij} \geq 0,\ i,j = 1, \cdots, N$, *such that*

1. $A_i^T P_i + P_i A_i - \alpha_i Q_i \leq 0,\ i = 1, \cdots, N.$ (7.3)
2. $(P_j - P_i) - \alpha_{ij} Q_{ij} \leq 0,\ i,j = 1, \cdots, N,\ i \neq j.$ (7.4)
3. $Q_i - \beta_{ij} Q_{ij} \leq 0,\ i,j = 1, \cdots, N,\ i \neq j.$ (7.5)
4. $Q_j - \gamma_{ij} Q_{ij} \leq 0,\ i,j = 1, \cdots, N,\ i \neq j.$ (7.6)

Conditions 3 and 4 in the above problem are posed because of the requirements that $S_{ij} \subseteq R_i$ and $S_{ij} \subseteq R_j$.

By solving this problem, we can find a stabilizing switching law for LTI switched systems. Yet because the matrix inequality problem is complicated and nonlinear in its variables, it is generally very difficult to solve. And if we cannot obtain a solution for the problem, we may not know whether the switched system is stabilizable or not. Even for second-order LTI switched systems discussed in this paper, it is very difficult to solve (7.3)-(7.6). One may need to associate multiple candidate Lyapunov functions to each subsystem in order to find a solution.

Therefore, although matrix inequality formulations for control design seem to be general for n-th order LTI switched systems, they are difficult to solve even for second-order switched systems. Our method in this paper can solve the design problem efficiently for second-order switched systems.

8 Conclusions

In this paper, we study the stabilization problem for switched systems consisting of second-order LTI subsystems. Necessary and sufficient conditions were given for the asymptotic stabilizability of such switched systems. Also conic switching laws which asymptotically stabilize the switched systems are obtained.

The method developed in this paper uses the geometric properties of vector fields of second-order LTI systems. The fundamental idea behind the methodology is to associate with each point on $\mathbb{R}^2$ plane a subsystem that has the potential to drive the system trajectory closer and closer to the origin. We mainly study switched systems consisting of two subsystems. The stabilization problem is studied according to the vector fields generated. In particular, it is studied according to the type of the origin (focus, node or saddle point) and the direction of the vector fields at a point (clockwise, or counterclockwise). The method does not directly use Lyapunov functions. Note that the computation to obtain the conic regions are easy to carry out. And also it is shown that the conic switching laws are robust.

Switched systems consisting of LTI subsystems with unstable foci are studied first in Section 3. All the necessary terminology and ideas for the stabilization are introduced there. For general switched systems consisting of subsystems with unstable nodes or saddle points, the design of stabilizing switching control laws are studied as extensions of the method for the case of unstable foci. The paper does not provide analysis of stabilizability of switched systems consisting of mix types of subsystems (e.g. one subsystem with unstable node and another subsystem with saddle point), yet similar analysis procedures can be carried out.

As stated in Section 7, while the matrix inequality formulation can be used to obtain a stabilizing switching law, our method is more suitable for the switched systems we consider in this paper. And the switching control law we derived will make the switched system asymptotically stable. In fact, by using the switching law presented in this paper, it can be shown that the switched system is globally exponentially stable (i.e., $\exists \alpha < 0, k > 0$ such that $\|x(t)\|_2 \leq ke^{\alpha t}\|x_0\|_2$).

Since the topological structures of $\mathbb{R}^2$ and $\mathbb{R}^n, n \geq 3$ are quite different, it may be difficult to extend the method to $\mathbb{R}^n$ space directly. Up to now, there has been no necessary and sufficient conditions for the asymptotic stabilizability of switched systems consisting of n-th ($n \geq 3$) order LTI subsystems. Note that a thorough study of switched systems consisting of second-order LTI subsystems is already quite complicated and so similar studies for more general n-th order switched systems, if possible, would require more work by researchers.

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